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TOPICS ON THE RESOLVENT OF NON-SELF-ADJOINT ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract: Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega \in C^\infty$. In this paper, we consider the linear operator

$$(Pu)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\omega^{2\alpha}(x)a_{ij}(x)Q(x)\frac{\partial u(x)}{\partial x_i}\right),$$

in the space $H_\ell = L^2(\Omega)^\ell = L^2(\Omega) \times \cdots \times L^2(\Omega)$ (ℓ-times) associated with the noncoercive bilinear form that defined by

$$\mathcal{P}[u,v] = \sum\nolimits_{i,j=1}^{n} \int_\Omega \left\langle \omega(x)a_{ij}(x)Q(x)\frac{\partial u(x)}{\partial x_i}, \omega(x)\frac{\partial v(x)}{\partial x_j}\right\rangle_{\mathbf{C}^\ell} dx.$$

In view of our ealier paper (see [10]), let the conditions made on the weighted function $\omega(x)$ be sufficiently more general than [10]. In this paper we investigate the resolvent of the operator P.

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1. Introduction

Let Ω be a bounded domain with smooth boundary in R^n. We introduce the space $\mathcal{H}_\ell = W^2_{2,\omega}(\Omega)^\ell = W^2_{2,\omega}(\Omega) \times \cdots \times W^2_{2,\omega}(\Omega)$ (ℓ-times) as the space of vector functions $u(x) = (u_1(x), \ldots, u_\ell(x))$ defined on Ω with the finite norm

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$$|u|_+ = \left(\sum_{i=1}^{n} \int \omega^{2\alpha}(x) |\frac{\partial u(x)}{\partial x_i}|^2_{\mathbf{C}^\ell} dx + \int_\Omega |u(x)|^2_{\mathbf{C}^\ell} dx \right)^{1/2},$$

the notions $|\frac{\partial u(x)}{\partial x_i}|^2_{\mathbf{C}^\ell}$, $|u(x)|^2_{\mathbf{C}^\ell}$ stand for the norm in space $\mathbf{C}^\ell$. The above definition of norm has been previously used. (See [9], [10], [12].) Of course this could also be done in matrix language, but at the cost of greater notational complexity. By $\overset{\circ}{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(\Omega)^\ell$ with respect to the above norm. $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. If $\ell = 1$, then, $H = H_1$, $\mathcal{H} = \mathcal{H}_1$, and $\overset{\circ}{\mathcal{H}} = \overset{\circ}{\mathcal{H}}_1$.

In view of space $H_\ell = W^2_{2,\alpha}(\Omega)^\ell$ above, for a closed extension of the operator P (for more explain see chapter 6 of [7]), we need to extend its domain to the closed domain

$$D(P) = \{u \in \overset{\circ}{\mathcal{H}}_\ell \cap W^2_{2,loc}(\Omega)^\ell : \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\omega^2 a_{ij} Q \frac{\partial u}{\partial x_i} \right) \in H_\ell \}.$$

Here $W^2_{2,\,loc}(\Omega)^\ell = W^2_{2,\,loc}(\Omega) \times \cdots \times W^2_{2,\,loc}(\Omega)$ ($\ell -$ times) and $W^2_{2,\,loc}(\Omega)$ is the space of the class of functions $u(x)$ $(x \in \Omega)$ in the form $W^2_{2,Loc}(\Omega) = \{u : \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty\}$, where J is an arbitrary open subset of Ω. Let $\omega(x) \in C^1(\bar{\Omega}, R^+)$, $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$, and $\rho(x)|\frac{\partial \omega(x)}{\partial x_i}| \le K\omega(x)$, $i = 1, \ldots, n$, and K is a constant number. And let $0 \le \alpha < 1$. We also assume that $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$, Moreover, suppose that for each $x \in \overline{\Omega}$, the matrix function $Q(x)$ has ℓ-simple non-zero eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ in the complex plane. Let $\mu_j(x) \in C^2(\bar{\Omega}, C)$, $j = 1, \ldots, \ell$. We assume that the eigenvalues $\mu_j(x)$, $(1 \le j \le \ell)$ are arranged in the complex plane in the form

$$\mu_1(x), \ldots, \mu_\nu(x) \in R^+,\ \mu_{\nu+1}(x), \ldots, \mu_\ell(x) \in \mathbf{C} \backslash \Phi,$$

where

$$\Phi = \{z \in \mathbf{C} : |\arg z| \le \varphi\} \ , \quad \varphi \in (0, \pi).$$

Let $a_{ij}(x) \in C^2(\bar{\Omega})$, $a_{ij}(x) = \overline{a_{ji}(x)}$ for $i, j = 1, 2, \ldots, n$ and the functions a_{ij} satisfy the uniformly elliptic condition: i.e., there exists $c > 0$ such that for every $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, $x \in \Omega$ we have

$$c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) s_i \overline{s_j}. \tag{1.1}$$

Here and in the sequel, the value of the function arg $z \in (-\pi, \pi]$, and $\|T\|$ denotes the norm of the bounded operator $T : H_\ell \longrightarrow H_\ell$.

To get a feeling for the history of the subject under study, refer to the papers [1], [2], [3], [10], [12]. In [1] the authors consider the differential operator

$$Pu = -(t^\alpha A(t)u'(t))' + Q(t)u(t), \ 0 \le \alpha < 2,$$

with Dirichlet-type boundary conditions in the space
$H = L^2((0,T);\ C)$, and find the distribution function of a series of eigenvalues of the operator P, see [1]. In [2] the authors consider certain matrix elliptic differential operator A and calculate the principle term in the asymptotic expansion of the function $N_\Phi(t)$ representing the distribution of eigenvalues of the operator A in the sector Φ, see [2]. In [3], the author studies the the distribution of eigenvalues of the operator in H^n defined by

$$(Py)(t) = -(t^\alpha A(t)y'(t))' + C(t)y(t),$$

with matrix coefficients $A(t) \in C^\infty([0,1],\ \mathbf{C}^n)$, $C(t) \in C([0,1],\ \mathbf{C}^n)$, see [3]. In [10], we consider the non-selfadjoint elliptic differential operator $(Au)(x) = -\sum_{i,j=1}^n \left(\rho^{2\alpha}(x)a_{ij}(x)q(x)u'_{x_i}(x)\right)'_{x_j}$ on the space $H_\ell = L^2(\Omega)^\ell$, where $\rho(x) = dist\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $a(x) \in C^2(\overline{\Omega},\ \mathbf{C}^\ell)$, then under Dirichlet-type boundary conditions, we determine the asymptotical formula for distribution of the eigenvalues of the operator A. In [12] we have generalized the conditions to the weight function and, in addition, we changed the structure of the sector. See [12] for more information. In this paper, we consider the weighted non-selfadjoint elliptic differential operator

$$(Pu)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\omega^{2\alpha}(x)a_{ij}(x)Q(x)\frac{\partial u(x)}{\partial x_i}\right),$$

defined in $H_\ell = L_2(\Omega)^\ell$. The conditions which we made on the weighted function $\omega(x)$ be sufficiently more general than [10], and then, we investigate the asymptotical formula for the distribution of the eigenvalues of the operator P, and so we will find the resolvent of P. In this article we use the following theorems. One can find the proof of these theorems in [7].

Theorem 1. *If $a[u,v]$ is a bounded form defined every where on H, then there is a bounded operator $T \in \mathcal{B}(H)$ such that $a[u,v] = (Tu, v)$.*

Theorem 2. (First representation theorem) *Let the form $a[u,v]$ be a densely defined, closed, sectorial and sesquilinear form in Hilbert space H.*

There exist an m-sectorial operator T such that:
i. $D(T) \subset D(a)$.
ii. $D(T)$ *is a core of* a.
iii. If $u \in D(a)$ *and* $w \in H$ *and* $a[u,v] = (w,v)$ *holds for every* v *belonging to a core ofa , then* $u \in D(T)$ *and* $Tu = w$. *The m-sectorial operator* T *is uniquely determined by the condition i.*

2. Resolvent Estimate

Theorem 3. *Let* P, $\omega(t)$ *be defined as in Section 1. Let* $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$, *Moreover, suppose that for each* $x \in \bar{\Omega}$, *the matrix function* $Q(x)$ *has* ℓ*-simple non-zero eigenvalues* $\mu_1(x), \ldots, \mu_\ell(x)$ *in the complex plane. Let* $\mu_j(x) \in C^2(\bar{\Omega},\ C)$, $j = 1, \ldots, \ell$. *We assume that the eigenvalues* $\mu_j(x)$, $(1 \le j \le \ell)$ *are arranged in the complex plane in the form*

$$\mu_1(x), \ldots, \mu_\nu(x) \in R^+,\ \mu_{\nu+1}(x), \ldots, \mu_\ell(x) \in \mathbf{C} \backslash \Phi,$$

where $\Phi = \{z \in \mathbf{C} :\ |\arg z| \le \varphi\}, \varphi \in (0, \pi)$. *Then under the above conditions, the operator* P *has discrete spectrum, and for sufficiently large in modulus* $\lambda \in \Phi_\psi$, *where*

$$\Phi_\psi = \{z \in \mathbf{C} :\ \psi \le |\arg\ z| \le \varphi\},\quad \psi \in (0, \varphi),\ \varphi \in (0, \pi), \tag{2.1}$$

the inverse $(P - \lambda I)^{-1}$ *exists and is continuous in space* $H_\ell = L^2(\Omega)^\ell$, *and the following estimate holds;*

$$\left\| (P - \lambda I)^{-1} \right\| \le M_\psi |\lambda|^{-1}\ \ (|\lambda| \ge C_\psi,\ C_\psi > 0,\ \ \lambda \in \Phi_\psi).$$

Proof. Let us first extend the operator P in view of space $\mathcal{H}_\ell = W^2_{2,\omega}(\Omega)^\ell$ above, to do this (i.e., for a closed extension of the operator P) we need to extend its domain to the closed domain

$$D(P) = \{u \in \overset{\circ}{\mathcal{H}}_\ell \cap W^2_{2,loc}(\Omega)^\ell :\ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\omega^2 a_{ij} Q \frac{\partial u}{\partial x_i} \right) \in H_\ell\}.$$

(For more details, see Chapter 6 of [7]). Here $W^2_{2,\,loc}(\Omega)^\ell = W^2_{2,\,loc}(\Omega) \times \cdots \times W^2_{2,\,loc}(\Omega)$ $(\ell - times)$ where $W^2_{2,\,loc}(\Omega)$ is the space of class of functions

$u(x)$ $(x \in \Omega)$ in the form

$$W^2_{2,loc}(\Omega) = \{u : \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty \ , \ J \ \subset \ \Omega\},$$

where J is an arbitrary open subset of Ω. To prove the discreteness of the spectrum of the operator P, the fact that the spectrum of P is discrete is quite well known (see [1]), or to prove this, by the definition of the operator P and from the extension of its domain in the closed set $D(P)$ above, we know that the space $W^2_{2,\alpha}(\Omega)^\ell$ is a compact space and from $D(P) \subset W^2_{2,\alpha}(\Omega)^\ell$, it implies that the closed extension $D(P)$ is a compact space, and since the imbedding of $D(P) \subset W^2_{2,\alpha}(\Omega)^\ell$, is a compact opertor, i.e., $P : D(P) \mapsto W^2_{2,\alpha}(\Omega)^\ell$ is a compact operator on the Hilbert spaces, consequently the compact operators on Hilbert spaces have discrete spectrum, therefore the operator P has discrete spectrum.

To prove the rest of the assertion of Theorem 3, by applying the eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ of the matrix function $Q(x)$, we defined the operators P_k $(k = 1, 2, \ldots, \ell)$ on $\mathcal{H} = \mathcal{H}_1$ by

$$(P_k y)(x) \ = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\omega^{2\alpha}(x) a_{ij}(x) \mu_k \frac{\partial y(x)}{\partial x_i} \right),$$

and

$$D(P_k) \ = \{y \in \overset{\circ}{\mathcal{H}} \cap W^{2\alpha}_{2,\ loc}(\Omega) : \ P_k y \in H\}.$$

To prove the existence of the resolvent of the operator P in space H_ℓ for sufficiently large in modulus $\lambda \in \Phi_\psi$, i.e., to estimate $(P - \lambda I)^{-1}$ in space $H_\ell = L^2(\Omega)^\ell$, we must first estimate the resolvent of the operators P_k for $k = 1, \ldots, \ell$, in space $H = L^2(\Omega)$, and then, by the diagonalization method of the matrix function $Q(x)$, we estimate $(P - \lambda I)^{-1}$ in space $H_\ell = L^2(\Omega)^\ell$. We now estimate the resolvent of the operators P_k for $k = 1, \ldots, \ell$, i.e., we must estimate $(P - \lambda I)^{-1}$, by assumption, the eigenvalues $\mu_j(x), (1 \le j \le \ell)$ are arranged in the complex plane in the form

$$\mu_1(x), \ldots, \mu_\nu(x) \in R^+, \ \ \mu_{\nu+1}(x), \ldots, \mu_\ell(x) \in \mathbf{C} \backslash \Phi_\psi.$$

Therefore to estimate the resolvent of the operators P_k for $k = 1, \ldots, \ell$, we consider two cases, one case for $k = 1, \ldots, \nu$ and then for $k = \nu + 1, \ldots, \ell$. In view of $\mu_1(x), \ldots, \mu_\nu(x) \in R_+$ (i.e., in this case the eigenvalues $\mu_j(x)$ are real value), therefore, for $k = 1, \ldots, \nu$, we have $\mu_k(x) = \overline{\mu_k(x)} > 0$ $(x \in \overline{\Omega})$, consequently, for $k = 1, \ldots, \nu$ we will have $P_k = P_k^* \geq 0$ (i.e., in this case P_k, for $k = 1, \ldots, \nu$ are selfadjoint), and from the well known theorems for selfajoint operators, it is easily to see that estimate $(P_k - \lambda I)^{-1}$ exists for $k = 1, \ldots, \nu$, i.e.,

$$\|(P_k - \lambda I)\| \leq M|\lambda|^{-\frac{1}{2}} \qquad (\lambda \in \overline{\Phi}, \ |\lambda| \geq 1).$$

For the case $k = \nu+1, \ldots, \ell$, by (2.2), and by below observations the estimate of the resolvent of P_k are obtained. Hence we can get the estimate of the resolvent of the operators P_k, i.e., for $k = 1, \ldots, \ell$ we will have

$$\|(P_k - \lambda I)\| \leq M|\lambda|^{-\frac{1}{2}} \qquad (\lambda \in \overline{\Phi}, \ |\lambda| \geq 1).$$

By the assumption

$$\mu_{\nu+1}(x), \ldots, \mu_\ell(x) \in \mathbf{C} \backslash \Phi_\psi,$$

and by assuming that $\varphi < \frac{\pi}{16}$ in the assumption Φ_ψ in the (2.1) above, since for $k \in \{\nu+1, \ldots, \ell\}$ the eigenvalue $\mu_{\nu+1}(x), \ldots, \mu_\ell(x)$ of $Q(x)$ lie outside of the closed sector Φ_ψ in (2.1), by $\varphi < \frac{\pi}{16}$, implies that the angles between the oscillation of variation of the functions of eigenvalues $\mu_{\nu+1}(x), \ldots, \mu_\ell(x)$ of $Q(x)$ are $< \frac{\pi}{16}$, i.e., we will have

$$|\arg\{\mu_k(x_1)\mu_k^{-1}(x_2)\}| < \frac{\pi}{16}, (k \in \{\nu+1, \ldots, \ell\}, \ x_1, \ x_2 \in \overline{\Omega}).$$

We now construct the nonnegative functions, $\varphi_{k_1}(x), \ldots, \varphi_{k_m}(x) \in C^\infty(\overline{\Omega})$ with the following properties:

$$\sum_{i=1}^{m} \varphi_{k_i}^2(x) \equiv 1 \ \ (x \in \overline{\Omega}), \ \ |\arg\{\mu_k(x_1)\mu_k^{-1}(x_2)\}| < \frac{\pi}{16} \ .$$

Here $x_1, x_2 \in$ *supp* φ_{k_i}. Now let us construct the functions $\mu_{k_r}(x) \in C^2(\overline{\Omega})$ such that

$$\mu_{k_r}(x) = \mu_k(x), \ \ (\forall\, x \in supp\, \varphi_{k_r}), \ \ \mu_{k_r}(x) \notin \overline{\Phi}, \ \ (\forall\, x \in \overline{\Omega}),$$

$$|\arg\{\mu_{k_r}(x_1)\mu_{k_r}^{-1}(x_2)\}| \leq \frac{\pi}{8}, \ \ (\forall\, x_1, x_2 \in \overline{\Phi}). \tag{2.2}$$

For $k = \nu+1, \ldots, \ell$ by applying the functions $\mu_{k_r}(x)$, let us define the operators P_{k_r} in H by setting

$$(P_{k_r} y)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\omega^{2\alpha}(x) a_{ij}(x) \mu_{k_r}(x) \frac{\partial y(x)}{\partial x_i} \right),$$

$$D(P_{k_r}) = \{y \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\,loc}(\Omega) : \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\omega^{2\alpha} a_{ij}\mu_{k_r}\frac{\partial y(x)}{\partial x_i}\right) \in H\}.$$

For $k = \nu + 1, \ldots, \ell$ and $r = 1, \ldots, m$, by (2.2) there exist a complex number $Z_{k_r} \in C$, $|Z_{k_r}| = 1$ such that $\forall\, x \in \overline{\Omega},\ \lambda \in \overline{\Phi}$ we have

$$c' \leq Re\{Z_{k_r}\mu_{k_r}(x)\},\ \ c'|\lambda| \leq -Re\{Z_{k_r}\lambda\}, \quad c' > 0. \tag{2.3}$$

In view of the uniformly elliptic condition in (1.1), we have

$$c|s|^2 = c\sum_{i=1}^{n}|s_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j}, \ \ (c > 0,\ s = (s_1, \ldots, s_n) \in \mathbf{C}^n\ ),$$

take $s_i = \frac{\partial y(x)}{\partial x_i}$ implies that

$$c\sum_{i=1}^{n}|\frac{\partial y(x)}{\partial x_i}|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial y(x)}{\partial x_i}\overline{\frac{\partial y(x)}{\partial x_j}},$$

and since for $y \in D(P_{k_r}),\ \ k = \nu + 1, \ldots, \ell$ in (2.3), we have

$$c' \leq Re\{Z_{k_r}\mu_{k_r}(x)\}.$$

We then multiply these two positive relations with each other, implies that

$$c_1\sum_{i=1}^{n}|\frac{\partial y(x)}{\partial x_i}|^2 \ \ \leq \ \ Re\{Z_{k_r}\sum_{i,j=1}^{n} a_{ij}\mu_{k_r}\frac{\partial y(x)}{\partial x_i}\overline{\frac{\partial y(x)}{\partial x_j}}\}.$$

For proof of others details, see [9]. Therefore we conclude that

$$\left\|\omega^{\alpha}\frac{\partial}{\partial x_i}(P_{k_r} - \lambda I)^{-1} f\right\|_H \leq M'|\lambda|^{-\frac{1}{2}}\|f\|_H,$$

and hence,

$$\|\omega^{\alpha}\frac{\partial}{\partial x_i}(P_{k_r} - \lambda I)^{-1}\|_H \leq M'|\lambda|^{-\frac{1}{2}}.$$

Now according to Hardy's inequality and (2.3), we conclude that

$$\|\omega^{2\alpha}\rho(P_{k_r} - \lambda I)^{-1}\| \leq M_2\sum_{i=1}^{n}\|\omega\frac{\partial}{\partial x_i}(P_{k_r} - \lambda I)^{-1}\| \leq M'|\lambda|^{-\frac{1}{2}}. \tag{2.4}$$

Let us now introduce in $\mathcal{H}_\ell$ the operator:

$$G_k(\lambda) = \sum_{r=1}^{n} \varphi_{k_r}(P_{k_r} - \lambda I)^{-1}\varphi_{k_r}$$

for $k = \nu + 1, \ldots, \ell$. Here φ_{k_r} is the operator of multiplication by the function $\varphi_{k_r}(x)$. It is easy to verify that

$$(P_k - \lambda I)G_k(\lambda) = I + \omega^{2\alpha}(x)\rho^{-1}(x)\sum_{r=1}^{m} \beta_{k_r}(x)(P_{k_r} - \lambda I)^{-1}\varphi_{k_r}$$

$$+\omega^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m} \gamma_{k_{ir}}(x)\frac{\partial}{\partial x_i}(P_{k_r} - \lambda I)^{-1}\varphi_{k_r}, \tag{2.5}$$

by taking $F_k(\lambda)$ equals to

$$\omega^{2\alpha}(x)\rho^{-1}(x)\sum_{r=1}^{m} \beta_{k_r}(x)(P_{k_r} - \lambda I)^{-1}\varphi_{k_r}$$
$$+\omega^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m} \gamma_{k_{ir}}(x)\frac{\partial}{\partial x_i}(P_{k_r} - \lambda I)^{-1}\varphi_{k_r}$$

we will have

$$(P_k - \lambda I)G_k(\lambda) = I + F_k(\lambda), \quad \|F_k(\lambda)\| \le M'|\lambda|^{-\frac{1}{2}},$$

where β_{k_r}, $\gamma_{k_{ir}}(x) \in L_\infty(\Omega), (x \in \Omega)$ are bounded functions, *supp* β_{k_r}, *supp* $\gamma_{k_{ir}}(x) \subset$ *supp* φ_{k_r}. Applying (2.2)-(2.5) we will have:

$$\|(P_k - \lambda I)G_k(\lambda) - I\| \le M'|\lambda|^{-\frac{1}{2}}, (\lambda \in \overline{\Phi}, \ |\lambda| \ge 1).$$

So we have the following representation

$$((P_k - \lambda I)(G_k(\lambda)))^{-1} = (I + F_k(\lambda))^{-1} = I + F'_k(\lambda),$$

where

$$\|F'_k(\lambda)\| \le M|\lambda|^{-\frac{1}{2}}, (\lambda \in \overline{\Phi}, \ |\lambda| \ge C_0).$$

Therefore

$$(P_k - \lambda I)^{-1} = (G_k(\lambda))(I + F'_k(\lambda)), \quad \|F'_k(\lambda)\| \le M|\lambda|^{-\frac{1}{2}}.$$

Consequently, for $k = 1, \ldots, \ell$ we will have

$$\|(P_k - \lambda I)^{-1}\| \le M|\lambda|^{-\frac{1}{2}}, \quad \lambda \in \overline{\Phi},\ |\lambda| \ge C_0. \tag{2.6}$$

By the assumption in Section 1, we now diagonalize the matrix function $Q(x)$ as follows:

$$Q(x) = U(x)\Lambda(x)U^{-1}(x), \text{ where } U(x), U^{-1}(x) \in C^2(\overline{\Omega}, End\, \mathbf{C}^\ell)$$

and

$$\Lambda(x) = diag\{\mu_1(x), \ldots, \mu_\ell(x)\}.$$

Put $\Gamma(\lambda) = UB(\lambda)U^{-1}$, where the operator $B(\lambda)$ has the representation

$$B(\lambda) = diag\{(P_1 - \lambda I)^{-1}, \ldots, (P_\ell - \lambda I)^{-1}\},$$

in the direct sum

$$H_\ell = H \oplus \cdots \oplus H \quad (\ell\text{-times}),$$

where $\lambda \in \overline{\Phi}\backslash R_+,\ |\lambda| \ge C_0$ and $(Uu)(x) = U(x)u(x),\ (u \in H_\ell)$. It is easy to verify the following equality

$$\begin{aligned}(A - \lambda I)\Gamma(\lambda) &= I + \omega^{2\alpha}(x)\rho^{-1}(x)q_0(x)B(\lambda)U^{-1} \\ &+ \omega^{2\alpha}(x)\sum_{i=1}^{n} q_i(x)\frac{\partial}{\partial x_i}B(\lambda)U^{-1}. \end{aligned} \tag{2.7}$$

By taking $F(\lambda)$ equals to

$$\omega^{2\alpha}(x)\rho^{-1}(x)q_0(x)B(\lambda)U^{-1} + \omega^{2\alpha}(x)\sum_{i=1}^{n} q_i(x)\frac{\partial}{\partial x_i}B(\lambda)U^{-1},$$

we will have

$$(A - \lambda I)\Gamma(\lambda) = I + F(\lambda) \qquad (\|F(\lambda)\| \le M|\lambda|^{-\frac{1}{2}}),$$

where $q_i(x) \in C(\overline{\Omega}, End\ \mathbf{C}^\ell),\quad i = 0, 1, \ldots, n$. Applying (2.5)-(2.7), we will have

$$((A - \lambda I)\Gamma(\lambda))^{-1} = (I + F(\lambda))^{-1} = I + F'(\lambda)$$

$$\|F'(\lambda)\| \le M'_\psi|\lambda|^{-\frac{1}{2}} \qquad (\lambda \in \Phi_\psi,\ |\lambda| \ge C_\psi,\ \forall\psi \in (0, \varphi)).$$

Thus

$$(A - \lambda I)^{-1} = \Gamma(\lambda)(I + F'(\lambda))$$

$$\|F'(\lambda)\| \leq M_\psi |\lambda|^{-\frac{1}{2}} \qquad (\lambda \in \Phi_\psi,\ |\lambda| \geq C_\psi,\ \forall \psi \in (0,\varphi)).$$

Thus $(A-\lambda I)^{-1}$ exists and so becomes continuous and we have

$$\|(A-\lambda I)^{-1}\| \leq M_\psi |\lambda|^{-1}.$$

Now the proof of Theorem 3 is complete. □

We can also prove the following results. For every natural number s, such that

$$s > \omega = \max\{\frac{n}{2}, \frac{n-1}{2-2\alpha}\}, \qquad s \in (\omega, \omega+1],$$

we will have

$$|tr(A-\lambda I)^{-s} - tr\ UB(\lambda)^s U^{-1}| \leq M|\lambda|^{-\frac{1}{2}}|\Gamma(\lambda)|_s^s,$$

where the symbols tr, $|\ \ |_s$ denote the trace of a trace-class operator and the σ_s-norm of the operator (see [4]).

By using the representation of the operator $B(\lambda)$ we obtain

$$tr\ UB(\lambda)^s U^{-1} = tr\ B(\lambda)^s = tr\ B_s(\lambda),$$

where $B_s(\lambda) = diag\{(P_1-\lambda I)^{-s}, \ldots, (P_\ell - \lambda I)^{-s}\}$. If we estimate $|\Gamma(\lambda)|_s^s$ by using (2.5) and (2.7) we obtain

$$|\sum_{i=1}^{+\infty}(\lambda'_i - \lambda)^{-s} - \sum_{j=1}^{\ell}\sum_{i=1}^{+\infty}(\lambda_{ij}-\lambda)^{-s}| \leq M_\psi |\lambda|^{\omega - s - \frac{1}{4}}, \qquad (\lambda \in \Phi_\psi\ ),$$

where $\lambda'_1, \lambda'_2, \ldots, \lambda_{1j}, \lambda_{2j}, \ldots$ denote the (ev) of the operators P and P_j, respectively. Using the contour integral method which is proved in Section 3, or as in [5,§4] for $\omega \neq 1, 2, \ldots$ we can again obtain the following relation; i.e.,

$$\sum_{i=1}^{+\infty}(\lambda'_i + \tau)^{-s} = \sum_{j=1}^{\nu}\sum_{i=1}^{+\infty}(\lambda_{ij}+\tau)^{-s} + O(\tau^{\omega - s - \frac{1}{4}}), \quad \tau \to \infty.$$

Keeping in mind that $\lambda_{ij} > 0$ $(i = 1, 2, \ldots,\ \ j = 1, \ldots, \nu)$, it is easy to establish the asymptotical formula;

$$\int_0^{+\infty} \frac{dN(t)}{(t+\tau)^s} \sim \sum_{i=1}^{\nu} \int_0^{+\infty} \frac{dN_i(t)}{(t+\tau)^s}, \qquad \tau \to +\infty,$$

where $N_i(t) = card\{j :\ \lambda_{ij} \leq t\},\ \ i = 1, \ldots, \nu$ which are well known asymptotical formulas for the function $N_i(t)$ (see [6]).

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