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UNIQUENESS OF DIFFERENCE DIFFERENTIAL POLYNOMIALS OF L-FUNCTIONS CONCERNING WEIGHTED SHARING

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Abstract: In this paper, we mainly investigate the value distributions of difference differential polynomials of L-functions. Concerning small and rational functions sharing we prove uniqueness theorems on difference differential polynomials of L-functions. The results improve some recent results of W.J. Hao, J.F. Chen [3], W.Q. Zhu, J.F. Chen [17] and N. Mandal, N.K. Datta [10].

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1. Introduction

The Riemann hypothesis and its extension to the general classes of L-functions is the most important open problem in today's pure mathematics. In the modern number theory the L-functions play very important role.

$L(z) = \sum_{n=1}^{\infty} a(n)/n^z$ is said to be an L-function in the Selberg class if it satisfies the following assumptions:

(i) $a(n) \ll n^{\epsilon}$, for every $\epsilon > 0$;

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(ii) There exists an integer $k \geq 0$ such that $(z-1)^k L(z)$ is a finite order entire function;

(iii) Every L-function satisfies the functional equation $\lambda_L(z) = \omega \overline{\lambda_L(1-\overline{z})}$, where $\lambda_L(z) = L(z) Q^z \prod_{i=1}^{k} \Gamma(\gamma_i z + \nu_i)$ with positive real numbers Q, γ_i and complex numbers ν_i, ω with $Re\nu_i \geq 0$ and $\mid \omega \mid = 1$;

(iv) $L(z) = \prod_p L_p(z)$, where $L_p(z) = exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfy only the assumptions (i)-(iii), then L is an L-function in the extended Selberg class. In this paper by an L function we always mean an L function in the extended Selberg class with $a(1) = 1$. Using Nevanlinna Value Distribution theory we study how uniquely difference differential polynomials of L-functions be determined in the extended Selberg class. We use the standard notations and definitions of the value distribution theory, [4].

Let $\alpha \in \mathbf{C} \cup \{\infty\}$ and ξ, ψ be meromorphic functions in the complex plane. The set of all the α points of ξ with multiplicities not exceeding l is denoted by $E_{l)}(\alpha;\xi)(\overline{E}_{l)}(\alpha;\xi))$, where l is a positive integer and we consider(ignore) the multiplicities of the α points. We define the hyper order $\rho_2(\xi)$ of ξ by $\rho_2(\xi) = \limsup_{r\to\infty} \frac{\log\log T(r,\xi)}{\log r}$. We denote by $S(r,\xi)$ any function satisfying $S(r,\xi) = o(T(r,\xi))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. We say that ξ and ψ share α CM if they have the same set of α points with the same multiplicities and if we do not consider the multiplicities then we say that ξ and ψ share α IM.

Definition 1. [7] Let ξ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbf{C} \cup \{\infty\}$. By $N(r,\alpha;\xi \mid\leq n)$ we denote the counting function of the α points of ξ with multiplicity $\leq n$ and by $\overline{N}(r,\alpha;\xi \mid\leq n)$ the reduced counting function. Also by $N(r,\alpha;\xi \mid\geq n)$ we denote the counting function of the α points of ξ with multiplicity $\geq n$ and by $\overline{N}(r,\alpha;\xi \mid\geq n)$ the reduced counting function. We define

$$N_n(r,\alpha;\xi) = \overline{N}(r,\alpha;\xi) + \overline{N}(r,\alpha;\xi \mid\geq 2) + \cdots + \overline{N}(r,\alpha;\xi \mid\geq n).$$

Definition 2. [7] Let ξ be a meromorphic function defined in the complex plane and $P(Z)$ be a small function of ξ or rational function. Then we denote by $N(r,P;\xi \mid\leq m)$, $\overline{N}(r,P;\xi \mid\leq m)$, $N(r,P;\xi \mid\geq m)$, $\overline{N}(r,P;\xi \mid\geq m)$, $N_m(r,P;\xi)$ etc. the counting functions $N(r,0;\xi - P \mid\leq m)$, $\overline{N}(r,0;\xi - P \mid\leq m)$, $N(r,0;\xi - P \mid\geq m)$, $\overline{N}(r,0;\xi - P \mid\geq m)$, $N_m(r,0;\xi - P)$ etc., respectively.

Definition 3. [5, 6] Let ξ and ψ be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity . we denote by $E_n(\alpha;\xi)$ the set of all zeros of $\xi - \alpha$ where $\alpha \in \mathbf{C} \cup \{\infty\}$ and a zero of multiplicity k is counted k times if $k \leq n$ and $n+1$ times if $k > n$. we say that ξ, ψ share α with weight n if $E_n(\alpha;\xi) = E_n(\alpha;\psi)$.

We say ξ, ψ share (α, n) to mean that ξ, ψ share α with weight n. Clearly ξ, ψ share α IM or CM if and only if ξ, ψ share $(\alpha, 0)$ or (α, ∞), respectively.

In 2010 Li [8] proved the following theorem.

Theorem 4. *[8] Let ξ be a nonconstant meromorphic function having finitely many poles and L be a nonconstant L-function. If ξ and L share (α, ∞) and $(\beta, 0)$ then $L \equiv \xi$, where α and β are two distinct finite values.*

Definition 5. [10] Let ξ be a meromorphic function defined in the complex plane and $P(z)$ be a rational function or a small function of ξ. Then we denote by $E_{m)}(P;\xi)$, $\overline{E}_{m)}(P;\xi)$ and $E_m(P;\xi)$ the sets $E_{m)}(0;\xi - P)$, $\overline{E}_{m)}(0;\xi - P)$ and $E_m(0;\xi - P)$, respectively.

We write ξ, ψ share (P, n) to mean that $\xi - P$, $\psi - P$ share the value 0 with weight n. Clearly if ξ, ψ share (P, n) then ξ, ψ share (P, m) for all integers $m, 0 \leq m < n$. Also we note that ξ, ψ share P IM or CM if and only if ξ, ψ share $(P, 0)$ or (P, ∞), respectively.

Considering differential monomial in 2017, Liu, Li and Yi [9] proved the following uniqueness theorems.

Theorem 6. *[9] Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and ξ be a nonconstant meromorphic function. If $\{\xi^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1, \infty)$, then $\xi \equiv \alpha L$ for some nonconstant α satisfying $\alpha^j = 1$.*

Theorem 7. *[9] Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and ξ be a nonconstant meromorphic function. If $\{\xi^j\}^{(k)}(z)$ and $\{L^j\}^{(k)}(z)$ share (z, ∞), then $\xi \equiv \alpha L$ for some nonconstant α satisfying $\alpha^j = 1$.*

Considering differential polynomials in 2018, W.J. Hao and J.F. Chen [3] obtained the following uniqueness results on L-function:

Theorem 8. *[3] Let ξ be a nonconstant meromorphic function and L be an L-function such that $[\xi^n(\xi-1)^m]^{(\tau)}$ and $[L^n(L-1)^m]^{(\tau)}$ share $(1,\infty)$, where $n,m,\tau \in \boldsymbol{Z}^+$. If $n > m+3\tau+6$ and $\tau \geq 2$, then, $\xi \equiv L$ or, $\xi^n(\xi-1)^m \equiv L^n(L-1)^m$.*

Theorem 9. *[3] Let ξ be a nonconstant meromorphic function and L be an L-function such that $[\xi^n(\xi-1)^m]^{(\tau)}$ and $[L^n(L-1)^m]^{(\tau)}$ share $(1,0)$, where $n,m,\tau \in \boldsymbol{Z}^+$. If $n > 4m+7\tau+11$ and $\tau \geq 2$, then, $\xi \equiv \mathcal{L}$ or, $\xi^n(\xi-1)^m \equiv L^n(L-1)^m$.*

In 2019 W.Q. Zhu and J.F. Chen [17] using truncated sharing proved the following uniqueness theorem.

Theorem 10. *[17] Let L be an L-function and ξ be a transcendental meromorphic function defined in the complex plane $\boldsymbol{C}$. Also let n, $k(\geq 2)$, $l(\geq 2)$ be positive integers such that $n \geq 7k+17$. If $\overline{E}_{l)}(1,(\xi^n(\xi-1))^{(k)}) = \overline{E}_{l)}(1,(L^n(L-1))^{(k)})$, then $f \equiv L$.*

Considering truncated sharing of small functions in 2020 Mandal and Datta [10] proved the following theorem.

Theorem 11. *[10] Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \not\equiv 0,\infty$. If $\overline{E}_{4)}(\rho;L) = \overline{E}_{4)}(\rho;(L^m)^{(k)})$, $E_{2)}(\rho;L) = E_{2)}(\rho;(L^m)^{(k)})$ and $2N_{2+k}(r,0;L^m) \leq (\sigma+o(1))T(r,L)$, where $m \geq 1$, $k \geq 1$ are integers and $0<\sigma<1$, then $L \equiv (L^m)^{(k)}$.*

Now the following questions come naturally.

Question 12. If we consider rational or small function sharing in Theorem 8, Theorem 9 and Theorem 10, then what happens?

Question 13. Can we take difference differential polynomials in place of differential polynomials in Theorem 8, Theorem 9, Theorem 10 and Theorem 11?

Definition 14. [5] Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}_*(r,\alpha;\xi,\psi)$ the counting function of the α-points of ξ and ψ with different multiplicities, where each α-point is counted only once.

Definition 15. Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}(r, \alpha; \xi| > \psi)$ the counting function of the α-points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ, where each α-point is counted once only.

Definition 16. Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}_E(r, \alpha; \xi, \psi| > m)$ the counting function of the α-points of ξ and ψ with multiplicities greater than m and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ, where each α-point is counted once only.

In this paper we try to solve Questions 12, 13 and prove the following theorems.

Theorem 17. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function. Let $\tau, n, \eta, \mu_j (j = 1, 2, ..., \eta)$, $\lambda = \sum_{j=1}^{\eta} \mu_j$ be positive integers such that $n > \lambda + \eta(2\tau + 4) + 4$ and $\omega_j \in \boldsymbol{C} - \{0\}$ $(j = 1, 2, ..., \eta)$ be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(\xi) < 1$, $[L^n(z) \prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}$ and $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$ share $(\rho(z), l)$ and ξ, L share $(\infty, 0)$, where $\rho(z)$ is a small function of ξ and L. If $l = 0$ and $n > \lambda + (\eta + 1)(5\tau + 7)$ or $l = 1$ and $n > \lambda + \frac{3}{2}(\eta + 1)(2\tau + 3)$, then one of the following holds:*

(i) $[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$,

(ii) $[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv \rho(z)^2$.

Theorem 18. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function. Let $\tau, n, \eta, \mu_j (j = 1, 2, ..., \eta)$, $\lambda = \sum_{j=1}^{\eta} \mu_j$ be positive integers such that $n > \lambda + \eta(2\tau + 4) + 4$ and $\omega_j \in \boldsymbol{C} - \{0\}$ $(j = 1, 2, ..., \eta)$ be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(\xi) < 1$, $[L^n(z) \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)}$ and $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$ share $(R(z), l)$ and ξ, L share $(\infty, 0)$, where $R(z)$ is a rational function. If $l = 0$ and $n > \lambda + (\eta + 1)(5\tau + 7)$ or $l = 1$ and $n > \lambda + \frac{3}{2}(\eta + 1)(2\tau + 3)$, then one of the following holds:*

(i) $[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$,

(ii) $[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv R(z)^2$.

2. Lemmas

In this section we present some necessary lemmas.

Henceforth we denote by Ω the function defined by

$$\Omega = (\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}).$$

Lemma 19. *[12] Let L be an L-function with degree q. Then $T(r, L) = \frac{q}{\pi} r \log r + O(r)$.*

Lemma 20. *[10] Let L be an L-function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.*

Lemma 21. *Let ξ be a nonconstant meromorphic function and L be an L-function. If ξ and L share $(\infty, 0)$, then $\overline{N}(r, \infty; \xi) = S(r, L) = O(\log r)$.*

Proof. Since ξ and L share $(\infty, 0)$, therefore by lemma 20 we have $\overline{N}(r, \infty; \xi) = \overline{N}(r, \infty; L) = S(r, L) = O(\log r)$. This completes the proof. □

Lemma 22. *[16] Let $\xi(z) = \dfrac{\alpha_0 + \alpha_1 z + ... + \alpha_n z^n}{\beta_0 + \beta_1 z + ... + \beta_m z^m}$ be a nonconstant rational function defined in the complex plane $\boldsymbol{C}$, where $\alpha_0, \alpha_1, ..., \alpha_n$ $(\neq 0)$ and $\beta_0, \beta_1, ..., \beta_m$ $(\neq 0)$ are complex constants. Then $T(r, \xi) = max\{m, n\} \log r + O(1)$.*

Lemma 23. *[13] Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. Then for any $\alpha \in \boldsymbol{C} - 0$:*

$$T(r, \xi(z + \alpha)) = T(r, \xi(z)) + S(r, \xi(z)),$$

$$N(r, \infty; \xi(z + \alpha)) = N(r, \infty; \xi(z)) + S(r, \xi(z)),$$

$$N(r, 0; \xi(z + \alpha)) = N(r, 0; \xi(z)) + S(r, \xi(z)).$$

Lemma 24. *[11] Let Φ and Ψ be two nonconstant meromorphic functions sharing $(1, 1)$ and $(\infty, 0)$. If $\Omega \not\equiv 0$, then*

$$T(r, \Phi) \leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + \frac{3}{2}\overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi)$$

$$+ \quad \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Phi) + S(r,\Phi) + S(r,\Psi),$$

$$\begin{aligned} T(r,\Psi) \quad &\leq \quad N_2(r,0;\Phi) + N_2(r,0;\Psi) + \frac{3}{2}\overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) \\ &+ \quad \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Psi) + S(r,\Phi) + S(r,\Psi). \end{aligned}$$

Lemma 25. *[11] Let Φ and Ψ be two nonconstant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $\Omega \not\equiv 0$, then*

$$\begin{aligned} T(r,\Phi) \quad &\leq \quad N_2(r,0;\Phi) + N_2(r,0;\Psi) + 3\overline{N}(r,\infty;\Phi) + 2\overline{N}(r,\infty;\Psi) \\ &+ \quad \overline{N}_*(r,\infty;\Phi,\Psi) + 2\overline{N}(r,0;\Phi) + \overline{N}(r,0;\Psi) \\ &+ \quad S(r,\Phi) + S(r,\Psi), \end{aligned}$$

$$\begin{aligned} T(r,\Psi) \quad &\leq \quad N_2(r,0;\Phi) + N_2(r,0;\Psi) + 3\overline{N}(r,\infty;\Psi) + 2\overline{N}(r,\infty;\Phi) \\ &+ \quad \overline{N}_*(r,\infty;\Phi,\Psi) + 2\overline{N}(r,0;\Psi) + \overline{N}(r,0;\Phi) \\ &+ \quad S(r,\Phi) + S(r,\Psi). \end{aligned}$$

Lemma 26. *[15] Let Φ be a nonconstant meromorphic function and k, p be two positive integers. Then*

$$T(r,\Phi^{(k)}) \leq T(r,\Phi) + k\overline{N}(r,\infty;\Phi) + S(r,\Phi),$$

$$N_p(r,0;\Phi^{(k)}) \leq T(r,\Phi^{(k)}) - T(r,\Phi) + N_{p+k}(r,0;\Phi) + S(r,\Phi),$$

$$N_p(r,0;\Phi^{(k)}) \leq N_{p+k}(r,0;\Phi) + k\overline{N}(r,\infty;\Phi) + S(r,\Phi),$$

$$N(r,0;\Phi^{(k)}) \leq N(r,0;\Phi) + k\overline{N}(r,\infty;\Phi) + S(r,\Phi).$$

Lemma 27. *[2] Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$ and $\phi(z) = \prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}$, where n, η, μ_j $(j = 1, 2, ..., \eta)$, $\lambda = \sum_{j=1}^{\eta} s_j$ are positive integers and $c_j \in \boldsymbol{C} - \{0\}$ $(j = 1, 2, ..., \eta)$ be distinct constants. Then*

$$(n-\lambda)T(r,\xi) + S(r,\xi) \leq T(r,\xi^n\phi) \leq (n+\lambda)T(r,\xi) + S(r,\xi).$$

3. Proof of the Main Results

Proof of Theorem 17.

Let $\phi(z)=\prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}$, $\psi(z)=\prod_{j=1}^{\eta}L(z+\omega_j)^{\mu_j}$, $\Phi(z)=\frac{(\xi(z)^n\phi(z))^{(\tau)}}{\rho(z)}$ and $\Psi(z)=\frac{(L(z)^n\psi(z))^{(\tau)}}{\rho(z)}$. Then Φ, Ψ share $(1,l)$ and Φ, Ψ share $(\infty,0)$ except for zeros and poles of $\rho(z)$.

Clearly by Lemma 19, L is a transcendental meromorphic function. We have by Lemma 26 and Lemma 27:

$$\begin{aligned}N_2(r,0;\Phi) &\leq N_2(r,0;(\xi^n\phi)^{(\tau)})+S(r,\xi)\\ &\leq T(r,(\xi^n\phi)^{(\tau)})-T(r,\xi^n\phi)+N_{2+\tau}(r,0;\xi^n\phi)+S(r,\xi)\\ &\leq T(r,\frac{(\xi^n\phi)^{(\tau)}}{\rho(z)})-(n-\lambda)T(r,\xi)+N_{2+\tau}(r,0;\xi^n\phi)\\ &+ S(r,\xi). \qquad (1)\end{aligned}$$

Hence we get from (1)

$$(n-\lambda)T(r,\xi)\leq T(r,\Phi)-N_2(r,0;\Phi)+N_{2+\tau}(r,0;\xi^n\phi)+S(r,\xi). \qquad (2)$$

Similarly we get

$$(n-\lambda)T(r,L)\leq T(r,\Psi)-N_2(r,0;\Psi)+N_{2+\tau}(r,0;L^n\psi)+S(r,L). \qquad (3)$$

Now we have to consider the following two cases:

Case 1. Let $\Omega\not\equiv 0$. In this case we have to consider the following two subcases.

Subcase 1.1. Let $l=0$. Hence by Lemma 20, Lemma 21 and Lemma 25, we have from (2):

$$\begin{aligned}&(n-\lambda)T(r,\xi)\\ \leq\ & N_2(r,0;\Phi)+N_2(r,0;\Psi)+3\overline{N}(r,\infty;\Phi)\\ +\ & 2\overline{N}(r,\infty;\Psi)+\overline{N}_*(r,\infty;\Phi,\Psi)+2\overline{N}(r,0;\Phi)\\ +\ & \overline{N}(r,0;\Psi)-N_2(r,0;\Phi)+N_{2+\tau}(r,0;\xi^n\phi)+S(r,\xi)+S(r,L)\end{aligned}$$

$$\leq N_2(r,0;\Phi)+N_2(r,0;\Psi)+2\overline{N}(r,0;\Phi)+\overline{N}(r,0;\Psi)$$

$$
\begin{aligned}
&- && N_2(r,0;\Phi) + N_{2+\tau}(r,0;\xi^n\phi) + S(r,\xi) + S(r,L) \\
&\leq && N_2(r,0;(\xi^n\phi)^{(\tau)}) + N_2(r,0;(L^n\psi)^{(\tau)}) + 2\overline{N}(r,0;(\xi^n\phi)^{(\tau)}) \\
&+ && \overline{N}(r,0;(L^n\psi)^{(\tau)}) - N_2(r,0;(\xi^n\phi)^{(\tau)}) + N_{2+\tau}(r,0;\xi^n\phi) \\
&+ && S(r,\xi) + S(r,L) \\
&\leq && N_2(r,0;(L^n\psi)^{(\tau)}) + 2\overline{N}(r,0;(\xi^n\phi)^{(\tau)}) + \overline{N}(r,0;(L^n\psi)^{(\tau)}) \\
&+ && N_{2+\tau}(r,0;\xi^n\phi) + S(r,\xi) + S(r,L) \\
&\leq && N_{2+\tau}(r,0;L^n\psi) + 2N_{1+\tau}(r,0;\xi^n\phi) + N_{1+\tau}(r,0;L^n\psi) \\
&+ && N_{2+\tau}(r,0;\xi^n\phi) + S(r,\xi) + S(r,L) \\
&\leq && (2+\tau)(1+\eta)T(r,L) + 2(\tau+1)(\eta+1)T(r,\xi) \\
&+ && (\tau+1)(\eta+1)T(r,L) + (2+\tau)(1+\eta)T(r,\xi) \\
&+ && S(r,\xi) + S(r,L) \\
&\leq && (3+2\tau)(1+\eta)T(r,L) + (3\tau+4)(\eta+1)T(r,\xi) \\
&+ && S(r,\xi) + S(r,L). \qquad (4)
\end{aligned}
$$

Similarly by Lemma 20, Lemma 21 and Lemma 25, we have from (3):

$$
\begin{aligned}
(n-\lambda)T(r,L) &\leq (3+2\tau)(1+\eta)T(r,\xi) + (3\tau+4)(\eta+1)T(r,L) \\
&+ S(r,\xi) + S(r,L). \qquad (5)
\end{aligned}
$$

Hence we get from (4) and (5)

$$
\begin{aligned}
(n-\lambda)\{T(r,L)+T(r,\xi)\} &\leq (7+5\tau)(1+\eta)\{T(r,L)+T(r,\xi)\} \\
&+ S(r,\xi) + S(r,L). \qquad (6)
\end{aligned}
$$

From (6) we arrive at a contradiction, since $n > \lambda + (7+5\tau)(1+\eta)$.

Subcase 1.2. Let $l = 1$.
By Lemma 20, Lemma 21 and Lemma 24 we have from (2):

$$
\begin{aligned}
& && (n-\lambda)T(r,\xi) \\
&\leq && N_2(r,0;\Psi) + \frac{3}{2}\overline{N}(r,\infty;\Phi) + \overline{N}(r,\infty;\Psi) \\
&+ && \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Phi) + N_{2+\tau}(r,0;\xi^n\phi) \\
&+ && S(r,\xi) + S(r,L) \\
&\leq && N_2(r,0;(L^n\psi)^{(\tau)}) + \frac{1}{2}N_{\tau+1}(r,0;\xi^n\phi) \\
&+ && \overline{N}(r,0;(L^n\psi)^{(\tau)}) + N_{2+\tau}(r,0;\xi^n\phi) + S(r,\xi) + S(r,L)
\end{aligned}
$$

$$\begin{aligned} &\leq (2+\tau)(1+\eta)T(r,L)+\frac{1}{2}(\tau+1)(\eta+1)T(r,\xi) \\ &+ (2+\tau)(1+\eta)T(r,\xi)+S(r,\xi)+S(r,L) \\ &\leq (2+\tau)(1+\eta)T(r,L)+\frac{1}{2}(3\tau+5)(\eta+1)T(r,\xi) \\ &+ S(r,\xi)+S(r,L). \end{aligned} \tag{7}$$

Similarly we have by Lemma 20, Lemma 21 and Lemma 24 from (3):

$$\begin{aligned} (n-\lambda)T(r,L) &\leq (2+\tau)(1+\eta)T(r,\xi)+\frac{1}{2}(3\tau+5)(\eta+1)T(r,L) \\ &+ S(r,\xi)+S(r,L). \end{aligned} \tag{8}$$

Using (7) and (8) we get

$$\begin{aligned} (n-\lambda)\{T(r,L)+T(r,\xi)\} &\leq \frac{1}{2}(9+5\tau)(1+\eta)\{T(r,L)+T(r,\xi)\} \\ &+ S(r,\xi)+S(r,L). \end{aligned} \tag{9}$$

Hence from (9) we arrive at a contradiction, since $n>\lambda+\frac{1}{2}(9+5\tau)(1+\eta)$.

Case 2. Let $\Omega\equiv 0$. Then $(\frac{\Phi''}{\Phi'}-\frac{2\Phi'}{\Phi-1})-(\frac{\Psi''}{\Psi'}-\frac{2\Psi'}{\Psi-1})\equiv 0$.
Hence we have

$$\Phi-1\equiv\frac{\Psi-1}{b-c(\Psi-1)}, \tag{10}$$

where $b(\neq 0)$ and c are constants.

Now we have to consider the following two cases.

Subcase 2.1. Let $c=0$. Then from (10) we have

$$\Phi-1\equiv\frac{(\Psi-1)}{b}. \tag{11}$$

If $b\neq 1$, then from (11)

$$\overline{N}(r,0;\Phi)=\overline{N}(r,1-b;\Psi). \tag{12}$$

By Lemma 20, Lemma 26 and the second fundamental theorem, we have from (3)

$$\begin{aligned} &(n-\lambda)T(r,L) \\ = \ &T(r,\Psi)-N_2(r,0;\Psi)+N_{\tau+2}(r,0;L^n\psi)+S(r,L) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,0;\Psi)+\overline{N}(r,1-b;\Psi)+\overline{N}(r,\infty;\Psi)\\
&- N_2(r,0;\Psi)+N_{\tau+2}(r,0;L^n\psi)+S(r,L)\\
&\leq \overline{N}(r,0;\Psi)+\overline{N}(r,0;\Phi)-N_2(r,0;\Psi)\\
&+ N_{\tau+2}(r,0;L^n\psi)+S(r,L)\\
&\leq \overline{N}(r,0;(\xi^n\phi)^{(\tau)})+\overline{N}(r,0;(L^n\psi)^{(\tau)})+N_{\tau+2}(r,0;L^n\psi)+S(r,L)\\
&\leq N_{\tau+1}(r,0;\xi^n\phi)+N_{\tau+1}(r,0;L^n\psi)+N_{\tau+2}(r,0;L^n\psi)+S(r,L)\\
&\leq (\tau+1)(\eta+1)T(r,L)+(\tau+1)(\eta+1)T(r,\xi)\\
&+ (\tau+2)(\eta+1)T(r,L)+S(r,L)\\
&\leq (2\tau+3)(\eta+1)T(r,L)+(\tau+1)(\eta+1)T(r,\xi)\\
&+ S(r,L)+S(r,\xi). \qquad (13)
\end{aligned}$$

Similarly we have from (2)

$$\begin{aligned}
(n-\lambda)T(r,\xi) &\leq (2\tau+3)(\eta+1)T(r,\xi)+(\tau+1)(\eta+1)T(r,L)\\
&+ S(r,L)+S(r,\xi). \qquad (14)
\end{aligned}$$

From (13) and (14) we have

$$\begin{aligned}
(n-\lambda)(T(r,L)+T(r,\xi)) &\leq (3\tau+4)(\eta+1)(T(r,L)\\
&+ T(r,\xi))+S(r,\xi)+S(r,L). \qquad (15)
\end{aligned}$$

From (15) we arrive at a contradiction since $n>\lambda+(3\tau+4)(\eta+1)$.

Hence $b=1$ and therefore we get from (11)

$$[L^n\prod_{j=1}^{\eta}L(z+\omega_j)^{\mu_j}]^{(\tau)}\equiv[\xi^n\prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}]^{(\tau)}.$$

Subcase 2.2. Let $c\neq 0$ and $b=-c$.

If $c=1$, then from (10) we have $\Phi\Psi\equiv 1$. Hence

$$[L^n\prod_{j=1}^{\eta}L(z+\omega_j)^{\mu_j}]^{(\tau)}[\xi^n\prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}]^{(\tau)}\equiv\rho(z)^2.$$

If $c\neq 1$, then from (10) we have $\frac{1}{\Phi}=\frac{-c\Psi}{(1-c)\Psi-1}$.

Hence $\overline{N}(r,0;\Phi)=\overline{N}(r,\frac{1}{1-c};\Psi)$.

Now proceeding as in Subcase 2.1. we arrive at a contradiction.

If $c = 1$, then from (10) we have

$$\Phi \equiv \frac{-b}{\Psi - b - 1}. \tag{16}$$

By Lemma 21 we have from (16)

$$\overline{N}(r, C+1; \Psi) = \overline{N}(r, \infty; \Phi) = \overline{N}(r, \infty; \xi) + S(r, L) = S(r, L).$$

Now proceeding as in Subcase 2.1. we arrive at a contradiction.

If $c \neq 1$, then from (10) we have

$$\Phi - (1 - \frac{1}{c}) \equiv \frac{-b}{c^2(\Psi - \frac{b+c}{c})}.$$

Therefore by Lemma 21 we have

$$\overline{N}(r, \frac{b+c}{c}; \Psi) = \overline{N}(r, \infty; \Phi) = \overline{N}(r, \infty; \xi) + S(r, L) = S(r, L).$$

Hence proceeding as in Subcase 2.1. we arrive at a contradiction.

This completes the proof of the theorem.

Proof of Theorem 18.

Since ξ and L are transcendental meromorphic functions and $R(z)$ is a rational function therefore $R(z)$ is a small function of ξ and L.

Hence by Theorem 17 we get the required result.

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