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THREE-LAYER HELE-SHAW DISPLACEMENT WITH AN INTERMEDIATE NON-NEWTONIAN FLUID

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Abstract: We study the displacement of two Stokes immiscible fluids in a porous medium, approximated by the Hele-Shaw horizontal model. An intermediate non-Newtonian polymer-solute, whose viscosity is depending on the velocity, is considered between the initial fluids. The linear stability problem of this three-layer displacement does not make sense. If the intermediate viscosity depends on velocity *and* on the polymer concentration, we can obtain a minimization of the Saffman-Taylor instability.

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1. Introduction

We consider a displacing fluid (water) with constant viscosity $\mu_W > 0$, which is pushing a second immiscible fluid (oil) with constant viscosity $\mu_O > \mu_W$, in a horizontal Hele-Shaw cell. A well-known result is given by Saffman and Taylor [15]: the flow is unstable.

An intermediate non-Newtonian liquid, with a *variable* viscosity depending on the velocity, is considered between water and oil. We study the effect of the intermediate liquid on the flow stability.

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The Hele-Shaw cell is parallel with the fix plane x_1Oy. The flow is from the left to the right, due to the water velocity U far upstream ($x_1 \to -\infty$). In the moving coordinate system $x = x_1 - Ut$, the displacing and displaced fluids are contained in the regions $x < -L$ and $x > 0$. The intermediate region is the interval $-L < x < 0$ and contains the non-Newtonian liquid. The Oz axis is orthogonal on the plates. The gravity effects are neglected.

As usual in the Hele-Shaw model, between the above three immisicble fluids we have two sharp interfaces. We prove that the system which governs the linear stability of the interfaces of this three-layer displacement does not make sense.

Some previous results were obtained when the middle region is filled by a polymer solution with variable viscosity, which is function of the polymer concentration c, or is verifying a diffusion equation. In the first case - see [3], [6], [7], [10] - it was proved that an "optimal" viscosity profile exists, which can give a significant improvement of the stability. In the second case - see [4], [5], [11] - it was proved that the diffusion is improving the stability, compared with the Saffman-Taylor case, when the viscosity profile is verifying some specific conditions.

In [14] we proved that several intermediate Stokes liquids with constant viscosities, inserted between the initial fluids, can not minimize the Saffman-Taylor instability.

The use of polymer-solute in the middle region is related with liquids which are non-Newtonian. The relation between tangential stress and deformation tensor is not linear, or/and the viscosity in the middle region depends on some parameters of the flow.

When an Oldroyd-B fluid is displaced by air or two Olrdroyd-B fluids are displacing in a Hele-Shaw cell, the interfaces are much more unstable compared with the Saffman-Taylor displacement - see [12] and [16].

In Section 2 we consider an intermediate polymer-solute whose viscosity is depending only on the displacing velocity U far upstream. The linear stability problem does not make sense.

In Section 3 we consider an intermediate viscosity depending on U *and* on the polymer concentration c. With some additional conditions, we get a stabilizing effect, compared with the Saffman-Taylor case.

2. When μ is depending on U

A non-Newtonian model of a liquid with viscosity depending on the velocity was proposed in [8]. A Darcy's law type was obtained for a shear-rate dependent viscosity. This model can be used to justify the flows studied in [1], [2].

Consider a horizontal Hele-Shaw cell with the gap b between the plates. The pressure, the averaged velocities (across the gap b) and the viscosity are denoted by $p, (u, v, w), \mu$. The component w is neglected. The following equations are considered (in the cited above papers) for the flow in the middle region:

$$\nabla_{x,y} p = -\mu(\mathbf{u}^2/b^2)\mathbf{u}, \quad p_z = 0,$$

$$\mathbf{u} = (u, v), \quad u_x + v_y = 0, \tag{1}$$

$$\mu(\mathbf{u}^2) = \mu_S \frac{1 + a\mathbf{u}^2\tau}{1 + \mathbf{u}^2\tau} \cdot \frac{b^2}{12},$$

$$d\mu/d\theta = \mu_S \frac{\tau(a-1)}{(1+\theta\tau)^2} \cdot \frac{b^2}{12}, \quad \theta = \mathbf{u}^2, \tag{2}$$

where τ is a characteristic time. The parameter a is governing the sign of $d\mu/d\theta$. We have shear thinning for $a < 1$ and shear thickening for $a > 1$.

In [8] is given the following explanation for the Darcy type law (1). A dependence of μ on the square of the trace of the rate-of-strain-tensor is assumed. For a Poiseuille flow, that means μ is function of $\mathbf{u_z}^2$.

The first step is to use the general form of the flow equations - see for example [9]. Thus the unsteady flow is governed by the equations

$$\mathbf{u}_t = -\nabla p + [\mu(\phi)\mathbf{u}_z]_z = -\nabla p + \eta \mathbf{u}_{zz},$$

$$\eta(\phi) = [\mu(\phi) + 2\phi \ d\mu/d\phi], \quad \phi = \mathbf{u}_z^2, \tag{3}$$

where η is the "effective viscosity". We need a positive η. To this end we impose the condition

$$\mu(\phi) + 2\phi \ d\mu/d\phi > 0. \tag{4}$$

Therefore, in the steady 2D case, the general flow equations with viscosity depending on the rate-of-strain tensor are

$$\nabla_2 p = \{\mu(\mathbf{u_z^2})\mathbf{u_z}\}_z, \quad \mathbf{u} = (u, v), \quad u_x + v_y = 0, \tag{5}$$

where the index 2 on the ∇ operator stands for x, y derivatives. We integrate with respect to z and get (recall p is not depending on z)

$$z\nabla_2 p = \mu(\mathbf{u_z^2})\mathbf{u_z}. \tag{6}$$

The second step is to find $\mathbf{u}_z$ as function of $\nabla_2 p$. The invertibility of the above relation near $\mathbf{u}_z = 0$, by using the implicit function theorem, is possible only if

$$h_\psi(0) > 0, \quad h(\psi) = \psi\mu(\psi^2), \ \ \psi = \mathbf{u_z}.$$

We have $h_\psi = \mu(\psi^2)+2\psi^2\mu_\psi(\psi^2)$, therefore the invertibility condition is exactly the inequality (4). The inversion procedure is performed in [8]. The last step for obtaining the Darcy type law (1) is the average procedure of the velocity across the Hele-Shaw gap. Thus $\mathbf{u}$ is obtained in terms of the pressure gradient.

We consider the basic solution with two straight interfaces:

$$x_L = -L, \quad x_R = 0;$$

$$u = U, v = 0; \quad P_x = -\mu U; P_y = 0;$$

$$\mu = \mu_W,\ x < -L; \quad \mu = \mu_O,\ x > 0;$$

$$\mu_W < \mu = \mu(U^2) < \mu_O, \quad x \in (-L, 0). \tag{7}$$

In this section we study the linear stability of the solution (7).

The small perturbations are denoted by u', v', p', μ'.

The viscosity perturbation μ' is obtained as follows. In the frame of the linear stability analysis, we neglect u'^2, v'^2 and get

$$E := \mu((U + u')^2 + v'^2) \approx \mu(U^2 + 2Uu').$$

The first-order Taylor expansion is giving

$$E = \mu(U^2) + 2Uu' \cdot \mu_\theta(U^2) := \mu(\theta) + \mu',$$

$$\mu' := 2\,U\,\mu_\theta\,u', \quad \theta = U^2, \quad \mu_\theta = d\mu/d\theta. \tag{8}$$

We insert u', v', p', μ' in the flow equations (1) and obtain

$$(P + p')_x = -(\mu + \mu')(U + u'),$$

$$(P + p')_y = -(\mu + \mu')(v'),$$

$$(u + u')_x + (v + v')_y = 0 \Rightarrow \tag{9}$$

$$u'_x + v'_y = 0, \tag{10}$$

$$p'_x = -\mu(\theta)u' - 2\theta\mu_\theta u', \tag{11}$$

$$p'_y = -\mu(\theta)v'. \tag{12}$$

We use a Fourier decomposition for the velocity perturbation

$$u'(x, y, t) = f(x)\exp(iky + \sigma t)$$

and from (10) and (12) we get

$$v' = -[f_x/ik]\exp(iky + \sigma t),$$
$$p' = -\mu(\theta)[f_x/k^2]\exp(iky + \sigma t). \tag{13}$$

By cross derivation of the pressure perturbations (11) and (12) it follows the amplitude equation:

$$[\mu(\theta)u' + 2\theta\mu_\theta u']_y = [\mu(\theta)v']_x$$
$$\Rightarrow -\mu f_{xx} + k^2(\mu + 2\theta\mu_\theta)f = 0. \tag{14}$$

The above equation is quite similar with the relation (11) of [14], but the coefficient of f is $k^2(\mu + 2\theta\mu_\theta)$ instead of k^2.

The relation (14) holds for all $x \in R\ x \neq -L,\ x \neq 0$.

We use the relation (2) and get

$$\mu + 2\theta\mu_\theta = \left[\frac{1 + a\theta\tau}{1 + \theta\tau} + 2\theta\frac{\tau(a-1)}{(1+\theta\tau)^2}\right]\frac{\mu_S b^2}{12} > 0$$
$$\Leftrightarrow (\theta\tau)^2 a + (\theta\tau)(3a - 1) + 1 > 0, \quad \forall(\theta\tau). \tag{15}$$

The last above inequality is verified only if we impose the condition

$$\Delta_1 = (3a-1)^2 - 4a < 0 \Leftrightarrow 9a^2 - 10a + 1 < 0$$
$$\Leftrightarrow a \in (1/9, 1). \tag{16}$$

Therefore we get the following result:

$$\mu + 2\theta\mu_\theta > 0 \Leftrightarrow a \in (1/9, 1). \tag{17}$$

The inequality $\mu + 2\theta\mu_\theta > 0$ is quite similar with the restriction (4) of [8]. In this paper we consider the viscosity profiles which verify the condition (17).

We introduce the notation

$$\gamma = k(1 + 2\theta\mu_\theta/\mu)^{1/2}, \tag{18}$$

thus from (17) it follows $\gamma > 0$. The relation (14) becomes

$$-f_{xx} + \gamma^2 f = 0, \quad x \neq \{-L, 0\}.$$

We need far decay perturbations, thus by using again the condition (17) (that means $\gamma > 0$) we obtain

$$f(x) = f(-L)\exp[(x+L)\gamma],\ x < -L,$$

$$f(x) = f(0)\exp(-\gamma x), \;\; x > 0,$$
$$f_x^-(-L) = \gamma f(-L), \quad f_x^+(0) = -\gamma f(0), \tag{19}$$

where the indices $^-,^+$ stands for "left" and "right" limits.

The solution of (14) inside the intermediate region is

$$f(x) = Ae^{\gamma x}, \tag{20}$$

because a term of the form $Be^{-\gamma x}$ becomes very large for large positive γ (recall $x < 0$) and the amplitudes f must be small in the frame of the linear stability analysis.

We study now the perturbed interfaces in $x = 0, x = -L$. Near the point $x = 0$ we consider the perturbed interface denoted by $g(x, y, t)$. As the interface is a material one, in the first approximation we get

$$g_t = u' \Rightarrow g(0, y, t) = [f(0)/\sigma]\exp(iky + \sigma t). \tag{21}$$

The *perturbed pressure* is obtained by using the Darcy law (1), the relation (13) and the first-order Taylor expansion for the basic pressure P near $x = 0$:

$$\begin{aligned} p^+(0) &= P(0, y, t) + P_x^+(0, y, t) \cdot g(0, y, t) + p'^+(0, y, t) \\ &= P(0) - \mu^+(0)\left[\frac{Uf(0)}{\sigma} + \frac{f_x^+(0)}{k^2}\right]\exp(iky + \sigma t). \end{aligned} \tag{22}$$

The same procedure is used to get the left limit of the pressure in $x = 0$ and it follows

$$p^-(0) = P(0) - \mu^-(0)\left[\frac{Uf(0)}{\sigma} + \frac{f_x^-(0)}{k^2}\right]\exp(iky + \sigma t). \tag{23}$$

A similar relation can be obtained for the point $x = -L$, which will be used in the relation (27) below.

On the interfaces $x = 0, x = -L$ we consider the surface tensions $T(0), T(-L)$. We use the Laplace law and get

$$p^+(0) - p^-(0) = T(0)g_{yy}(0, y, t),$$
$$p^+(-L) - p^-(-L) = T(-L)g_{yy}(-L, y, t). \tag{24}$$

For simplicity, we will use notations

$$f_0 = f(0), \quad f_L = f(-L), \quad T_0 = T(0), \quad T_L = T(-L),$$
$$f_{xo}^{+,-} = f_x^{+,-}(0), \quad f_{xL}^{+,-} = f_x^{+,-}(-L),$$

$$\mu_0^{+,-} = \mu^{+,-}(0), \quad \mu_L^{+,-} = \mu^{+,-}(-L). \tag{25}$$

From relations (22) - (24) it follows

$$\mu_0^- \left[\frac{Uf_0}{\sigma} + \frac{f_{x0}^-}{k^2}\right] - \mu_0^+ \left[\frac{Uf_0}{\sigma} + \frac{f_{x0}^+}{k^2}\right] = -T_0 \frac{f_0}{\sigma} k^2, \tag{26}$$

$$\mu_L^- \left[\frac{Uf_L}{\sigma} + \frac{f_{xL}^-}{k^2}\right] - \mu_L^+ \left[\frac{Uf_L}{\sigma} + \frac{f_{xL}^-}{k^2}\right] = -T_L \frac{f_L}{\sigma} k^2. \tag{27}$$

By direct calculations, as in [14], we obtain

$$\mu_0^- f_x^-(0) - \mu_0^+ f_x^+(0) = \frac{k^2 U[\mu_0^+ - \mu_0^-] - k^4 T_0}{\sigma} f_0, \tag{28}$$

$$\mu_L^- f_x^-(-L) - \mu_L^+ f_x^+(-L) = \frac{k^2 U[\mu_L^+ - \mu_L^-] - k^4 T_L}{\sigma} f_L. \tag{29}$$

Therefore the interfaces stability is governed by the equation (14) and the boundary conditions (28), (29). We recall that μ is not depending on x; we consider

$$\mu_W < \mu^-(0) = \mu^+(-L) = \mu(U) < \mu_O.$$

All growth rates σ must verify the equations (28) - (29).

We use the far filed conditions (19) for f. Thus all limit values of f_x in $x = -L, x = 0$ contains the factor γ given by (18).

We compute σ from both equations (28) - (29), we simplify with γ in the denominators, and obtain

$$\frac{kU(\mu - \mu_W) - k^3 T_0}{\mu_W - \mu} = \frac{kU(\mu_O - \mu) - k^3 T_L}{\mu + \mu_O}. \tag{30}$$

We equate the coefficient of k, k^3, thus it follows

$$\mu_O = 0, \quad \mu = \mu_W T_L/(T_L + T_0). \tag{31}$$

Both above relations are in contradiction with our hypothesis:

i) The oil viscosity must be strictly positive.

ii) μ is depending on U and is not depending on T_0, T_L.

As a consequence, the growth rates σ can not exists. The stability problem (14), (28), (29) does not make sense.

3. When μ is depending on U and c

Let us consider a polymer-solute in the middle region, with a variable viscosity μ of the type

$$\mu_W < \mu = \mu(c, U^2) < \mu_O,$$

where $c = c(x)$ is the polymer concentration. In [13] was pointed out that, for diluted polymer-solutes, μ is invertible with respect to c. Thus the viscosity μ in the middle region is depending also on x.

The perturbed viscosity is obtained as follows. We have

$$\mu(c + c', (U + u')^2 + v'^2)$$

$$\approx \mu(c, U^2) + \mu_c\ c' + \mu_\theta\ 2Uu', \tag{32}$$

therefore

$$\mu' = \mu_c\ c' + \mu_\theta\ 2Uu', \tag{33}$$

where c is the basic concentration profile and c' is the perturbed concentration.

As in [6], [7], we consider the following equation for c in the fix coordinate system x_1Oy

$$c_t + Uc_{x_1} = 0. \tag{34}$$

We use the moving reference $x = x_1 - Ut$ and in the first approximation we get

$$c'_t = -c_x u', \quad c' = -\frac{1}{\sigma}u'\ c_x, \tag{35}$$

(see [6], relations (2.8) and (2.12) with c instead of μ). The equations (33) and (35) give us

$$\mu' = \{-\mu_c\ c_x/\sigma + +\mu_\theta\ 2U\}u'$$

$$= \{-\mu_x/\sigma + +\mu_\theta\ 2U\}u'. \tag{36}$$

The equations (9)-(10) *still hold. We use the expression* (36) *and obtain*

$$p'_x = -\mu u' - U\mu'$$

$$= -\mu u' - U(-\mu_x/\sigma + 2\mu_\theta U)u',$$

$$p'_y = -\mu v'. \tag{37}$$

The relation $(p'_x)_y = (p'_y)_x$ gives us the amplitude equation inside the middle region $(-L, 0)$:

$$k^2(-\mu + U\mu_x/\sigma - 2\theta\mu_\theta)f = -(\mu f_x)_x, \tag{38}$$

$$-(\mu f_x)_x + k^2(\mu + 2\theta\mu_\theta)f = k^2 U\mu_x f/\sigma. \tag{39}$$

This time we do not know the exact expression of f inside the middle region, but we have the same relations (19) for f in the far field (where $\mu_x = 0$). The boundary conditions (26), (27) still hold.

We multiply the relation (39) with f, we integrate on $(-L, 0)$ and obtain

$$-(\mu^- f_x^- f)(0) + (\mu^+ f_x^+ f)(-L) + \int \mu f_x^2 dx$$

$$+ k^2 \int (\mu + 2\theta\mu_\theta)dx = k^2 \frac{U}{\sigma} \int \mu_x f^2 dx, \tag{40}$$

where we used the notations

$$(FGH)(x) = F(x)H(x)F(x), \qquad \int F dx = \int_{-L}^{0} F(x)dx.$$

The boundary conditions (28), (29) and (40) give us:

$$-[kE_0 f_0^2/\sigma - \gamma\mu_0^+ f_0^2] + [\gamma\mu_L^- f_L^2 - kE_L f_L^2/\sigma]$$

$$+ \int \mu f_x^2 dx + k^2 \int (\mu + 2\theta\mu_\theta)dx = k^2 \frac{U}{\sigma} \int \mu_x f^2 dx,$$

$$\mu_0^{+,-} = \mu^{+,-}(0), \quad \mu_L^{-,+} = \mu^{-,+}(-L),$$

$$E_0 = kU[\mu_O - \mu_0^-]k^3 T_0,$$

$$E_L = kU[\mu_L^+ - \mu_W] - k^3 T_L. \tag{41}$$

From the above relations (41) we obtain

$$\sigma = \frac{kE_0 f_0^2 + kE_L f_L^2 + k^2 U \int \mu_x f^2 dx}{\mu_0 \gamma f_0^2 + \mu_W \gamma f_L^2 + \int \mu f_x^2 dx + k^2 \int (\mu + 2\ \theta\mu_\theta) f^2 dx}. \tag{42}$$

We consider now a continuous viscosity and zero surfaces tensions, that means

$$E_0 = E_L = T_0 = T_L = 0. \tag{43}$$

For a Newtonian intermediate liquid depending only on c, the following estimate of the growth rates (say, σ_c) is given in formula (44) of [13]:

$$\sigma_c \leq \frac{U \int \mu_x f^2 dx}{\int \mu f^2 dx}. \tag{44}$$

From (42) we get the upper estimate

$$\sigma \leq \frac{U \int \mu_x f^2 dx}{\int (\mu + 2\theta\mu_\theta) f^2 dx}. \tag{45}$$

The upper limit (45) is less than the upper bound (44) when $\mu_\theta > 0$. Therefore we get an improved stability in the non-Newtonian case, if the hypothesis (43) is fulfilled.

Remark. For small enough μ_x and large enough μ_θ (recall $\theta = U^2$), the upper bound (45) becomes arbitrary small. Thus we can almost suppress the Saffman-Taylor instability, even if the surface tensions are zero. We recall the Saffman-Taylor growth rate σ_{ST} for water displacing oil with a surface tension T:

$$\sigma_{ST} = \frac{kU(\mu_O - \mu_W) - k^3 T}{\mu_O + \mu_W}.$$

We have

$$T = 0 \Rightarrow \lim_{k \to \infty} \sigma_{ST} = \infty.$$

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References

[1] D. Bonn, H. Kellay, M. Brunlich, M. Amar, J. Meunier, Viscous fingering in complex fluids, *Physica A: Statistical Mechanics and its Applications*, **220**, No 1-2 (1995), 60-73; doi: 10.1016/0378-4371(95)00114-M.

[2] D. Bonn, H. Kellay, M. Amar, J. Meunier, Viscous finger widening with surfactants and polymers. *Physical Review Letters*, **75**, No 11 (1995), 2132-2135; doi: 10.1103/PhysRevLett.75.2132.

[3] C. Carasso and G. Paşa, An optimal viscosity profile in the secondary oil recovery, *Mod. Math. et Analyse Numérique*, **32**, No 2 (1998), 211-221; id=M2AN-1998-32-2-211-0.

[4] P. Daripa and G. Pasa, Stabilizing effect of diffusion in enhanced oil recovery and three-layer Hele-Shaw flows with viscosity gradient, *Transp. in Porous Media*, **70**, No 1 (2007), 11-23; doi: 10.1007/s11242-007-9122-7.

[5] P. Daripa and G. Pasa, On diffusive slowdown in three-layers Hele-Shaw flows, *Quarterly of Appl. Math.*, **LXVIII**, No 33 (2010), 591-606; doi: 10.1090/S0033-569X-2010-01174-3.

[6] S.B. Gorell and G.M. Homsy, A theory of the optimal policy of oil recovery by secondary displacement process, *SIAM J. Appl. Math.*, **43**, No 1 (1983), 79-98; doi: 10.1137/0143007.

[7] S.B. Gorell and G.M. Homsy, A theory for the most stable variable viscosity profile in graded mobility displacement process, *AIChE Journal*, **31**, No 9 (1985), 1598-1503; doi: 10.1002/aic.690310912.

[8] L. Kondic, P. Palffy-Muhoray, and M.J. Shelley, Models of non-Newtonian Hele-Shaw flow, *Phys. Rev. E*, **54** No 5 (1996), 4536-4539; doi: 10.1103/physreve.54r4536.

[9] R. Niu, H. Zheng, B. Zhang, Navier-Stokes equations the half-space in variable exponent spaces of Clifford-valued functions, *Electron. J. Differential Equations*, **2017**, No 98 (2017), 1-21; doi: 10.1186/s13661-015-0291-y.

[10] G. Pasa and O. Titaud, A class of viscosity profiles for oil displacement in porous media or Hele-Shaw cell, with Olivier Titaud, *Transp. in Porous Media*, **58**, No 3 (2005), 269-286; doi: 10.1007/s11242-004-0773-3.

[11] G. Pasa, Stability analysis of diffusive displacement in three-layer Hele-Shaw cell or porous medium, *Transport in Porous Media*, **85**, No 1 (2010), 317-332; doi: 10.1007/s11242-010-9564-1.

[12] G. Pasa, n the displacement of two immiscible Oldroyd-B fluids in a Hele-Shaw cell, *Ann. Univ. Ferrara*, **65**, No 2 (2019), 337-359; doi: 10.1007/s11565-019-00320-7.

[13] G. Pasa, A paradox in Hele-Shaw displacements, *Ann. Univ. Ferrara*, **66**, No 1 (2020), 99-111; doi: 10.1007/s11565-020-00339-1.

[14] G. Pasa, The multi-layer Hele-Shaw model with constant viscosity fluids can not minimize the Saffman-Taylor instability, *Internaional Journal of Applied Mathematics*, **33**, No 4 (2020), 697-708; doi: 10.12732/ijam.v33i4.13.

[15] P.G. Saffman and G.I. Taylor, The penetration of a fluid in a porous medium or Helle-Shaw cell containing a more viscous fluid, *Proc. Roy. Soc. A*, **245**, No 1242 (1958), 312-329; doi: 10.1098/rspa.1958.0085.

[16] S.D.R. Wilson, The Taylor-Saffman problem for a non-Newtonian liquid, *J. Fluid Mech.* **220** (1990), 413-425; doi: 10.1017/S0022112090003329.