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POSITIVITY OF THE DIFFERENCE NEUTRON TRANSPORT OPERATOR

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Abstract: In the present study, a two-dimensional difference neutron transport operator is considered. The resolvent equation for this neutron transport operator is constructed. The positivity of this difference neutron transport operator in $L_1\left(\mathbb{R}^2_{(r,h)}\right)$ is provided. The structure of fractional spaces generated by the two-dimensional difference neutron transport operator is studied. It is established that the norms in the spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right), A_{r,h}\right)$ and $W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)$ are equivalent. This result enabled us to prove the positivity of the difference neutron transport operator in the Slobodeckij space. In practice, the theorem on the stability of the Cauchy problem for the difference neutron transport equation in Banach spaces is presented.

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1. Introduction

The neutron transport theory is one of the most important fields of reactor theory.

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The neutron transport equation is used in many applications of nuclear physics such as the kinetic theory of gases, neutron transport, radiative transfer, etc. The neutron transport equation describes the distribution of neutrons produced in and out of a specific region in terms of their positions in space and time, their energies, and their travel directions. (see, [1], [2], [3], [4], and the references therein). Neutron transport theory has powerful connections to functional analysis, semigroups theory, spectral theory, and positive operators. Several researchers have studied the positivity property of the differential and difference operators in Banach spaces (see, [5], [7], [8], [9], [15], [20], [21], [22], [23], and the references therein).

Many researchers have investigated the structure of fractional spaces generated by differential and difference operators and their applications (see, [10], [11], [12], [13], [14], [15], [16], [17], and the references therein). Eventually, some important results in fractional spaces generated by positive operators and their applications to partial differential equations are given in [18].

In this paper, the structure of fractional spaces generated by the two-dimensional difference neutron transport operator $A_{r,h}$ is investigated. It is established that for any $0<\alpha<1$ the norms in the spaces $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),A_{r,h}\right)$ and $W_1^\alpha\left(\mathbb{R}^2_{(r,h)}\right)$ are equivalent. This result permits us to prove the positivity of the difference neutron transport operator in Slobodeckij space $W_1^\alpha\left(\mathbb{R}^2_{(r,h)}\right)$. In applications, new theorems on the stability of Cauchy problem for the difference neutron transport equation in Banach spaces are presented. Finally, some of these statements were formulated in [19] without proof.

2. Preliminaries

An operator A densely defined in a Banach space E with the domain $D(A)$ is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0<\varphi<\pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi)=\{\rho e^{i\varphi}:0\le\rho\le\infty\}$ and $S_2(\varphi)=\{\rho e^{-i\varphi}:0\le\rho\le\infty\}$, and outside of the sector the resolvent $(\lambda I-A)^{-1}$ is subject to the bound [5]

$$\left\|(A-\lambda I)^{-1}\right\|_{E\to E}\le\frac{M}{1+|\lambda|}.$$

Here, I is the identity operator and $\lambda\in\mathbb{C}$ is a regular value of the operator A. Throughout the present paper, M denotes positive constants, which may differ in time, and thus, is not a subject of precision. However, we will use $M(\alpha,\ \beta,\ ...)$ to stress the fact that the constant depends only on α, β, ...

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$. The operator A is said to be strongly positive in a Banach space E, if $\varphi(A, E) < \frac{\pi}{2}$.

3. The Difference Neutron Transport Operator

We first introduce the resolvent equation for the two-dimensional difference neutron transport operator $A_{r,h}$ acting on the space of grid functions $u^{r,h}$,

$$\lambda u_{m-1}^{n-1} - A_{r,h}u^{r,h} = f\left(x_{n-1}, y_{m-1}, \bar{\omega}\right), (x_n, y_m) \in \mathbb{R}^2_{(r,h)}. \tag{1}$$

Here, $\mathbb{R}^2_{(r,h)} = \{(x_n, y_m)|x_n = nr, y_m = mh, n, m = 0, \pm 1, \pm 2, ...\}$ is the grid space, the numbers r and h are the steps of the grid spaces, and $A_{r,h}$ is defined by the formula

$$A_{r,h}u^{r,h} = \left\{\omega_1 \frac{u_m^n - u_m^{n-1}}{r} + \omega_2 \frac{u_m^{n-1} - u_{m-1}^{n-1}}{h}\right\}_{n,m=0,\pm 1,\pm 2,...}. \tag{2}$$

For the solution of (1) the following formula holds

$$\begin{aligned} &(\lambda I - A_{r,h})^{-1} f\left(x_n, y_m, \bar{\omega}\right) \\ &= \frac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}} \sum_{i=1}^{\infty} R^i f\left(x_n + \frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, y_m + \frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right), \end{aligned} \tag{3}$$

where the constant coefficients ω_1 and ω_2 are directional cosines of neutrons with respect to the x and y axes, $\bar{\omega} = (\omega_1, \omega_2)$,

$\cos\varphi = \frac{\omega_1}{\sqrt{\omega_1^2+\omega_2^2}}$, $\sin\varphi = \frac{\omega_2}{\sqrt{\omega_1^2+\omega_2^2}}$, $x_i = s_i \cos\varphi + x_0$,

$y_i = s_i \sin\varphi + y_0$, $R = \left(1 + \frac{\lambda \Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\right)^{-1}$, $x_i - x_{i-1} = \frac{\omega_1 \Delta s}{\sqrt{\omega_1^2+\omega_2^2}} = r$,

$y_i - y_{i-1} = \frac{\omega_2 \Delta s}{\sqrt{\omega_1^2+\omega_2^2}} = h$, $\Delta s = s_i - s_{i-1}$, $\quad -\infty < s_i < \infty$,

$1 \le i < \infty$, $\quad x_n = nr$, $\quad y_m = mh$, $-\infty < n, m < \infty$.

Let us denote $B_{r,h} = -A_{r,h}$. Then, using formula (3), we obtain

$$u_m^n = (\lambda I + B_{r,h})^{-1} f(x_n, y_m, \bar{\omega})$$
$$= \frac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}} \sum_{i=1}^{\infty} R^i f\left(x_n + \frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, y_m + \frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right), \tag{4}$$

where $n, m = 0, \pm 1, \pm 2, \ldots$. For any $\lambda > 0$, we have the equality

$$B_{r,h} (\lambda I + B_{r,h})^{-1} f(x_n, y_m, \bar{\omega})$$
$$= \lambda \left[\frac{1}{\lambda} - (\lambda I + B_{r,h})^{-1}\right] f(x_n, y_m, \bar{\omega}). \tag{5}$$

Applying formulas (4) and (5), we get

$$\lambda^\alpha B_{r,h} (\lambda I + B_{r,h})^{-1} f(x_n, y_m, \bar{\omega}) = \frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}$$
$$\times \sum_{i=1}^{\infty} s_{i-1}^\alpha R^i \Delta s \frac{\left[f(x_n, y_m, \bar{\omega}) - f\left(x_n + \frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, y_m + \frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right)\right]}{s_{i-1}^\alpha}, \tag{6}$$

where $s_{i-1}^\alpha = \left(\sqrt{\left(\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}\right)^2 + \left(\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}\right)^2}\right)^\alpha$.

4. Positivity of the Difference Neutron Transport Operator $B_{r,h}$ in $L_1\left(R^2_{(r,h)}\right)$

The space $L_1\left(\mathbb{R}^2_{(r,h)}\right)$ is the space of all bounded grid functions, $f^{r,h} = \{f(x_n, y_m, \bar{\omega})\}_{n,m=0,\ \pm 1,\ \pm 2,\ldots}$ defined on $\mathbb{R}^2_{(r,h)}$ for which the following norm is finite:

$$\left\|f^{r,h}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right)} = \sum_{(x_n, y_m) \in \mathbb{R}^2_{(gr,h)}} |f(x_n, y_m, \bar{\omega})|\, rh. \tag{7}$$

Theorem 1. *For all $\lambda > 0$, the resolvent $(\lambda I + B_{r,h})^{-1}$ satisfies the following estimate:*

$$\left\|(\lambda I + B_{r,h})^{-1}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right) \to L_1\left(\mathbb{R}^2_{(r,h)}\right)} \leq \frac{1}{\lambda}. \tag{8}$$

Proof. Using formula (7) and Minkowski's inequality, we obtain

$$\left\|(\lambda I+B_{r,h})^{-1}f^{r,h}\right\|_{L_1\left(R^2_{(r,h)}\right)}\leq\frac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\sum_{i=2}^{\infty}|R|^i$$
$$\times\sum_{(x_n,y_m)\in R^2_{(r,h)}}\left|f\left(x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|rh. \tag{9}$$

Making the change of variables $\bar{x}_n=x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}$ and $\bar{y}_m=y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}$, we have

$$\left\|(\lambda I+B_{r,h})^{-1}f^{r,h}\right\|_{L_1\left(R^2_{(r,h)}\right)}\leq\frac{1}{\lambda}\left\|f^{r,h}\right\|_{L_1\left(R^2_{(r,h)}\right)}$$

for any $\lambda>0$. This completes the proof of Theorem 1. □

In this section, we study the structure of the fractional space $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),B_{r,h}\right)$, $0<\alpha<1$ of all grid functions $f^{r,h}=\{f(x_n,y_m,\bar{\omega})\}_{n,m=0,\ \pm1,\ \pm2,\dots}$ defined on $\mathbb{R}^2_{(r,h)}$ for which the following norm is finite:

$$\left\|f^{r,h}\right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)}=\left\|f^{r,h}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right)}$$
$$+\int_0^{\infty}\left\|\lambda^{\alpha}B_{r,h}(\lambda I+B_{r,h})^{-1}f^{r,h}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right)}\frac{d\lambda}{\lambda}. \tag{10}$$

Recall that the Sobolev-Slobodeckij space $W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)$, $0<\alpha<1$, of all grid functions $f^{r,h}$ defined on $\mathbb{R}^2_{(r,h)}$ for which the following norm is finite:

$$\left\|f^{r,h}\right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)}=\left\|f^{r,h}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right)} \tag{11}$$
$$+\sum_{(x_n,y_m)\in\mathbb{R}^2_{(r,h)}}\sum_{i=2}^{\infty}\frac{\left|f(x_n,y_m,\bar{\omega})-f\left(x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|}{s_{i-1}^{\alpha+1}}rh\Delta s.$$

Applying the definition of $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)$, we get the following estimate:

$$\left\|(\lambda I+B_{r,h})^{-1}\right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)\to E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)}$$
$$\leq\left\|(\lambda I+B_{r,h})^{-1}\right\|_{L_1\left(\mathbb{R}^2_{(r,h)}\right)\to L_1\left(\mathbb{R}^2_{(r,h)}\right)}. \tag{12}$$

From (12) and Theorem 1 it follows the following theorem:

Theorem 2. *Let* $0 < \alpha < 1$. *Then,* $B_{r,h}$ *is a positive operator in* $W_1^\alpha\left(\mathbb{R}^2_{(r,h)}\right)$.

The proof of this statement is based on the following theorem:

Theorem 3. *The spaces* $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)$ *and* $W_1^\alpha\left(\mathbb{R}^2_{(r,h)}\right)$ *are identical for* $0 < \alpha < 1$ *and their norms are equivalent uniformly in* r *and* h.

Proof. First, let us prove that

$$\left\|f^{r,h}\right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)} \leq \frac{M\left(\omega_1,\omega_2\right)}{(1-\alpha)}\left\|f^{r,h}\right\|_{W_1^\alpha\left(\mathbb{R}^2_{(r,h)}\right)}. \tag{13}$$

Using formula (6), the definition of $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)$ and triangle inequality, we obtain

$$\left\|f^{r,h}\right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)}$$

$$\leq \sum_{i=2}^{\infty}\sum_{(x_n,y_m)\in\mathbb{R}^2_{(r,h)}} J_i \frac{\left|f\left(x_n,y_m,\bar{\omega}\right)-f\left(x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|}{s_{i-1}^{\alpha+1}}rh,$$

where

$$J_i=\int_0^\infty \frac{\lambda^\alpha}{\sqrt{\omega_1^2+\omega_2^2}}\frac{s_{i-1}^{\alpha+1}\Delta s}{\left(1+\frac{\lambda\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\right)^i}d\lambda.$$

The change of variable $\frac{\lambda\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}=y$ yields (see [6])

$$J_i=\left(\sqrt{\omega_1^2+\omega_2^2}\right)^\alpha\int_0^\infty t^{i-1}e^{-t}\frac{(i-1)^{\alpha+1}}{(i-1)!}\int_0^\infty y^\alpha e^{-ty}dydt.$$

The change of variables $ty=z$ yields

$$J_i=\left(\sqrt{\omega_1^2+\omega_2^2}\right)^\alpha\Gamma\left(\alpha+1\right)\int_0^\infty e^{-t}t^{i-1-\alpha-1}\frac{(i-1)^\alpha}{(i-2)!}dt$$

$$= \left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha} \frac{(i-1)^{\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(i-\alpha-1)}{\Gamma(i)}.$$

Recall Gautschi's inequality (see [24]) which says that for any
$0 < s < 1$ and $n = 1, 2, 3, ...$

$$\frac{\Gamma(n+s)}{\Gamma(n+1)} \le n^{s-1}.$$

Since

$$\Gamma(i-\alpha-1) = \frac{\Gamma(i-\alpha)}{(i-\alpha-1)},$$

and making substitution $n = i-1$ and $s = 1-\alpha$ in Gautschi's inequality, we obtain

$$\frac{\Gamma(i-\alpha)}{\Gamma(i)} \le (i-1)^{-\alpha}.$$

So, we can write

$$J_i \le \left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha} \Gamma(\alpha+1)\,\frac{(i-1)}{(i-\alpha-1)} \le \frac{M(\omega_1,\omega_2)}{(1-\alpha)}$$

for $i \ge 2$ and $0 < \alpha < 1$. The last inequality yields

$$\left\| f^{r,h} \right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)} \le \frac{M(\omega_1,\omega_2)}{(1-\alpha)} \left\| f^{r,h} \right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)}.$$

Thus, we have proved that

$$E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right) \subset W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right). \tag{14}$$

Let us prove the reverse inclusion

$$W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right) \subset E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right). \tag{15}$$

Applying formula (6), we can write

$$\begin{aligned} & f(x_n, y_m, \bar{\omega}) \\ &= \int_0^{\infty} \frac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}} \sum_{i=1}^{\infty} R^i \tilde{f}\left(x_n + \frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, y_m + \frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right) d\lambda, \end{aligned} \tag{16}$$

where $\tilde{f}\left(x_n,y_m,\bar{\omega}\right)=B_{r,h}\left(\lambda I+B_{r,h}\right)^{-1}f\left(x_n,y_m,\bar{\omega}\right)$.

From (16) it follows that

$$f\left(x_n+\tfrac{\omega_1 s_j}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_j}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)=\int\limits_0^{\infty}\tfrac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\sum_{i=1}^{\infty}R^i$$
$$\times\tilde{f}\left(x_n+\tfrac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}+\tfrac{\omega_1 s_j}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}+\tfrac{\omega_2 s_j}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)d\lambda. \tag{17}$$

Using identities (16) and (17), we obtain

$$f\left(x_n,y_m,\bar{\omega}\right)-f\left(x_n+\tfrac{\omega_1 s_j}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_j}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)$$
$$=\int\limits_0^{\infty}\tfrac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\left\{\sum_{i=1}^{\infty}R^i\tilde{f}\left(x_n+\tfrac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right.$$
$$\left.-\sum_{i=1}^{\infty}R^i\tilde{f}\left(x_n+\tfrac{\omega_1 s_{i+j-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{i+j-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right\}d\lambda.$$

The change of variable $i=k$, $i=k-j$, and $k=i+j$ yields

$$f\left(x_n,y_m,\bar{\omega}\right)-f\left(x_n+\tfrac{\omega_1 s_j}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_j}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)$$
$$=\int\limits_0^{\infty}\tfrac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\left\{\sum_{k=1}^{j}R^k\tilde{f}\left(x_n+\tfrac{\omega_1 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right.$$
$$\left.-\sum_{k=j+1}^{\infty}\left[R^k-R^{k-j}\right]\tilde{f}\left(x_n+\tfrac{\omega_1 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right\}d\lambda.$$

Performing the change of variable $j=i+1$, we get

$$f\left(x_n,y_m,\bar{\omega}\right)-f\left(x_n+\tfrac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)$$
$$=\int\limits_0^{\infty}\tfrac{\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\left\{\sum_{k=1}^{i-1}R^k\tilde{f}\left(x_n+\tfrac{\omega_1 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right.$$
$$\left.-\sum_{k=i}^{\infty}\left[R^k-R^{k-i+1}\right]\tilde{f}\left(x_n+\tfrac{\omega_1 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\tfrac{\omega_2 s_{k-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right\}d\lambda.$$

Taking sum, using the triangle inequality, and performing change of

$\tilde{f}(x_n, y_m, \bar{\omega}) = B_{r,h}(\lambda I + B_{r,h})^{-1} f(x_n, y_m, \bar{\omega})$, we have

$$\sum_{(x_n,y_m)\in\mathbb{R}^2_{(r,h)}} \sum_{i=1}^{\infty} \frac{\left|f(x_n,y_m,\bar{\omega})-f\left(x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right)\right|}{s_{i-1}^{\alpha+1}} rh\Delta s$$

$$\leq (M_1+M_2)\int_0^{\infty} \sum_{(\bar{x_n},\bar{y_m})\in\mathbb{R}^2_{(r,h)}} \left|\lambda^{\alpha} B_{r,h}(\lambda I + B_{r,h})^{-1} f(\bar{x_n}, \bar{y_m}, \bar{\omega})\right| rh\frac{d\lambda}{\lambda}, \tag{18}$$

where

$$M_1 = \left|\sum_{i=1}^{\infty} s_{i-1}^{-\alpha-1} \frac{(\Delta s)^2}{\sqrt{\omega_1^2+\omega_2^2}} \lambda^{1-\alpha} \sum_{k=1}^{i-1} R^k\right|,$$

$$M_2 = \left|\sum_{i=1}^{\infty} s_{i-1}^{-\alpha-1} \frac{(\Delta s)^2}{\sqrt{\omega_1^2+\omega_2^2}} \lambda^{1-\alpha} \sum_{k=i}^{\infty} \left[R^k - R^{k-i+1}\right]\right|.$$

Let us estimate M_1 and M_2, separately. First, let us estimate M_1:

$$M_1 = \left|\sum_{i=1}^{\infty} (i\lambda\Delta s)^{-\alpha-1} \lambda\Delta s \left(1-\left(1+\frac{\lambda\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\right)^{-i}\right)\right|.$$

The change of variables $\lambda\Delta s = \Delta\rho$ yields

$$M_1 = \left|\sum_{i=1}^{\infty} (i\Delta\rho)^{-\alpha-1} \left(1-\left(1+\frac{\lambda\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\right)^{-i}\right)\Delta\rho\right|$$

$$\leq \left|\int_0^{\infty} \rho^{-\alpha-1}\left(1-e^{-\rho}\right) d\rho\right| \leq \frac{1}{\alpha(1-\alpha)}. \tag{19}$$

Actually, applying the inequality $e^{-\tau} \geq 1-\tau$, we obtain

$$\int_0^{\infty} \frac{1-e^{-\tau}}{\tau^{\alpha+1}} d\tau \leq \int_0^1 \tau^{-\alpha} d\tau + \int_1^{\infty} \tau^{-\alpha-1} d\tau = \frac{1}{\alpha} + \frac{1}{1-\alpha} = \frac{1}{\alpha(1-\alpha)}.$$

Now, let us estimate M_2:

$$M_2 = \left|\sum_{i=1}^{\infty} s_{i-1}^{-\alpha-1} \frac{(\Delta s)^2}{\sqrt{\omega_1^2+\omega_2^2}} \lambda^{1-\alpha} \left(\frac{R^i}{1-R} - \frac{R}{1-R}\right)\right|$$

$$= \left|\sum_{i=1}^{\infty} (i\lambda\Delta s)^{-\alpha-1} \left(1-\left(1+\frac{\lambda\Delta s}{\sqrt{\omega_1^2+\omega_2^2}}\right)^{-i}\right)\lambda\Delta s\right|. \tag{20}$$

The change of variables $\lambda\Delta s = \Delta\rho$ yields

$$M_2 = \left|\sum_{i=1}^{\infty}(i\Delta\rho)^{-\alpha-1}\left(1-\left(1+\frac{\Delta\rho}{\sqrt{\omega_1^2+\omega_2^2}}\right)^{-i}\right)\Delta\rho\right|$$
$$\leq \left|\int_0^{\infty}\rho^{-\alpha-1}\left(1-e^{-\rho}\right)d\rho\right| \leq \frac{1}{\alpha(1-\alpha)}. \tag{21}$$

Using identity (18) and combining estimates (19) and (21), we get

$$\sum_{(x_n,y_m)\in\mathbb{R}^2_{(r,h)}}\sum_{i=1}^{\infty}\frac{\left|f\left(x_n,y_m,\bar{\omega}\right)-f\left(x_n+\frac{\omega_1 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},y_m+\frac{\omega_2 s_{i-1}}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|}{s_{i-1}^{\alpha+1}}rh\Delta s$$
$$\leq \frac{2}{\alpha(1-\alpha)}\int_0^{\infty}\sum_{\left(\bar{x_n},\bar{y_m}\right)\in\mathbb{R}^2_{(r,h)}}\left|\lambda^{\alpha}B_{r,h}\left(\lambda I+B_{r,h}\right)^{-1}f\left(\bar{x_n},\bar{y_m},\bar{\omega}\right)\right|rh\frac{d\lambda}{\lambda}. \tag{22}$$

This means that the following inequality holds:

$$\left\|f^{r,h}\right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)} \leq \frac{2}{\alpha\left(1-\alpha\right)}\left\|f^{r,h}\right\|_{E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right),\ B_{r,h}\right)}. \tag{23}$$

This completes the proof of Theorem 3. □

5. Applications

In this section we consider the application of results of Sections 3 and 4. We consider the difference scheme

$$\begin{cases} \frac{u^k_{n,m}-u^{k-1}_{n,m}}{\tau}-\omega_1\frac{u^k_{n,m}-u^k_{n,m-1}}{r}-\omega_2\frac{u^k_{n-1,m}-u^k_{n-1,m-1}}{h} \\ = f\left(t_k,x_n,y_m\right), \\ t_k=k\tau,\ 1\leq k\leq N,\ N\tau=T,\ \left(x_n,y_m\right)\in\mathbb{R}^2_{(r,h)} \\ u(0,x_n,y_m)=\varphi\left(x_n,y_m\right),\ \left(x_n,y_m\right)\in\mathbb{R}^2_{(r,h)} \end{cases} \tag{24}$$

for the numerical solution of initial value problem

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t}-\omega_1\frac{\partial u(t,x,y)}{\partial x}-\omega_2\frac{\partial u(t,x,y)}{\partial y}=f\left(t,x,y\right), \\ (x,y)\in\mathbb{R}^2,\ 0\leq t\leq T, \\ u(0,x,y)=\varphi\left(x,y\right),\ (x,y)\in\mathbb{R}^2. \end{cases} \tag{25}$$

Theorem 4. *Let* $0 < \alpha < 1$. *Then, for the solution of difference scheme (24) we have the following stability inequality:*

$$\sum_{k=1}^{N} \tau \left\| \left(u^k\right)^{r,h} \right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)} \\ \leq M(\alpha) \left[\left\| \varphi^{r,h} \right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)} + \sum_{k=1}^{N} \tau \left\| \left(f^k\right)^{r,h} \right\|_{W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)} \right]. \tag{26}$$

The proof of Theorem 4 is based on Theorem 1 on the positivity of the difference neutron transport operator $B_{(r,h)} = -A_{(r,h)}$ defined by the formula (2), on Theorem 3 on the structure of fractional space $W_1^{\alpha}\left(\mathbb{R}^2_{(r,h)}\right)$, and on the following theorem on stability of difference scheme (24) for the approximate solution of abstract initial value problem (25).

Theorem 5. *Let* $B_{(r,h)}$ *be a positive operator in a Banach space* $E_{\alpha,1}\left(L_1\left(\mathbb{R}^2_{(r,h)}\right), B_{r,h}\right)$. *Then, for the solution of difference scheme (24) the following stability inequality holds:*

$$\sum_{k=1}^{N} \tau \left\| u^k \right\|_{E_{\alpha,1}\left(\mathbb{R}^2_{(r,h)}\right)} \leq M \left[\left\| \varphi \right\|_{E_{\alpha,1}\left(\mathbb{R}^2_{(r,h)}\right)} + \sum_{k=1}^{N} \tau \left\| f^k \right\|_{E_{\alpha,1}\left(\mathbb{R}^2_{(r,h)}\right)} \right].$$

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