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CRANK-NICOLSON DIFFERENCE SCHEME FOR REVERSE PARABOLIC NONLOCAL PROBLEM WITH INTEGRAL AND NEUMANN BOUNDARY CONDITIONS

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Abstract: In this paper, we study Crank-Nicholson difference scheme for approximate solutions of parabolic nonlocal reverse problem with integral and Neumann boundary conditions. Stability estimates for its solution are established. Via Mathlab framework, we give numerical example with explanation on computer realization.

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1. Introduction

Reverse parabolic (RP) equations appear in many applications such as mean field equilibria in economics, plasma physics, fluid dynamics, investigation of electron beam propagation through the solar corona (see [1, 8, 9, 10, 12, 13, 16] and references therein). Well-posedness of nonlocal boundary value problems (BVPs) for RP equation and difference schemes (DSs) for their approximations were investigated in [2, 4, 5, 6, 7].

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Well-posedness of the RP problem with the corresponding nonlocal integral condition

$$\begin{cases} u_t(t) - Au(t) \ = \ f(t), \ \ 0 \le t \le 1, \\ u(1) \ = \ \int\limits_0^1 \mu(\gamma) u(\gamma)\, d\gamma \ + \ \varphi, \end{cases} \tag{1}$$

was established in paper [4]. Here and in future, H is an arbitrary Hilbert space, $A : H \to H$ is a self-adjoint positive definite operator (SAPDO) such that $A > \delta I$ for identity operator $I : H \to H$ and some positive number $\delta > 0$, $\mu : [0,1] \to R,\ \ f : [0,1] \to H$ are given functions and $\varphi \in H$ is known.

In paper [5], stability estimates (SEs) for solution of Rothe DS for the approximate solution RP problem (1) were established. In [6], second order of accuracy difference scheme (ADS) by using A^2 was studied. In this work, we will study Crank-Nicolson DS for approximately solving of RP problem (1) and establish SEs for its solution.

Denote by $[0,1]_\tau = \left\{ t_i |\ t_i = i\tau, \tau = \frac{1}{N} \right\}$ the uniform grid space for natural number N, and $C_\tau(H) = C([0,1]_\tau, H)$, Banach space of grid functions $v^\tau = \{v_i\}_{i=1}^{i=N}$ with $v_i = v(t_i) \in H$ and norm $\|v^\tau\|_{C_\tau(H)} = \max\limits_{1 \le i \le N} \|v_i\|_H$. In addition, by $C_\tau^\alpha(H) = C^\alpha([0,1]_\tau, H)$,
and $C_1^\alpha(H) = C_1^\alpha([0,1]_\tau, H)$ we will denote Banach spaces of grid functions v^τ with the corresponding norms

$$\|v^\tau\|_{C_\tau^\alpha(H)} = \|v^\tau\|_{C_\tau(H)} + \max_{1 \le i < i+j \le N} \frac{\|v_{i+j} - v_i\|_H}{(j\tau)^\alpha},$$

$$\|v^\tau\|_{C_1^\alpha(H)} = \|v^\tau\|_{C_\tau^\alpha(H)} + \max_{1 \le i < i+j \le N} (j\tau)^{-\alpha} \left((N-i)\tau\right)^\alpha \ \|v_{i+j} - v_i\|_H. \tag{2}$$

Let us take $t_{j-\frac{1}{2}} = t_j - \frac{\tau}{2},\ j = 1, ..., N.$ Assume that

$$\tau \left| \mu(t_{N-\frac{1}{2}}) \right| + \tau \left| \mu(\frac{\tau}{2}) \ \right| + \tau \sum_{j=1}^{N-1} \left| \mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}}) \right| < 2. \tag{3}$$

Lemma 1. *Under assumption (3), there exists bounded inverse* $Q_\tau = T_\tau^{-1}$ *for operator*

$$T_\tau = \left(1 - \frac{\tau}{2}\mu(t_{N-\frac{1}{2}}\ )\right) I - \frac{\tau}{2}\mu(\frac{\tau}{2})\ P^N - \frac{\tau}{2} \sum_{j=1}^{N-1} \left[\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right] P^{N+j},$$

where $P = \left(I + \frac{A\tau}{2}\right)^{-1} \left(I - \frac{A\tau}{2}\right)$.

Proof. It is known that $\|P\|_{H\to H} \leq 1$ ([3]). Denote by

$$q_1 = \frac{\tau}{2}\left|\mu(\frac{\tau}{2})\right|\left(\frac{1-\frac{\tau\lambda}{2}}{1+\frac{\tau\lambda}{2}}\right)^N, \; q_2 = \frac{\tau}{2}\sum_{j=1}^{N-1}\left[\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right]\left(\frac{1-\frac{\tau\lambda}{2}}{1+\frac{\tau\lambda}{2}}\right)^{N+j}.$$

Applying the definition of function's norm for SAPDO (see [11]), we have

$$\begin{aligned} \|Q_\tau\|_{H\to H} &\leq \sup_{\lambda\geq\delta} \frac{1}{\left|1-\frac{\tau}{2}\left|\mu(t_{N-\frac{1}{2}})\right|-q_1-q_2\right|} \\ &\leq \sup_{\lambda\geq\delta} \frac{1}{1-\frac{\tau}{2}\left|\mu(t_{N-\frac{1}{2}})\right|-\frac{\tau}{2}\left|\mu(\frac{\tau}{2})\;\right|-\sum\limits_{j=1}^{N-1}\left|\mu(t_{j-\frac{1}{2}})+\mu(t_{j+\frac{1}{2}})\right|\frac{\tau}{2}} \leq M. \end{aligned} \tag{4}$$

Lemma 1.1 is proved. □

2. Crank-Nicolson DS

Nonlocal integral condition of (1) can be replaced by the second order approximation

$$u_N = \sum_{j=1}^{N} t_{j-\frac{1}{2}}\left(\frac{u_j + u_{j-1}}{2}\right)\tau + \varphi.$$

By using Crank-Nicolson DS, we get DS in the following form

$$\begin{aligned} &\tfrac{u_k-u_{k-1}}{\tau} - \tfrac{1}{2}\left(Au_k + Au_{k-1}\right) = \theta_k, \theta_k = f\left(t_{k-\frac{1}{2}}\right), k = 1,..,N, \\ &-\tfrac{\tau}{2}\mu(\tfrac{\tau}{2})u_0 - \sum_{j=1}^{N-1}\tfrac{\tau}{2}\left[\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right]u_j + \left(1-\tfrac{\tau}{2}\mu(t_{N-\frac{1}{2}})\right)u_N = \varphi. \end{aligned} \tag{5}$$

Theorem 2. *Assume that $\theta^\tau \in C_\tau(H), \varphi \in D(A)$, and (3) is satisfied. Then, DS (5) is uniquely solvable and for the solutions the following SE*

$$\max_{0\leq j\leq N}\|u_j\|_H \leq M(\mu,\delta)\left(\|\varphi\|_H + \|\theta^\tau\|_{C_\tau(H)}\right) \tag{6}$$

is valid for some positive real constant $M(\mu,\delta)$ which depends on μ, δ but it is independent of τ, φ, θ^τ.

Proof. Let us take $C = \left(I - \frac{A\tau}{2}\right)^{-1}$. From DS (5) it follows

$$\left(I + \frac{A}{2}\right)u_{k-1} = \left(I - \frac{A\tau}{2}\right)u_k - \tau\theta_k\;, u_{k-1} = Pu_k - \tau C\theta_k,\; k = 1, ..., N.$$

Thus, we can conclude that

$$u_q = P^{N+q} u_N - \tau \sum_{i=N}^{q+1} P^{N-2-i} C\theta_i, q = N-1, ..., 0. \tag{7}$$

By using (7) in nonlocal condition of DS (5), we get

$$\begin{aligned} &-\tfrac{\tau}{2}\mu(\tfrac{\tau}{2}) \left(P^N u_N - \tau \sum_{i=N}^{1} P^{N-2-i} C\theta_i \right) + \left(1 - \tfrac{\tau}{2}\mu(t_{N-\frac{1}{2}})\right) u_N \\ &- \sum_{j=1}^{N-1} \tfrac{\tau}{2} \left[\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right] \left(P^{N+j} u_N - \tau \sum_{i=N}^{j+1} P^{N-2-i} C\theta_i \right) = \varphi. \end{aligned}$$

Thus, we have

$$\begin{aligned} u_N \ &= Q_\tau \left\{ \varphi - \tfrac{\tau^2}{2}\mu(\tfrac{\tau}{2}) \sum_{i=N}^{1} P^{N-2-i} C\theta_i \right. \\ &\left. -\tfrac{\tau^2}{2} \sum_{j=1}^{N-1} \left[\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right] \sum_{i=N}^{j+1} P^{N-2-i} C\theta_i \right\}. \end{aligned} \tag{8}$$

Triangle inequality, estimate (4), formulas (7), (8) yield estimate (6). Theorem 2.1 is proved. □

Applying formulas (7), (8) in Theorem 2.2 of paper [5], we have the next theorem on coercive SE for solution of DS (5).

Theorem 3. *Assume $\varphi \in D(A), \theta^\tau \in C_1^\alpha(H)$, and inequality (3) is valid. Then, the solution of DS (5) satisfies the coercive SE*

$$\begin{aligned} &\left\| \left\{ \tfrac{u_i - u_{i-1}}{\tau} \right\}_1^N \right\|_{C_1^\alpha(H)} + \left\| \left\{ \tfrac{1}{2}\left(Au_i + Au_{i-1}\right) \right\}_1^N \right\|_{C_1^\alpha(H)} \\ &\leq \ M(\mu,\delta) \left(\tfrac{1}{\alpha(1-\alpha)} \left\|\theta^\tau\right\|_{C_1^\alpha(H)} + \left\|A\varphi\right\|_H \right) \end{aligned} \tag{9}$$

for some positive real constant $M(\mu,\delta)$ which is independent of τ, φ, θ^τ, but depends on μ, δ.

3. DS for multidimensional problem

Let $\Omega=(0,l)^n$ be open cube in $R^n, S=\partial\Omega$, $\overline{\Omega}=\Omega\cup S$, and
$a_r:\Omega\to R, \varphi:\overline{\Omega}\to R,\ \mu:[0,1]\to R,\ f:(0,1)\times\Omega\to R$ be given functions, σ be known positive real number. In addition,
$\forall r=1,...,n,\ \forall x=(x_1,..,x_n)\in\Omega,\ a_r(x)\geq a_0>0.$

In the work [6], BVP for multidimensional RP equation with nonlocal integral and second kind of boundary conditions

$$\begin{cases} u_t\ (t,x)+\sum\limits_{r=1}^{n}\ (a_r(x)\ u_{x_r}(t,x))\ _{x_r}-\sigma\ u(t,x)=f(t,x), \\ t\in(0,1)\,, x\in\Omega, \frac{\partial u}{\partial\overrightarrow{n}}(t,x)=0,\ x\in S,\ t\in[0,1]\,, \\ u(1,x)\ =\ \int\limits_0^1\mu(\gamma)u(\gamma,x)d\gamma+\varphi(x),\ x\in\overline{\Omega} \end{cases} \tag{10}$$

was investigated on well-posedness.

Now, we will construct the second order of ADS to solve BVP (10). Introduce space of grid points

$$\widetilde{\Omega}_h=\{x_j=(h_1m_1,\cdots,h_nm_n),\ m=(m_1,\cdots,m_n),m_r=0,\cdots,N_r,$$
$$h_rN_r=l,r=1,\cdots,n\},\Omega_h=\Omega\cap\widetilde{\Omega}_h,S_h=\widetilde{\Omega}_h\cap S.$$

Denote by A_h^x the operator $A_h^xu^h=-\sum_{r=1}^n\left(a_ru_{\overline{x}_r}^h\right)_{x_r,j_r}+\sigma u^h$ acting in the space of grid functions $u^h(x)$ which satisfies the condition $D^hu^h(x)=0$ on $x\in S_h$. Here $D^hu^h(x)$ means the second order of approximation of $\frac{\partial u}{\partial\overrightarrow{n}}(x)$. Then, for problem (10) we have the second order of ADS

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-\frac{1}{2}\left(A_h^xu_k^h(x)+A_h^xu_{k-1}^h(x)\right)=\theta_k(x), \\ \theta_k(x)=f^h(t_{k-\frac{\tau}{2}},x),t_k=k\tau,\ k=1,...,N,\ N\tau=1,x\in\widetilde{\Omega}_h, \\ u_N^h(x)=\sum\limits_{j=1}^{N}\mu(t_{j-\frac{1}{2}})\frac{\tau}{2}\ \left[u_j^h(x)+u_{j-1}^h(x)\right]+\varphi^h(x),x\in\widetilde{\Omega}_h. \end{cases} \tag{11}$$

Here, $|h|=\left(\sum\limits_{r=1}^{n}h_r^2\right)^{\frac{1}{2}}$ and τ are sufficiently small positive real numbers.

Let L_{2h} and W_{2h}^2 be spaces of the grid functions $\varphi^h(x)$ defined on grid space $\widetilde{\Omega}_h$, equipped with the appropriate norms

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{x\in\widetilde{\Omega}_h}|\varphi^h(x)|^2h_1\cdots h_n\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^2}=\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|(\varphi^h(x))_{x_r\overline{x_r},\ m_r}\right|^2h_1\cdots h_n\right)^{1/2}.$$

Theorem 4. *For the solution of DS (11), the SE*

$$\left\| \left\{ u_k^h \right\}_1^N \right\|_{\mathcal{C}_\tau(L_{2h})} \leq M(\delta,\mu) \left[\left\| \varphi^h \right\|_{L_{2h}} + \left\| \left\{ \theta_k^h \right\}_1^N \right\|_{\mathcal{C}_\tau(L_{2h})} \right]$$

is fulfilled, where $M(\delta,\mu)$ *is independent of* τ, $\varphi^h(x)$, *and* $\left\{ \theta_k^h \right\}_1^N$.

Theorem 5. *Solution of DS (11) satisfies the following coercivity SE:*

$$\begin{aligned} &\left\| \left\{ \tau^{-1}(u_k^h - u_{k-1}^h) \right\}_1^N \right\|_{C_1^\alpha(L_{2h})} + \left\| \left\{ \tfrac{1}{2}\left(A_h^x u_k^h + A_h^x u_{k-1}^h \right) \right\}_1^N \right\|_{C_1^\alpha(L_{2h})} \\ &\leq M(\delta,\mu,\alpha) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \tfrac{1}{\alpha(1-\alpha)} \left\| \left\{ \theta_k^h \right\}_1^N \right\|_{C_1^\alpha(W_{2h}^2)} \right], \end{aligned}$$

where $M(\delta,\mu,\alpha)$ *is independent of* τ, $\left\{ \theta_k^h \right\}_1^N$, *and* $\varphi^h(x)$.

Proof. The proofs of Theorems 4 and 5 are based on estimates (6), (9), assumption (3), and the theorem on the coercivity stability property ([15]) for the solution of the elliptic difference problem in L_{2h}. □

4. Numerical example

To illustrate test example, we consider the following BVP with nonlocal integral condition

$$\begin{cases} u_t(t,x) + (1+x)^2 u_{xx}(t,x) + 2(1+x)u_x(t,x) - u(t,x) \\ = -4e^{-3t}\left\{ \left(x^2 + 2x + 2 \right)\cos 2x + (1+x)\sin 2x \right\}, \\ 0 \leq t \leq 1, x \in \left(0, \frac{\pi}{2} \right), \\ u(1,x) = \int\limits_0^1 e^{-s} u(s,x)ds + \left(\frac{e^{-4} + 4e^{-3} - 1}{4} \right)\cos 2x, \ x \in \left[0, \frac{\pi}{2} \right], \\ u_x(t,0) = 0, \ u_x(t, \frac{\pi}{2}) = 0, \ t \in [0,1]. \end{cases} \tag{12}$$

Exact solution of problem(12) is $u(t,x) = e^{-3t}\cos 2x$.

Applying (11) to this problem, we get the second order of ADS in t and x

for BVP (12):

$$\begin{aligned}
&\tfrac{u_n^k-u_n^{k-1}}{\tau}-\tfrac{1}{2}\Bigg(-(1+x_n)^2\tfrac{\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right)}{h^2}-2(1+x_n)\tfrac{\left(u_{n+1}^k-u_{n-1}^k\right)}{2h}\\
&+u_n^k-(1+x_n)^2\tfrac{\left(u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}\right)}{h^2}-2(1+x_n)\tfrac{\left(u_{n+1}^{k-1}-u_{n-1}^{k-1}\right)}{2h}+u_n^{k-1}\Bigg)\\
&=f(t_{k-1/2}),\ \ -3u_1^k+4u_2^k-u_3^k=0,\\
&10u_1^k-15u_2^k+6u_3^k-u_4^k=0,-3u_{M+1}^k+4u_M^k-u_{M-1}^k=0,\\
&10u_{M+1}^k-15u_M^k+6u_{M-1}^k-u_{M-2}^k=0,k=0,...,N,\\
&-\tfrac{1}{2}\mu(\tfrac{\tau}{2})\tau\ u_n^0-\sum_{j=1}^{N-1}\tfrac{1}{2}\left[\mu(t_j-\tfrac{\tau}{2})+\mu(t_{j+1}-\tfrac{\tau}{2})\right]\ \tau\ u_n^j\\
&+\left(1-\tfrac{1}{2}\mu(t_N-\tfrac{\tau}{2})\tau\ \right)u_n^N=\varphi_n,\ \ n=\overline{0,M}.
\end{aligned} \tag{13}$$

One can write (13) in the following matrix form

$$\begin{cases}
A_nu_{n+1}+B_nu_n+C_nu_{n-1}=R\psi_n,n=3,...,M-3,\\
-3u_1+4u_2-u_3=0,10u_1-15u_2+6u_3-u_4=0\\
-3\ u_{M+1}+4\ u_M\ -\ u_{M-1}=0,\\
10\ u_{M+1}-\ 15\ u_M\ +\ 6\ u_{M-1}\ -\ u_{M-2}=0.
\end{cases} \tag{14}$$

Here, R is identity matrix, ψ_n , u_s,
$s=n-1,n,n+1$ are $(N+1)\times 1$ column matrices

$$\begin{aligned}
&\psi_n=\left[\begin{array}{cccc} \psi_n^0 & \psi_n^1 & \cdots & \psi_n^{N-1} & \psi_n^N\end{array}\right]^t,\\
&u_s=\left[\begin{array}{cccc} u_s^0 & u_s^1 & \cdots & u_s^{N-1} & u_s^N\end{array}\right]^t,s=n-1,n,n+1,
\end{aligned}$$

and A_n, B_n, C_n are $(N+1)\times(N+1)$ matrices

$$A_n=\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0\\
a_n & a_n & \cdots & 0 & 0\\
\vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \ddots & a_n & 0\\
0 & 0 & \cdots & a_n & a_n
\end{array}\right],C_n=\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0\\
d_n & d_n & \cdots & 0 & 0\\
\vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \ddots & d_n & 0\\
0 & 0 & \cdots & d_n & d_n
\end{array}\right],$$

$$B_n=\left[\begin{array}{ccccc}
s_0 & s_1 & \cdots & s_{N-1} & s_N\\
b_n & c_n & \cdots & 0 & 0\\
\vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \ddots & c_n & 0\\
0 & 0 & \cdots & b_n & c_n
\end{array}\right],$$

with elements

$$
\begin{array}{l}
a_n = \frac{(1+x_n)^2}{2h^2} + \frac{(1+x_n)}{2h}, b_n = \frac{1}{\tau} - \frac{(1+x_n)^2}{h^2} - \frac{1}{2}, c_n = -\frac{1}{\tau} - \frac{(1+x_n)^2}{h^2} - \frac{1}{2},\\
d_n = \frac{(1+x_n)^2}{2h^2} - \frac{(1+x_n)}{2h}, s_0 = -\frac{\tau}{2}\mu(\frac{\tau}{2}), s_N = 1 - \frac{\tau}{2}\mu(t_N - \frac{\tau}{2}),\\
s_j = -\frac{\tau}{2}\left(\mu(t_{j-\frac{1}{2}}) + \mu(t_{j+\frac{1}{2}})\right), j = 1, ..., N-1.
\end{array}
$$

To solve matrix equation modified Gauss elimination method [14] is used.

We seek solution in the form $u_n = \alpha_n u_{n+1} + \beta_n u_{n+2} + \gamma_n$, $n = 0, ..., M-2$. It is easy to see that

$$
\begin{array}{l}
\alpha_0 = \frac{4}{3}R, \beta_0 = -\frac{1}{3}R, \alpha_1 = \frac{8}{5}R, \beta_1 = -\frac{3}{5}R, \gamma_0 = \gamma_1 = O,\\
\alpha_n = -D_n(A_n + C_n\beta_{n-1}), \beta_n = 0, \gamma_n = D_n(R\psi_n - C_n\gamma_{n-1}),\\
D_n = (B_n + C_n\alpha_{n-1})^{-1} \ n = 2, ..., M-2.
\end{array}
$$

For unknowns u_M and u_{M+1} we get the system of equation

$$
Q_{11}u_M + Q_{12}\ u_{M+1} = G_1, Q_{21}u_M + Q_{22}u_{M+1} = G_2, \tag{15}
$$

where

$$
\begin{array}{l}
Q_{11} = B_M + 4C_M, Q_{12} = A_M - 3C_M, Q_{21} = 4\alpha_{M-2} + \beta_{M-2} - 9R,\\
Q_{22} = -3\alpha_{M-2} + 8R, G_1 = R\psi_M, G_2 = -R\gamma_{M-2}.
\end{array}
$$

Solution of (15) can be derived by formulas

$$
\begin{array}{l}
u_{M+1} = Q_{22}^{-1}\left(G_2 - \ Q_{21}u_M\right),\\
u_M = \left(Q_{11} - Q_{12}\ Q_{22}^{-1}Q_{21}\right)^{-1}\left(G_1 - Q_{12}\ Q_{22}^{-1}G_2\right).
\end{array}
$$

Table 1 shows error computed by

$$
\begin{array}{l}
E_1 = \max_{1\le k\le N-1}\max_{0\le n\le M} \ \left|u(x_n, t_k) - u_n^k\right|,\\
E_2 = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}(u(x_n, t_k) - u_n^k)^2 h\right)^{\frac{1}{2}}
\end{array}
$$

for different values of (N, M).

Numerical results presented in Table 1 shows good agreement with obtained theoretical results on stability of solution of proposed DS to RP BVP with second kind boundary and integral conditions. We observe that DS (13) has the second order convergence as it is expected to be.

Table 1. The errors between the exact and the numerical solutions of BVP (12) for different values of **N** and **M**.

$(\mathbf{N},\mathbf{M})$	$\mathbf{E}_1$	$\mathbf{E}_2$
$(10,10)$	0.13019	0.12041
$(20,20)$	0.02412	0.02384
$(40,40)$	0.00437	0.00437
$(80,80)$	0.00085	0.00085
$(160,160)$	0.00021	0.00021
$(320,320)$	0.00005	0.00005

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