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FIRST ORDER DIFFERENTIAL EQUATION SUDORDINATION ASSOCIATED WITH CASSINI CURVE

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Abstract: We denote $p(z)$ as analytic functions defined on the open unit disk with $p(0) = 1$. In this paper, we determined the condition for β so that the results hold for the expressions $1+\beta zp'(z)$, $1+\beta zp'(z)/p(z)$ and $1+\beta zp'(z)/p^2(z)$ are subordinate to $\sqrt{1+cz}$.

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1. Introduction

Let A be the class of all the analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in D),$$

in a unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. We denote S as the subset of A of univalent functions. Also, we denoted C as the class of convex functions and S^* as the class of starlike functions. An analytic function f is subordinate to an analytic function g, we

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write $f(z) \prec g(z)$ for $z \in D$, if there exists an analytic function w in D such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in D, we say that $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

Goluzin [1] found that if the first order differential subordination $zp'(z) \prec zq'(z)$ holds and $zq'(z)$ is convex, then $p(z) \prec q(z)$ holds where q is the best dominant. Eventually, researchers continued to study about this and the general theory is discussed detailed by Miller and Mocanu in [2]. Nunokawa et al. [3] proved that if $1 + zp'(z) \prec 1 + z$ hold, the subordination $p(z) \prec 1 + z$ also hold. There are more results obtained by many other researchers, see [4], [5], [6], [7], [8], [9] and [10].

Sokól and Stankiewicz [11] introduced a class called S_L^* which consists the function of $f \in A$ such that $w(z) := zf'(z)/f(z)$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. This class is associated with the function $\sqrt{1+z}$.

Besides, Aouf et al. [12] defined the class $S^*(q_c)$ for $c \in (0, 1]$ as:

$$S^*(q_c) = \left\{ f \in A : \left| \left[\frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < c, z \in D \right\}.$$

It can be established that

$$f \in S^*(q_c) \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1+cz} \quad (z \in D).$$

We also denoted θ_c as the set of all points in the right half-plane such that the product of the distances from each point to the focuses -1 and 1 is less than c:

$$\theta_c := \left\{ w \in \mathbb{C} : Re\ w > 0, \left| w^2 - 1 \right| < c \right\},$$

thus the boundary $\partial\theta_c$ is the right loop of the Cassinian ovals $\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) = c^2 - 1$ and for $c = 1$, $S^*(q_1) \equiv S_L^*$.

For an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, we determine the condition of β so that $p(z) \prec P(z)$ where $P(z)$ is a function with positive real part like $\sqrt{1+z}$ and $\varphi_0(z) := 1 + \frac{z}{k}\left((k+z)/(k-z)\right)$ $(k = \sqrt{2} + 1)$, whenever $1 + \beta zp'(z)/p^j(z) \prec \sqrt{1+cz}$, where $j = 0, 1, 2$ (please see [13] for more about $\varphi_0(z)$).

2. Preliminary results and definitions

Our results deal with classes of $S^*(q_c)$ associated with S_L^* and $\varphi_0(z)$ respectively. Some sufficient conditions with for functions belong to the above defined classes can be obtained by applying the application on starlike functions with positive real part. The first result gives a bound of β so that $1 + \beta zp'(z) \prec \sqrt{1+cz}$ implies that the function p is subordinate to the $\sqrt{1+z}$ function.

Before to get our result, we need the following lemma to prove the theorems.

Lemma 1. *([14]) Let q be analytic in D and let ψ and v be analytic in domain U containing $q(D)$ with $\psi(w) \neq 0$ when $w \in q(D)$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := v(q(z)) + Q(z)$. Suppose that:*

i. either h is convex or Q is starlike univalent in D, and

ii. Re $(zh'(z)/Q(z)) > 0$ for $z \in D$.

If p is analytic in D, with $p(0) = q(0)$, $p(D) \subseteq U$ and

$$v(p(z)) + zp'(z)\psi(p(z)) \prec v(q(z)) + zq'(z)\psi(q(z)),$$

then $p(z) \prec q(z)$ and q is best dominant.

3. Main results

Theorem 1. *Let the function p be analytic in D, $p(0) = 1$ and $1 + \beta zp'(z) \prec \sqrt{1+cz}, c \in (0, 1]$. Then the following subordination results hold:*

$$(a)\ \text{If}\ \beta \geq \frac{2\left[\sqrt{1+c} - \ln\left(1+\sqrt{1+c}\right) + \ln 2 - 1\right]}{\sqrt{2}-1},$$

then $p(z) \prec \sqrt{1+z}$.

$$(b)\ \text{If}\ \beta \geq \frac{2\left[\sqrt{1-c} - \ln\left(1-\sqrt{1-c}\right) + \ln 2 - 1\right]}{\sqrt{2}-3},$$

then $p(z) \prec \varphi_0(z)$.

Proof. The function $q_B : \overline{D} \to \mathbb{C}$ defined by

$$q_B(z) = 1 + \frac{2}{\beta}\left[\sqrt{1+cz} - \ln(1+\sqrt{1+cz}) + \ln 2 - 1\right]$$

is analytic and it is the solution of $1+\beta zp'(z)=\sqrt{1+cz}$. Let $v(w)=1$ and $\psi(w)=\beta$. So the function $Q:\overline{D}\to\mathbb{C}$ is defined by $Q(z)=zq_B'(z)\psi(q_B(z))=\beta zq_B'(z)$. Since $\sqrt{1+cz}-1$ is starlike function in D, it follows that function Q is starlike. Besides, the function $h(z)=v(q_B(z))+Q(z)$ satisfies $Re(zh'(z)/Q(z))>0$ for $z\in D$. Thus, by using Lemma 1, it shows $1+\beta zp'(z)\prec 1+\beta zq_B'(z)$ implies $p(z)\prec q_B(z)$. We can say that $p(z)\prec P(z)$ for appropriate P and this holds if the subordination $q_B(z)\prec P(z)$ holds. If $q_B(z)\prec P(z)$, then $P(-1)<q_B(-1)<q_B(1)<P(1)$. This gives a necessary condition for $p\prec P$ hold. This necessary condition is sufficient.

(a). By taking $P(z)=\sqrt{1+z}$, the inequalities $q_B(-1)\geq P(-1)$ and $q_B(1)\leq P(1)$ reduce to $\beta\geq\beta_1$ and $\beta\geq\beta_2$, where

$$\beta_1=2\left[\ln(1+\sqrt{1-c})+1-\ln 2-\sqrt{1-c}\right]$$

and

$$\beta_2=\frac{2\left[\sqrt{1+c}-\ln(1+\sqrt{1+c})+\ln 2-1\right]}{\sqrt{2}-1},$$

respectively. The subordination
$q_B(z)\prec\sqrt{1+z}$ holds if $\beta\geq\max\{\beta_1,\beta_2\}=\beta_2$.

(b). Consider $P(z)=\varphi_0(z)$, then the inequalities $q_B(-1)\geq\varphi_0(-1)$ and $q_B(-1)\leq\varphi_0(1)$ reduce to $\beta\geq\beta_1$ and $\beta\geq\beta_2$, where

$$\beta_1=\frac{2\left[\sqrt{1-c}-\ln(1+\sqrt{1-c})+\ln 2-1\right]}{2\sqrt{2}-3}$$

and

$$\beta_2=2\left[\sqrt{1+c}-\ln(1+\sqrt{1+c})+\ln 2-1\right],$$

respectively. Thus, the subordination $q_B(z)\prec\varphi_0(z)$ holds if $\beta\geq\max\{\beta_1,\beta_2\}=\beta_1$.

When $c=1$, we may get Corollary 5. The next result gives bound on β so that $1+\beta zp'(z)/p(z)\prec\sqrt{1+cz}$ implies p is subordinate to $\varphi_0(z)$ function. □

Theorem 2. *Let the function p be analytic in D, $p(0)=1$ and $1+\beta zp'(z)/p(z)\prec\sqrt{1+cz}$, $c\in(0,1]$. Then the following subordination result holds:*

$$\text{If } \beta\geq\frac{2\left[\sqrt{1-c}-\ln(\sqrt{1-c}+1)+\ln 2-1\right]}{\ln(2\sqrt{2}-2)},\ \text{then } p(z)\prec\varphi_0(z).$$

Proof. The function $q_B : \overline{D} \to \mathbb{C}$ defined by

$$q_B(z) = \exp\left\{1 + \frac{2}{\beta}\left[\sqrt{1+cz} - \ln(1+\sqrt{1+cz}) + \ln 2 - 1\right]\right\}$$

is analytic and is the solution of $1 + \beta z p'(z)/p(z) = \sqrt{1+cz}$. Define $v(w) = 1$ and $\psi(w) = \beta/w$. The function $Q : \overline{D} \to \mathbb{C}$ defined by $Q(z) := z q_B'(z)\psi(q_B(z)) = \beta z q_B'(z)/q_B(z) = \sqrt{1+cz} - 1$ is starlike in D. The function $h(z) := v(q_B(z)) + Q(z) = 1 + Q(z)$ satisfies $Re(zh'(z)/Q(z)) > 0$ for $z \in D$. Therefore, by using Lemma 1, we get

$$1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \beta \frac{zq_B'(z)}{q_B(z)}$$

that implies $p(z) \prec q_B(z)$. In the similar lines of the proof of Theorem 2, the proof of the result is completed. By substituting $c = 1$, we get the result in Corollary 6. □

Next, we determine a bound on β so that $1 + \beta z p'(z)/p^2(z) \prec \sqrt{1+cz}$ implies p is subordinate to $\varphi_0(z)$.

Theorem 3. *Let the function p be analytic in D, $p(0) = 1$ and $1 + \beta z p'(z)/p^2(z) \prec \sqrt{1+cz}, c \in (0,1]$. Then the following subordination results hold:*

$$\text{If } \beta \geq \frac{4\left(\sqrt{2}-1)(\sqrt{1-c} - \ln(\sqrt{1-c}+1) + \ln 2 - 1\right)}{2\sqrt{2}-3},$$

then $p(z) \prec \varphi_0(z)$.

Proof. The function $q_B : \overline{D} \to \mathbb{C}$ defined by

$$q_B(z) = \left(1 + \frac{2}{\beta}\left[\sqrt{1+cz} - \ln(1+\sqrt{1+cz}) + \ln 2 - 1\right]\right)^{-1}$$

is analytic. It is the solution of $1 + \beta z p'(z)/p^2(z) = \sqrt{1+cz}$. Define $v(w) = 1$ and $\psi(w) = \beta/w^2$. The function $Q : \overline{D} \to \mathbb{C}$ defined by

$$Q(z) := z q_B'(z)\psi(q_B(z)) = \beta z q_B'(z)/q_B^2(z) = \sqrt{1+cz} - 1$$

is starlike in D, so Q is starlike function. The function $h(z) := v(q_B(z)) + Q(z) = 1 + Q(z)$ satisfies $Re(zh'(z)/Q(z)) > 0$ for $z \in D$. Therefore, by using Lemma 1, we get that

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq_B'(z)}{q_B^2(z)}$$

implies $p(z) \prec q_B(z)$. As the similar lines of the proof of Theorem 2 the proof of the result is completed. □

Also, let $c = 1$, we have the result in Corollary 7.

4. Corollaries

Corollary 4. *([10]) Let the function p be analytic in D, $p(0) = 1$ and $1 + \beta z p'(z) \prec \sqrt{1 + cz}$. Then the following subordination results hold:*

$$(a) \text{ If } \beta \geq \frac{2\left[\sqrt{2} - 1 + \ln 2 - \ln\left(1 + \sqrt{2}\right)\right]}{\sqrt{2} - 1} \approx 1.09116,$$

then $p(z) \prec \sqrt{1 + z}$.

$$(b) \text{ If } \beta \geq \frac{2(1 - \ln 2)}{3 - 2\sqrt{2}} \approx 3.57694, \quad \text{then } p(z) \prec \varphi_0(z).$$

Corollary 5. *([10]) Let the function p be analytic in D, $p(0) = 1$ and $1 + \beta z p'(z)/p(z) \prec \sqrt{1 + cz}$. Then the following subordination results hold:*

$$\text{If } \beta \geq \frac{2(\ln 2 - 1)}{\ln(2\sqrt{2} - 2)} \approx 3.26047, \quad \text{then } p(z) \prec \varphi_0(z).$$

Corollary 6. *([10]) Let the function p be analytic in D, $p(0) = 1$ and $1 + \beta z p'(z)/p^2(z) \prec \sqrt{1 + cz}$. Then the following subordination results hold:*

$$\text{If } \beta \geq 4(1 + \sqrt{2})(1 - \ln 2) \approx 2.96323, \quad \text{then } p(z) \prec \varphi_0(z).$$

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References

[1] G.M. Goluzin, On the majorization principle in function theory, *Dokl. Akad. Nauk. SSSR*, **42** (1935), 647-650.

[2] S.S. Miller, P.T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.*, **32**, No 2 (1985), 185-195.

[3] M. Nunokawa, M. Obradović, S. Owa, On criterion for univalency, *Proc. Amer. Math. Soc.*, **106**, No 4 (1989), 1035-1037.

[4] R.M. Ali, V. Ravichandran, N. Seenivasagan, Sufficient conditions for Janowski starlikeness, *Int. J. Math. Math. Sci.*, **2007** (2007), Art. ID 62927, 7 pp.

[5] R. Omar, S.A. Halim, Differential subordinations properties of Sokół-Stankiewicz starlike functions, *Kyungpook Math. J.* **53**, No 3 (2013), 459-465.

[6] R. Omar, S.A. Halim, R.W. Ibrahim, Differential subordination properties of certain analytic functions, *Int. J. of Math.*, **21**, No 6 (2013), 7 pp.

[7] N.E. Cho, H.J. Lee, J. H. Park, R. Srivastava, Some application of the first-order differential subordinations, *Filomat*, **30**, No 6 (2016), 1465-1474.

[8] V. Ravichandran, K. Sharma, Sufficient conditions for starlikeness, *J. Korean Math. Soc.*, **52**, No 4 (2015), 727-749.

[9] K. Sharma, V. Ravichandran, Applications of subordination theory to starlike functions, *Bull. Iranian Math. Soc.*, **42**, No 3 (2016), 761-777.

[10] P. Ahuja, S. Kumar, V. Ravichandran, Application of first order differential subordination for functions with positive real part, *Stud. Univ. Babeş-Bolyai Math.*, **63** (2018), 303-311.

[11] J. Sokół, J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, No 19 (1996), 101-105.

[12] M.K. Aouf, J. Dziok, J. Sokół, On a subclass of strongly starlike functions, *Appl. Math. Lett.*, **24** (2011), 27-32.

[13] S. Kumar, V. Ravichandran, A subclass of starlike functions associated with a rational function, *Southcast Asian Bull. Math.*, **40**, No 2 (2016), 199-212.

[14] S.S. Miller, P.T. Mocanu, *Differential Subordination*, Dekker, New York (2000).