

STRONGLY RIGHT SINGULAR CLEAN RINGS

Fatemeh Rashedi

Department of Mathematics, Velayat University
Iranshahr – 99111313, IRAN

Abstract: Let R be an associative ring with identity. Then R is said to be strongly right singular clean, if every element of R can be expressed as a sum of a right singular element and an idempotent that commute. In this paper, we study various properties of the strongly right singular clean rings.

AMS Subject Classification: 16U99, 16N40

Key Words: associative ring; right singular clean ring; strongly right singular clean ring

1. Introduction

Throughout, all rings R examined in the current paper shall be assumed associative, containing the identity element 1 which possibly differs from the zero element 0. Standardly, $U(R)$ denotes the group of all units of R , $Id(R)$ the set of all idempotent elements of R and $N(R)$ is the nil-radical of R . Recall that an element $a \in R$ is said to be a full element if there exist $s, t \in R$ such that $sat = 1$ [8]. An element a of a ring R is right (left) singular if $ann_r(a)$ ($ann_l(a)$) is an essential right (left) ideal of R . The set of all right (left) singular elements of R is an ideal of R which is called the right (left) singular ideal of R and is denoted by $\mathcal{Z}(R_R)$ ($\mathcal{Z}({}_R R)$).

An element in a ring R is called clean if it is the sum of an idempotent and a unit in R , and R is called a clean ring if every element is clean [2]. Obviously, invertible elements are all full elements. Therefore f -clean (resp. strongly f -

clean) rings are clean (resp. strongly clean) [8]. An element r in a ring is called nil clean if there is an idempotent $e \in Id(R)$ and a nilpotent $n \in N(R)$ such that $r = e + n$. A ring is called nil clean if every one of its elements is nil clean [3]. After that, some generalizations of clean rings such as strongly clean [6], uniquely clean [7] and weakly clean rings [1], have been considered.

A ring R is called right singular clean, if every element of R can be expressed as a sum of a right singular element and an idempotent. Left singular clean rings are defined similarly. A ring R is called singular clean, if it is both right and left singular clean [5]. In this paper, we introduce a generalization of right singular clean, which is called strongly right singular clean. A ring R is called strongly right singular clean, if every element of R can be expressed as a sum of a right singular element and an idempotent that commute. We study various properties of the strongly right singular clean rings. The direct products of strongly right singular clean rings, is also strongly right singular clean. Finally, we show that, if R is a ring with central idempotents, then R is strongly right singular clean if and only if the ring $\frac{R[x]}{\langle x \rangle^n}$ is strongly right singular clean for every $n \in \mathbb{N}$.

2. Strongly right singular clean rings

Definition 1. A ring R is called strongly right singular clean, if every element of R can be expressed as a sum of a right singular element and an idempotent that commute. Left singular clean rings are defined similarly. A ring R is called singular clean, if it is both right and left singular clean.

Clearly, the rings \mathbb{Z}_2 and \mathbb{Z}_4 are strongly singular clean but the ring \mathbb{Z}_3 is not. Every singular clean ring R with $Id(R) = \{0, 1\}$ is strongly singular clean.

Example 2. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Then

$$Id(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

and

$$\mathcal{Z}(R_R) = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Hence R is right singular clean, by [5, Example 2.7]. But R is not strongly right singular clean.

Proposition 3. *A direct product $R = \prod_{i \in I} R_i$ of rings R_i is strongly right singular clean if and only if the same is true for each R_i .*

Proof. Let $x = (x_i) \in R$. Then each R_i is strongly right singular clean if and only if for every i ,

$$x_i = e_i + z_i$$

such that $e_i \in Id(R_i)$, $z_i \in \mathcal{Z}(R_i)$ and $e_i z_i = z_i e_i$ if and only if $x = e + z$, where $e = (e_i) \in Id(R)$ and $z = (z_i) \in \mathcal{Z}(R)$ such that

$$ez = (e_i)(z_i) = (z_i)(e_i) = ze$$

if and only if R is strongly right singular clean. □

If e is a central idempotent in R , then R is a strongly right singular clean ring if and only if eRe and $(1 - e)R(1 - e)$ are so. Note that if $e \in Id(R)$ is not central, then the above statement is not valid in general. For example, in the ring $M_2(\mathbb{Z}_2)$, let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $eRe \cong \mathbb{Z}_2$ and $(1 - e)R(1 - e) \cong \mathbb{Z}_2$ are strongly right singular clean. But the ring R is not strongly right singular clean.

Lemma 4. *Let R be a strongly right singular clean ring and $e \in Id(R)$ such that $ef = fe$ for all $f \in Id(R)$ and $eR(1 - e) = 0$. Then the ring $S = eRe$ is strongly right singular clean.*

Proof. Suppose R be a strongly right singular clean ring. Hence S is right singular clean, [5, Proposition 2.14]. Let $x \in S$ and $x = efe + eze$, which $f \in Id(R)$, $z \in \mathcal{Z}(R)$, $fz = zf$, $efe \in Id(S)$ and $eze \in \mathcal{Z}(S)$. Then

$$efeeze = efze = efze = ezfe = ezfee = ezeefe.$$

Therefore S is strongly right singular clean. □

Proposition 5. *Let R be a strongly right singular clean ring and $2 \in U(R)$ be a central element of R . Then every element of R is the sum of a right singular element and a square root of 1 that commute.*

Proof. Suppose that $x \in R$. Since R is strongly right singular clean, $\frac{1}{2}(x + 1) = e + z$, where $e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $ez = ze$. Then $x = 2z + (2e - 1)$ is the desired decomposition. \square

Given a ring R and group G we denote the group ring of R over G by RG . A monomial in the group ring is written as rg for $r \in R$ and $g \in G$. An arbitrary element of RG , say $\alpha \in RG$, is of the form $\alpha = \sum_{g \in G} r_g g$ where it is assumed that the set $Supp(\alpha) = \{g \in G \mid r_g \neq 0\}$, called the support of α , is finite [4].

Proposition 6. *Let R be a strongly right singular clean ring, $2 \in U(R)$ and $G = \{1, g\}$. Then RG is strongly strongly right singular clean.*

Proof. Since $2 \in U(R)$, we have $RG \cong R \times R$ via the map $\psi : x + yg \mapsto (x + y, x - y)$, by [3]. Hence RG is strongly strongly right singular clean, by Proposition 3. \square

Let R be a ring and let ${}_R M_R$ be an R - R -bimodule which is a general ring (possibly with no unity) in which $(nm)x = n(mx)$, $(nx)m = n(xm)$ and $(xn)m = x(nm)$ hold for all $n, m \in M$ and $r \in R$. Then the ideal-extension $I(R; M)$ of R by M is defined to be the additive abelian group $I(R; M) = R \oplus M$ with multiplication $(x, n)(y, m) = (xy, xm + ny + nm)$. Note that if S is a ring and $S = R \oplus A$, where R is a subring and A is an ideal of S , then $S \cong I(R; A)$.

Theorem 7. *An ideal-extension $S = I(R; M)$ is strongly right singular clean, if the following conditions are satisfied:*

- (1) R is strongly right singular clean.
- (2) If $e^2 = e \in R$, then $en = ne$ for all $n \in M$.
- (3) If $z \in \mathcal{Z}(R_R)$, then $(z, n) \in \mathcal{Z}(S_S)$ for all $n \in M$.

Proof. Let $s = (x, n) \in S$. Then $x = e + z$ where $e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $ez = ze$. Then $s = (e, 0) + (z, n)$. It is clear that $(e, 0)$ is an idempotent in S . $(e, 0)(z, n) = (z, n)(e, 0)$, by (2) and $(z, n) \in \mathcal{Z}(S_S)$, by (3). Therefore S is strongly right singular clean. \square

A ring R is called strongly right almost clean, if for every $x \in R$ there exist $y \in R$ and $e \in Id(R)$ such that $x = y + e$, where $ann_r(y) = 0$ and $ye = ey$.

Proposition 8. *Every strongly right singular clean ring is strongly right almost clean.*

Proof. Let R be a strongly right singular clean ring and $x \in R$. Then there exist $z \in \mathcal{Z}(R_R)$ and $e \in Id(R)$ such that $x - 1 = e + z$ and $ez = ze$. So $x = (z + 1) + e$, $ann_r(z + 1) = 0$ and

$$(z + 1)e = e(z + 1).$$

Therefore R is a strongly right almost clean ring. □

Proposition 9. *Let R be a local ring. Then R is a strongly right singular clean ring if and only if $R/\mathcal{Z}(R_R) \cong \mathbb{Z}_2$.*

Proof. Follows from [5, Proposition 2.15]. □

A ring R is called weakly right singular clean, if for every $x \in R$, there exist $z \in \mathcal{Z}(R_R)$ and $e \in Id(R)$ such that $x = z + e$ or $x = z - e$ [5].

Clearly every strongly right singular clean ring is weakly right singular clean, but the converse is not true. For example, the ring \mathbb{Z}_3 is weakly singular clean but is not strongly singular clean.

Theorem 10. *Let R be a ring such that idempotent elements of R are central. Then R is strongly right singular clean if and only if R is weakly right singular clean and $2 \in \mathcal{Z}(R_R)$.*

Proof. Assume that R is strongly right singular clean. Hence R is weakly right singular clean and $2 \in \mathcal{Z}(R_R)$, by [5, Corollary 2.4]. Conversely, Suppose R is weakly right singular clean and $2 \in \mathcal{Z}(R_R)$. If $x = z - e$, $z \in \mathcal{Z}(R_R)$ and $e \in Id(R)$, $x = (z - 2e) + e$, which $z - 2e \in \mathcal{Z}(R_R)$ and $(z - 2e)e = e(z - 2e)$. Then R is strongly right singular clean. □

Lemma 11. *Let R be a ring. Then the following are equivalent:*

- (1) R is strongly right singular clean.
- (2) For any $x \in R$, there exist an idempotents $e \in R$ and $z \in \mathcal{Z}(R_R)$ that commute such that $x = e - z$.

Proof. (1) \implies (2) Suppose that $x \in R$. Hence $1 - x = e + z$, which $e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $ez = ze$. Then $x = (1 - e) - z$, as desired.

(2) \implies (1) Assume that $x \in R$. Then there exist an idempotents $e \in R$ and $z \in \mathcal{Z}(R_R)$ that commute such that $1 - x = e - z$. Then $x = (1 - e) + z$, which $1 - e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $(1 - e)z = z(1 - e)$. Therefore R is strongly right singular clean. \square

Lemma 12. *Let R be a ring. Then $x \in R$ is strongly right singular clean if and only if $1 - x$ is strongly right singular clean.*

Proof. Let $x \in R$ is strongly right singular clean. Then $x = e + z$, which $e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $ez = ze$. Hence

$$1 - x = 1 - (e + z) = (1 - e) + (-z),$$

which $1 - e \in Id(R)$, $-z \in \mathcal{Z}(R_R)$ and $(1 - e)(-z) = (-z)(1 - e)$. Conversely, assume $1 - x$ is strongly right singular clean. Then $1 - x = e + z$ which $e \in Id(R)$, $z \in \mathcal{Z}(R_R)$ and $ez = ze$. Hence $x = (1 - e) + (-z)$, as desired. \square

Proposition 13. *Let R be a ring and I be an ideal of R such that $I \subseteq \mathcal{Z}(R_R)$ and for every $y \in I$, $e \in Id(R)$, $ey = ye$. Then the ring R is strongly right singular clean if and only if the ring $S = \begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$ is strongly right singular clean.*

Proof. Since $\mathcal{Z}(S_S) = \begin{pmatrix} \mathcal{Z}(R_R) & I \\ 0 & \mathcal{Z}(R_R) \end{pmatrix}$, is clear. \square

Proposition 14. *Let R be a ring and I be an ideal of R containing an element y with $ann_l(y) = 0$ and for every $y \in I$, $e \in Id(R)$, $ey = ye$. Then the ring R is strongly right singular clean if and only if the ring $S = \{ \begin{pmatrix} a & y \\ 0 & a \end{pmatrix} \mid a \in R, y \in I \}$ is strongly right singular clean.*

Proof. By [5, Lemma 2.25],

$$\mathcal{Z}(S_S) = \{ \begin{pmatrix} a & y \\ 0 & a \end{pmatrix} \mid a \in \mathcal{Z}(R_R), y \in I \}.$$

Then the ring R is strongly right singular clean if and only if the ring $S = \{ \begin{pmatrix} a & y \\ 0 & a \end{pmatrix} \mid a \in R, y \in I \}$ is strongly right singular clean. \square

In Proposition 14, if $I = R$, then $S \cong \frac{R[x]}{\langle x \rangle^2}$. So we have the following corollary.

Corollary 15. *Let R be a ring with central idempotents. Then R is strongly right singular clean if and only if the ring $\frac{R[x]}{\langle x \rangle^2}$ is strongly right singular clean.*

In general for a ring R , the ring

$$S = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R \right\},$$

we have $S \cong \frac{R[x]}{\langle x \rangle^n}$ and

$$\mathcal{Z}(S_S) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \in S \mid a_i \in \mathcal{Z}(R_R) \right\}.$$

Then we have the following theorem.

Theorem 16. *Let R be a ring with central idempotents. Then the following are equivalent:*

- (1) R is strongly right singular clean.
- (2) The ring $\frac{R[x]}{\langle x \rangle^n}$ is strongly right singular clean for some $n \in \mathbb{N}$.
- (3) The ring $\frac{R[x]}{\langle x \rangle^n}$ is strongly right singular clean for every $n \in \mathbb{N}$.

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