ON PERIODIC BOUNDARY VALUE PROBLEMS
WITH AN OBLIQUE DERIVATIVE FOR
A SECOND ORDER ELLIPTIC EQUATION

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Abstract: In this paper, we study solvability of new classes of nonlocal boundary value problems for a second-order elliptic type equation. The considered problems are multidimensional analogues (in the case of circular domains) of classical periodic boundary value problems in rectangular domains.

To study the main problem, first, an auxiliary boundary value problem with inclined derivative is considered for the second order elliptic equation. The main problems are solved by reducing them to a sequential solution of the Dirichlet problem and the problem with inclined derivative. Theorems on the existence and uniqueness of a solution of considered problems are proved.

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1. Introduction

Let $\tilde{x} = (x_1, ..., x_{n-1})$, $\Omega_m = \left\{ x \in R^n : |\tilde{x}|^2 + |x_n|^m < 1 \right\}$, $n \geq 3$, $m > 1$, $\partial\Omega_m = \left\{ x \in R^n : |\tilde{x}|^2 + |x_n|^m = 1 \right\}$ be a boundary of the domain Ω_m.

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In Ω_m we consider an uniformly elliptic operator

$$A\left(x, D\right) = \frac{\partial^2}{\partial x_n^2} + \sum_{p,q=1}^{n-1} a_{pq} \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial x_j} + c,$$

where the coefficients a_{pq}, b_j, c depend only on $\tilde{x} = (x_1, x_2, ..., x_{n-1})$ and they are smooth enough, and $c \leq 0$.

For any point $x \in \Omega_m$ we put in conformity a point $x^* = (x_1, x_2, ..., -x_n)$. It is obvious that if $x \in \partial\Omega_m$, then $x^* \in \partial\Omega_m$.

Denote

$$\partial\Omega_m^+ = \{x \in \partial\Omega_m : x_n \geq 0\}, \partial\Omega_m^- = \{x \in \partial\Omega_m : x_n \leq 0\},$$

$$S = \{x \in \partial\Omega_m : x_n = 0\} \equiv \partial\Omega_m^+ \cap \partial\Omega_m^-.$$

We introduce the operator $I_*[u](x) = u(z)\,|_{z=x^*}$. Let a parameter k take one of the values $k = 1, 2$. In Ω_m we consider the following problem:

$$A(x, D)u(x) = f(x), \ x \in \Omega, \tag{1}$$

$$u(x) + (-1)^k u(x^*) = g_0(x), \ x \in \partial\Omega_+, \tag{2}$$

$$\frac{\partial u(x)}{\partial x_n} + (-1)^k \frac{\partial u(x^*)}{\partial x_n} = g_1(x), \ x \in \partial\Omega_+, \tag{3}$$

$$u(x) = 0, \ x \in S. \tag{4}$$

Here $u(x^*)$ and $\frac{\partial u}{\partial x_n}(x^*)$ mean:

$$u\left(x^*\right) = I_*[u](x), \frac{\partial u}{\partial x_n}\left(x^*\right) = I_*\left[\frac{\partial u}{\partial x_n}\right](x).$$

As a solution of problem (1)-(4) we call a function $u(x)$ from the class $C^2\left(\Omega_m\right) \cap C^1\left(\bar{\Omega}_m\right)$ which satisfies conditions (1)-(4) in the classical sense.

Problem (1)-(4) is an analogue of the periodic and antiperiodic problems for circular domains.

It should be noted that

$$\frac{\partial u}{\partial x_n}\left(x^*\right) = I_*\left[\frac{\partial u}{\partial x_n}\right](x) \neq \frac{\partial}{\partial x_n} I_*[u](x),$$

i.e. operator I_* and differentiation operator $\frac{\partial}{\partial x_n}$ do not commute. Moreover, since

$$\frac{\partial^2 u}{\partial x_n^2}\left(x^*\right) = I_*\left[\frac{\partial^2 u}{\partial x_n^2}\right](x) = \frac{\partial^2}{\partial x_n^2} I_*[u](x),$$

then the following equality holds

$$A(x, D)u(x^*) \equiv A(x, D)I_*u(x) = I_*A(x, D)u(x). \tag{5}$$

Further, if $x \in S$, then $x = (\tilde{x}, 0)$ and the corresponding points $x^* = (x_1, x_2, ..., x_{n-1}, 0)$ also belong to the set S. Therefore, $u(x)$ belongs to the class $C^1(\bar{\Omega}_m)$, if the following condition holds:

$$g_0(\tilde{x}, 0) = u(\tilde{x}, 0) + (-1)^k u(\tilde{x}^*, 0)$$

$$= (-1)^k \left[u(\tilde{x}^*, 0) + (-1)^k u(\tilde{x}, 0) \right] = (-1)^k g_0(\tilde{x}^*, 0), \ \tilde{x} \in S.$$

However, due to condition (4), the equality $u(\tilde{x}, 0) = 0$, $\tilde{x} \in S$ holds, and therefore, the last condition can be rewritten in the form $g_0(\tilde{x}, 0) = g_0(\tilde{x}^*, 0) = 0$, $\tilde{x} \in S$, i.e.

$$g_0(x) = 0, \ x \in S. \tag{6}$$

The following condition is also necessary

$$\frac{\partial g_0}{\partial x_j}(x) - (-1)^k \frac{\partial g_0}{\partial x_j}(x^*) = 0, \ x \in S, \ j = 1, 2, ..., n, \tag{7}$$

and

$$g_1(x) + (-1)^k g_1(x^*) = 0, \ x \in S. \tag{8}$$

Furthermore, we will assume that conditions (6) - (8) are satisfied. Note that similar problems for the Laplace and Poisson equations with normal derivatives of integer and fractional orders were studied in [1],[2],[3]. Moreover, in [4] similar problem was studied for a boundary operator with inclined derivative without degeneracy. We also note that degenerate boundary value problems with inclined derivative were studied in [5],[6],[7],[8].

2. Uniqueness of Solution

We give the uniqueness theorem for the solution of problem (1)-(4).

Theorem 1. *Let k take one of the values $k = 1, 2$. If a solution of problem (1)-(4) exists, then it is unique.*

Proof. Let $k = 1$ and $u(x)$ be a solution of the homogenous problem (1)-(4). Then from condition (2) it follows that

$$u(x) = u(x^*) \equiv I_*[u](x), \ x \in \partial \Omega_m^+.$$

If $x \in \partial\Omega_m^-$, then it is obvious that $x^* \in \partial\Omega_m^+$. Therefore, from the boundary value condition (2) for the points $x \in \partial\Omega_m^-$ we get

$$u(x) = u(x^*) \equiv I_*[u](x), \ x \in \partial\Omega_m^-.$$

Consequently, for all $x \in \partial\Omega_m$ we have

$$u(x) - u(x^*) = 0, \ x \in \partial\Omega_m.$$

Denote $v(x) = u(x) - u(x^*)$. Then applying the operator $A(x, D)$ to $v(x)$, according to (5), we have

$$A(x, D) v(x) = A(x, D) u(x) - A(x, D) u(x^*) = 0, \ x \in \Omega_m.$$

Hence, the function $v(x)$ is the solution of the following Dirichlet problem:

$$A(x, D) v(x) = 0, \ x \in \Omega_m, \ v(x)|_{\partial\Omega_m} = 0.$$

Then, due to the uniqueness of the solution of the Dirichlet problem, $v(x) = 0, x \in \bar{\Omega}_m$. Therefore $u(x) = u(x^*) \equiv I_*[u](x), x \in \bar{\Omega}_m$. By the condition of the problem, $u(x)$ belongs to $C^1\left(\bar{\Omega}_m\right)$. Thus, from the condition $u(x) = u(x^*), x \in \bar{\Omega}_m$, due to the equality:

$$\frac{\partial}{\partial x_n} I_*[u](x) = \frac{\partial}{\partial x_n} u(x_1, ..., x_{n-1}, -x_n)$$

$$= -I_*\left[\frac{\partial u}{\partial x_n}\right](x_1, ..., x_{n-1}, x_n) \equiv -\frac{\partial u(x^*)}{\partial x_n},$$

we obtain

$$\frac{\partial u}{\partial x_n}(x) = -\frac{\partial u}{\partial x_n}(x^*) \equiv -I_*\left[\frac{\partial u}{\partial x_n}\right](x), \ x \in \bar{\Omega}_m. \tag{9}$$

On the other hand, by the boundary value condition (3), we have

$$\frac{\partial u}{\partial x_n}(x) = \frac{\partial u}{\partial x_n}(x^*) \equiv I_*\left[\frac{\partial u}{\partial x_n}\right](x), \ x \in \partial\Omega_m^+,$$

and

$$\frac{\partial u}{\partial x_n}(x) = \frac{\partial u}{\partial x_n}(x^*) \equiv I_*\left[\frac{\partial u}{\partial x_n}\right](x), \ x \in \partial\Omega_m^-.$$

Consequently, for all $x \in \partial\Omega_m$ we have

$$\frac{\partial u}{\partial x_n}(x) = \frac{\partial u}{\partial x_n}(x^*) \equiv I_*\left[\frac{\partial u}{\partial x_n}\right](x). \tag{10}$$

Then, adding the left-hand and right-hand sides of equalities (9) and (10), we obtain

$$\frac{\partial u}{\partial x_n}(x) = 0, \ x \in \partial \Omega_m.$$

Therefore, the solution of homogeneous problem (1)-(4) is also solution to the following problem

$$A(x, D) u(x) = 0, \ x \in \Omega_m, \ \left.\frac{\partial u}{\partial x_n}(x)\right|_{\partial \Omega_m} = 0, \ u(x)|_S = 0. \qquad (11)$$

In [5] it is proved that the solution of the problem (11) is unique and therefore, $u(x) \equiv 0, x \in \bar{\Omega}_m$.

Let now $k = 2$. In this case, if $u(x)$ is a solution of the homogenous problem (1)-(4), then from condition (2) for all point $x \in \partial \Omega_m^+$ we get

$$u(x) = -u(x^*), \ x \in \partial \Omega_m^+.$$

In its turn, for points $\partial \Omega_m^-$:

$$u(x) = -u(x^*), \ x \in \partial \Omega_m^-.$$

Consequently, for all $x \in \partial \Omega_m$ we have

$$u(x) + u(x^*) = 0, \ x \in \partial \Omega_m.$$

Further, if we denote $v(x) = u(x) + u(x^*)$, then the function $v(x)$ is a solution of the Dirichlet problem:

$$A(x, D) v(x) = 0, \ x \in \Omega_m, \ v(x)|_{\partial \Omega_m} = 0,$$

and due to the uniqueness of the solution of the Dirichlet problem, $v(x) = 0, x \in \bar{\Omega}_m$. Therefore, $u(x) = -u(x^*) \equiv -I_*[u](x)$, $x \in \bar{\Omega}_m$. Then

$$\frac{\partial u}{\partial x_n}(x) = -\frac{\partial u}{\partial x_n}(x^*) \equiv -\frac{\partial}{\partial x_n} I_*[u](x), \ x \in \bar{\Omega}_m.$$

Further, as in the case $k = 1$ from the boundary value condition (3) we obtain:

$$\frac{\partial u}{\partial x_n}(x) = -\frac{\partial u}{\partial x_n}(x^*) \equiv -I_*\left[\frac{\partial}{\partial x_n} u\right](x), \ x \in \partial \Omega_m^+,$$

$$I_*\left[\frac{\partial}{\partial x_n} u\right](x) \equiv \frac{\partial u}{\partial x_n}(x^*) = -\frac{\partial u}{\partial x_n}(x), \ x \in \partial \Omega_m^-,$$

and consequently, we have

$$\frac{\partial u}{\partial x_n}(x) = -\frac{\partial u}{\partial x_n}(x^*) \equiv -I_*\left[\frac{\partial}{\partial x_n}u\right](x) = \frac{\partial}{\partial x_n}I_*[u](x), \ x \in \partial\Omega_m.$$

Hence

$$\frac{\partial u}{\partial x_n}(x) = 0, \ x \in \partial\Omega_m.$$

In this case the function $u(x)$ satisfies the conditions of the problem (11). Then $u(x) \equiv 0, \ x \in \bar{\Omega}_m.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Box$

3. Existence of Solution

In this section, we prove theorem on the existence and smoothness of a solution of the problem (1)-(4). Consider the following auxiliary problem

$$A(x, D)z(x) = 0, \ x \in \Omega_m, \ \frac{\partial z}{\partial x_n}(x)\bigg|_{\partial\Omega_m} = h(x), \ z(x)|_S = 0. \qquad (12)$$

In [5] the following proposition is proved.

Lemma 2. Let $1 - \frac{1}{m} < \lambda$, moreover let the number $\lambda + \frac{1}{m}$ be not integer. For any function $h(x) \in C^\lambda(\partial\Omega_m)$ a solution of problem (12) exists, is unique and belongs to the class $C^{\lambda+\frac{1}{m}}(\bar{\Omega}_m)$.

The following statement is true for the main problem.

Theorem 3. Let k take one of the values $k = 1, 2,$

$$1 - \frac{1}{m} < \lambda < 1, f(x) \in C^\lambda(\bar{\Omega}_m), g_0(x) \in C^{\lambda+1}(\partial\Omega_m^+), g_1(x) \in C^\lambda(\partial\Omega_m^+)$$

and the matching conditions (6)-(8) be satisfied. Then a solution of the problem (1)-(4) exists, is unique and belongs to $C^{\lambda+\frac{1}{m}}(\bar{\Omega}_m).$

Proof. Let $k = 1$ and $u(x)$ be a solution of the problem (1)-(4). Consider the following function:

$$v(x) = \frac{u(x) - u(x^*)}{2}, \quad w(x) = \frac{u(x) + u(x^*)}{2}. \qquad (13)$$

Note that the functions $v(x)$ and $w(x)$ have the following properties:

$$v(x) = -v(x^*) \equiv -I_*[v](x); w(x) = w(x^*) \equiv I_*[w](x),$$

$$\frac{\partial v(x)}{\partial x_n} = -\frac{\partial}{\partial x_n} I_*[v](x) = I_*\left[\frac{\partial}{\partial x_n}v\right](x) \equiv \frac{\partial v(x^*)}{\partial x_n},$$

$$\frac{\partial w(x)}{\partial x_n} = \frac{\partial}{\partial x_n} I_*[w](x) = -I_*\left[\frac{\partial}{\partial x_n}w\right](x) \equiv -\frac{\partial w(x^*)}{\partial x_n}.$$

We find the conditions that the functions $v(x)$ and $w(x)$ will satisfy. Applying the operator $A(x, D)$ to the function $v(x)$, we have

$$A(x, D)v(x) = \frac{1}{2}[A(x, D)u(x) - A(x, D)u(x^*)]$$

$$= \frac{1}{2}[A(x, D)u(x) - I_*A(x, D)u(x)]$$

$$= \frac{1}{2}[f(x) - f(x^*)], \ x \in \Omega_m.$$

Further, from boundary condition (2) it follows that

$$v(x)|_{\partial\Omega_m^+} = \frac{1}{2}[u(x) - u(x^*)]\bigg|_{\partial\Omega_m^+} = \frac{1}{2}g_0(x).$$

If the point x belongs to the lower part of the boundary, i.e. $x \in \partial\Omega_m^-$, then the corresponding point x^* belongs to the upper part of the boundary, and therefore, again from the boundary condition (2), we obtain

$$v(x)|_{\partial\Omega_m^-} = \frac{1}{2}[u(x) - u(x^*)]\bigg|_{\partial\Omega_m^-} = -\frac{1}{2}[u(x^*) - u(x)]\bigg|_{x^*\in\partial\Omega_m^+}$$

$$= -\frac{1}{2}g_0(x^*).$$

Moreover, condition (4) implies

$$v(x)|_S = \frac{1}{2}[u(x) - u(x^*)]\bigg|_S = 0.$$

Further, making similar actions with respect to $w(x)$ from equality (13), we obtain

$$A(x, D)w(x) = \frac{1}{2}[A(x, D)u(x) + A(x, D)u(x^*)]$$

$$= \frac{1}{2}[f(x) + f(x^*)], x \in \Omega_m,$$

$$\frac{\partial w\left(x\right)}{\partial x_n}\bigg|_{\partial\Omega_+} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} + \frac{\partial}{\partial x_n}I_*[u]\left(x\right)\right]\bigg|_{\partial\Omega_+}$$

$$= \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} - I_*\left[\frac{\partial u}{\partial x_n}\right]\left(x\right)\right]\bigg|_{\partial\Omega_+} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} - \frac{\partial u}{\partial x_n}\left(x^*\right)\right]\bigg|_{\partial\Omega_+}$$

$$= \frac{1}{2}g_1(x), x \in \partial\Omega_m^+,$$

$$\frac{\partial w\left(x\right)}{\partial x_n}\bigg|_{\partial\Omega_-} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} + \frac{\partial}{\partial x_n}I_*[u]\left(x\right)\right]\bigg|_{\partial\Omega_-}$$

$$= \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} - I_*\left[\frac{\partial u}{\partial x_n}\right]\left(x\right)\right]\bigg|_{\partial\Omega_-} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} - \frac{\partial u\left(x^*\right)}{\partial x_n}\right]\bigg|_{\partial\Omega_-}$$

$$= -\frac{1}{2}\left[\frac{\partial u\left(x^*\right)}{\partial x_n} - \frac{\partial u\left(x\right)}{\partial x_n}\right]\bigg|_{x^*\in\partial\Omega_+} = -\frac{1}{2}g_1(x^*), \ x \in \partial\Omega_m^-.$$

Moreover, condition (4) implies

$$w\left(x\right)|_S = \frac{1}{2}\left[u\left(x\right) + u\left(x^*\right)\right]\bigg|_S = 0.$$

Denote,

$$f^{\pm}(x) = \frac{1}{2}\left[f\left(x\right) \pm f\left(x^*\right)\right], 2\tilde{g}_0\left(x\right) = \begin{cases} g_0\left(x\right), x \in \partial\Omega_m^+ \\ -g_0\left(x^*\right), x \in \partial\Omega_m^- \end{cases},$$

$$2\tilde{g}_1\left(x\right) = \begin{cases} g_1\left(x\right), x \in \partial\Omega_m^+ \\ -g_1\left(x^*\right), x \in \partial\Omega_m^- \end{cases}.$$

Let us examine the smoothness of these functions. Let $f(x) \in C^{\lambda}\left(\bar{\Omega}_m\right)$. Then it is obvious that the functions $f^{\pm}(x)$ also belong to the class $C^{\lambda}\left(\bar{\Omega}_m\right)$. Further, by the hypothesis of the theorem, the function $g_0\left(x\right)$ belongs to the class $C^{\lambda+1}\left(\partial\Omega_m^+\right)$ and the matching conditions (6) and (7) are satisfied for it. Then the function $\tilde{g}_0(x)$ belongs to the class $C^{\lambda+1}\left(\partial\Omega_m\right)$. Similarly, the function $g_1\left(x\right)$ belongs to the class and the matching condition (8) is satisfied for it. Then the function $\tilde{g}_1\left(x\right)$ belongs to the class $C^{\lambda}\left(\partial\Omega_m\right)$. Thus, if $u(x)$ is a solution of problem (1)-(4), then the functions $v(x)$ and $w\left(x\right)$ satisfy the conditions of the following problems:

$$A\left(x, D\right)v\left(x\right) = f^-(x), \ x \in \Omega, \ v\left(x\right)|_{\partial\Omega} = \tilde{g}_0\left(x\right), \tag{14}$$

with the additional condition

$$v\left(x\right)|_S = 0 \tag{15}$$

and

$$A(x, D) w(x) = f^+(x), x \in \Omega_m,$$

$$\frac{\partial w}{\partial x_n}(x)\bigg|_{\partial \Omega_m} = \tilde{g}_1(x), \ w(x)|_s = 0. \quad (16)$$

We study the problem (14). We will look for a solution to the problem in the form $v(x) = v_1(x) + v_2(x)$, where the functions $v_1(x)$ and $v_2(x)$ satisfy the conditions of the following problems

$$A(x, D) v_1(x) = f^-(x), \ x \in \Omega, \ v_1(x)|_{\partial \Omega} = 0, \quad (17)$$

$$A(x, D) v_2(x) = 0, \ x \in \Omega, \ v_2(x)|_{\partial \Omega} = \tilde{g}_0(x). \quad (18)$$

The problems (17) and (18) are classical Dirichlet problems, and for smooth data, solutions to these problems always exist. We need to clarify the smoothness of the solutions to these problems. It was proved in [9] that if $0 < \lambda < 1$, $f^-(x) \in C^\lambda(\bar{\Omega}_m)$, then solution to the problem (17) belongs to the class $C^{\lambda+2}(\bar{\Omega}_m)$. The exact smoothness order of the solution to the problem (18) in the case $\tilde{g}_0(x) \in C^{\lambda+1}(\partial \Omega_m)$ is given in [5] and the order has the form $C^{\lambda+1}(\bar{\Omega}_m)$. In addition, due to the matching condition (6), the equality $v_2(x)|_S = 0$ holds. Thus, if $f^-(x) \in C^\lambda(\bar{\Omega}_m)$, $\tilde{g}_0(x) \in C^{\lambda+1}(\partial \Omega_m)$, then the solution to the problem (14) exists and condition (15) holds for it. Further, we will look for a solution to the problem (16) in the form $w(x) = w_1(x) + w_2(x)$, where the functions $w_1(x)$ and $w_2(x)$ are solutions of the following problems:

$$A(x, D) w_1(x) = f^+(x), \ x \in \Omega_m, w_1(x)|_{\partial \Omega_m} = 0, \quad (19)$$

$$A(x, D) w_2(x) = 0, \ x \in \Omega_m;$$

$$\frac{\partial w_2(x)}{\partial x_n}\bigg|_{\partial \Omega_m} = \tilde{g}_1(x) - \frac{\partial w_1(x)}{\partial x_n}, \ w_2(x)|_S = 0. \quad (20)$$

As we have already noticed, under the conditions of the theorem and the matching conditions (8), the function $f^+(x)$ belongs to the class $C^\lambda(\bar{\Omega}_m)$. Then the solution to the problem (19) exists, is unique and belongs to the class $C^{\lambda+2}(\bar{\Omega}_m)$. Consequently, $\frac{\partial w_1(x)}{\partial x_n} \in C^{\lambda+1}(\bar{\Omega}_m)$. Further, if $g_1(x) \in C^\lambda(\partial \Omega_m^+)$, then the function $\tilde{g}_1(x) - \frac{\partial w_1(x)}{\partial x_n}$ at least belongs to the class $C^\lambda(\partial \Omega_m)$.

With these data, according to Lemma 1, the solution to problem (20) exists, is unique, and belongs to the class $C^{\lambda+\frac{1}{m}}\left(\bar{\Omega}_m\right)$. Therefore, the solution to problem (16) exists, is unique and also belongs to the class $C^{\lambda+\frac{1}{m}}\left(\bar{\Omega}_m\right)$. Thus, we will the functions $v(x)$ and $w(x)$ from equalities (13). We show that if $v(x)$ and $w(x)$ are the solutions to problems (13) and (16), then the function $u(x) = v(x) + w(x)$ satisfies all the conditions of the problem (1)-(4). Indeed, applying the operator $A(x, D)$ to the function $u(x) = v(x) + w(x)$, we have

$$A\left(x, D\right)u(x) = A\left(x, D\right)v(x) + A\left(x, D\right)w(x)$$

$$= f^+(x) + f^-(x) = f(x), \ x \in \Omega.$$

Further, according to the properties of the functions $v(x)$ and $w(x)$, we have

$$u(x) - u(x^*)|_{\partial\Omega_m^+} = v(x) + w(x) - (v(x^*) + w(x^*))|_{\partial\Omega_m^+}$$

$$= v(x) - v(x^*)|_{\partial\Omega_m^+} = 2v(x)|_{\partial\Omega_m^+} = 2\tilde{g}_0(x)|_{\partial\Omega_m^+} = g_0(x).$$

Similarly,

$$\frac{\partial u(x)}{\partial x_n} - I_*\left[\frac{\partial u}{\partial x_n}\right](x)\bigg|_{\partial\Omega_m^+} = \frac{\partial v(x)}{\partial x_n} + \frac{\partial w(x)}{\partial x_n}\bigg|_{\partial\Omega_m^+}$$

$$-I_*\left[\frac{\partial v}{\partial x_n}\right](x) - I_*\left[\frac{\partial w}{\partial x_n}\right](x)\bigg|_{\partial\Omega_m^+} = \frac{\partial w(x)}{\partial x_n} - I_*\left[\frac{\partial w}{\partial x_n}\right](x)\bigg|_{\partial\Omega_m^+}$$

$$= 2\frac{\partial w(x)}{\partial x_n}\bigg|_{\partial\Omega_m^+} = 2\tilde{g}_0(x)|_{\partial\Omega_m^+} = g_1(x).$$

Thus, the theorem is proved for the case $k = 1$.

Let now, $k = 2$ and $u(x)$ be a solution of problem (1)-(4) for this case. Applying the operator $A(x, D)$ to the function $v(x)$, as in the case $k = 1$ from equality (13), we obtain

$$A\left(x, D\right)v\left(x\right) = \frac{1}{2}\left[A\left(x, D\right)u\left(x\right) - A\left(x, D\right)u\left(x^*\right)\right]$$

$$= \frac{1}{2}\left[f\left(x\right) - f\left(x^*\right)\right] = f^-(x), \ x \in \Omega_m.$$

Further, from the boundary value condition (2) we obtain

$$\frac{\partial v\left(x\right)}{\partial x_n}\bigg|_{\partial\Omega_+} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} - \frac{\partial}{\partial x_n}I_*[u]\left(x\right)\right]\bigg|_{\partial\Omega_+}$$

$$= \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} + I_*\left[\frac{\partial u}{\partial x_n}\right](x)\right]\bigg|_{\partial\Omega_+} = \frac{1}{2}\left[\frac{\partial u\left(x\right)}{\partial x_n} + \frac{\partial u}{\partial x_n}\left(x^*\right)\right]\bigg|_{\partial\Omega_+}$$

$$= \frac{1}{2}g_1(x), \ x \in \partial\Omega_m^+.$$

Similarly, for $x \in \partial\Omega_m^-$ we have

$$\left.\frac{\partial v(x)}{\partial x_n}\right|_{\partial\Omega_m^-} = \frac{1}{2}\left[\frac{\partial u(x)}{\partial x_n} - \frac{\partial}{\partial x_n}I_*[u](x)\right]\Big|_{\partial\Omega_m^-}$$

$$= \frac{1}{2}\left[\frac{\partial u(x)}{\partial x_n} + I_*\left[\frac{\partial u}{\partial x_n}\right](x)\right]\Big|_{\partial\Omega_m^-} = \frac{1}{2}\left[\frac{\partial u(x)}{\partial x_n} + \frac{\partial u(x^*)}{\partial x_n}\right]\Big|_{\partial\Omega_m^-}$$

$$= \frac{1}{2}\left[\frac{\partial u(x^*)}{\partial x_n} + \frac{\partial u(x)}{\partial x_n}\right]\Big|_{x^*\in\partial\Omega_m^+} = \frac{1}{2}g_1(x^*), \ x \in \partial\Omega_m^-.$$

Moreover, condition (4) yields that

$$v(x)|_S = \frac{1}{2}\left[u(x) - u(x^*)\right]\Big|_S = 0.$$

Further, for the function $w(x)$ from (13), we get

$$A(x, D)w(x) = \frac{1}{2}\left[A(x, D)u(x) + A(x, D)u(x^*)\right]$$

$$= \frac{1}{2}\left[f(x) + f(x^*)\right] = f^+(x), \ x \in \Omega_m,$$

$$w(x)|_{\partial\Omega_+} = \frac{1}{2}\left[u(x) + u(x^*)\right]\Big|_{\partial\Omega_+} = \frac{1}{2}g_0(x),$$

$$w(x)|_{\partial\Omega_m^-} = \frac{1}{2}\left[u(x) + u(x^*)\right]\Big|_{\partial\Omega_m^-} = \frac{1}{2}\left[u(x^*) + u(x)\right]\Big|_{x^*\in\partial\Omega_m^+}$$

$$= \frac{1}{2}g_0(x^*).$$

Moreover, from (4) it follows that

$$w(x)|_S = \frac{1}{2}\left[u(x) + u(x^*)\right]\Big|_S = 0.$$

We introduce the following functions

$$2\tilde{g}_0(x) = \left\{\begin{array}{l} g_0(x), x \in \partial\Omega_m^+ \\ g_0(x^*), x \in \partial\Omega_m^- \end{array}\right., 2\tilde{g}_1(x) = \left\{\begin{array}{l} g_1(x), x \in \partial\Omega_m^+ \\ g_1(x^*), x \in \partial\Omega_m^- \end{array}\right..$$

Thus, in this case for the functions $v(x)$ and $w(x)$ we obtain the following problems:

$$A(x, D) v(x) = f^-(x), x \in \Omega;$$

$$\left. \frac{\partial v(x)}{\partial x_n} \right|_{\partial \Omega_m} = \tilde{g}_1(x), \; v(x)|_S = 0, \quad (21)$$

$$A(x, D) w(x) = f^+(x), x \in \Omega_m;$$

$$w(x)|_{\partial \Omega_m} = \tilde{g}_0(x), \; w(x)|_s = 0. \quad (22)$$

The problems (21) and (22) are investigated in the same way as in the case $k = 1$. Under the conditions of the theorem, the solutions to these problems exist, are unique and belong to the class $C^{\lambda + \frac{1}{m}}(\bar{\Omega}_m)$. Further, exactly as in the case $k = 1$, it can be shown that the function $u(x) = v(x) + w(x)$ satisfies all the conditions of problem (1)-(4). $\square$

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References

[1] M.A. Sadybekov, B.Kh. Turmetov, On analogues of periodic boundary value problems for the Laplace operator in a ball, *Eurasian Mathematical Journal*, **3**, No 1 (2012), 143-146.

[2] M.A. Sadybekov, B.Kh. Turmetov, On an analog of periodic boundary value problems for the Poisson equation in the disk, *Differential equations*, **50**, No 2 (2014), 268-273; doi:10.1134/S0012266114020153.

[3] B.Kh. Turmetov, M.D.Koshanova, K.I. Usmanov, About solvability of some boundary value problems for Poisson equation in the ball conditions, *Filomat*, **32**, No 3 (2018), 939-946.

[4] B. Turmetov, M. Koshanova, M. Muratbekova, On some analogues of periodic problems for Laplace equation with an oblique derivative under boundary conditions, *e-Journal of Analysis and Applied Mathematics*, **3** (2020), 13-27.

[5] Sh.A. Alimov, On a problem with an oblique derivative, *Differ. Uravn.*, **17**, No 10 (1981), 1738-17511.

[6] Sh.A. Alimov, On a boundary value problem, *Dokl. Akad. Nauk SSSR*, **252**, No 5 (1980), 1033-1034.

[7] D.I. Boyarkin, A boundary value problem with degeneration on the boundary along the manifold of codimension $k > 2$, *Zhurnal SVMO*, **18**, No 2 (2016), 7-10.

[8] P. Popivanov, Boundary value problems for the biharmonic operator in the unit ball, In: *AIP Conf. Proc.* **2159** (6th Intern. Conf. "New Trends in the Applications of Differential Equations in Sciences" (NTADES 2019), Bulgaria, July 1-4, 2019) (Ed. by: A. Slavova), Amer. Inst. Phys. (2019), 1–10.

[9] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (1977).