SOLUTION OF TWO-PHASE CYLINDRICAL DIRECT STEFAN PROBLEM BY USING SPECIAL FUNCTIONS IN ELECTRICAL CONTACT PROCESSES

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Abstract: In this work two-phase Stefan problem for the cylindrical heat equation is considered. One of the phase turns to zero at an initial time. In this case, it is difficult to solve it by radial heat polynomials because the equations are singular. The solution is represented in linear combination series of special functions, the Laguerre polynomial and confluent hypergeometric function. The free boundary is known. The undetermined coefficients of the heat in two phases and the heat flux are found. The convergence of special functions series is proved.

1. Introduction

The cylindrical heat equation is important in mathematical modeling of heat transfer in bodies which cylindrical domain. The solution, in other words is

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a temperature distribution, in such model can take form of series of a special function (Laguerre polynomials). In the development of heat processes models, the partial differential equations play necessary role (see [1], [3]). The theory of free boundaries made tremendous progress in the last half century. We refer to Chen et al. [5] and Friedman [6], [7] in which the realization of some models can be expressed as free boundary problems are given. In the process of heat arcing, the phase transformation takes place. Therefore, we consider Stefan-type problems [8], [9]. Present study is devoted to such Stefan problems in which temperature functions are considered in forms of special functions with unknown coefficients that have close link introduced by Rosenbloom and Widder [10]. To represent the solution form of the problem we consider the following equation

$$x\frac{d^2\phi}{dx^2} + \left(\frac{\nu+1}{2} - x\right)\frac{d\phi}{dx} + \frac{\beta}{2}\phi = 0, \quad \nu = 0, \quad -\infty < \beta < \infty. \tag{1}$$

It is well known that this equation has two linearly independent solutions

$$\begin{aligned}
\phi_1(x) &= \Phi\left(-\frac{\beta}{2}, \frac{\nu+1}{2}; x\right), \\
\phi_2(x) &= x^{\frac{1-\nu}{2}}\Phi\left(\frac{1-\beta-\nu}{2}, \frac{3-\nu}{2}; x\right),
\end{aligned} \tag{2}$$

where $\Phi(a, b; x)$ is the confluent (degenerate) hypergeometric function. Setting $T(z) = \phi(x)$, where $x = -z^2$, one can find that $T(z)$ satisfies the equation

$$\frac{d^2T}{dz^2} + \left(\frac{\nu}{z} + 2z\right)\frac{dT}{dz} - 2\beta T(z) = 0.$$

Using this equation one can check up that the function

$$\theta(z, t) = (2a\sqrt{t})^\beta T\left(\frac{z}{2a\sqrt{t}}\right)$$

satisfies the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial z^2} + \frac{\nu}{z}\frac{\partial\theta}{\partial z}\right). \tag{3}$$

Hence, the functions

$$\begin{aligned}
S_{\beta,\nu}^{(1)}(z, t) &= (2a\sqrt{t})^\beta \Phi\left(-\frac{\beta}{2}, \frac{\nu+1}{2}; -\frac{z^2}{4a^2t}\right), \\
S_{\beta,\nu}^{(2)}(z, t) &= (2a\sqrt{t})^\beta \left(\frac{z^2}{4a^2t}\right)^{\frac{1-\nu}{2}} \Phi\left(-\frac{\beta}{2}, \frac{\nu+1}{2}; -\frac{z^2}{4a^2t}\right)
\end{aligned} \tag{4}$$

satisfy the equation (3). If β is an even integer, $\beta = 2n$, the function $S_{\beta,\nu}(z,t)$ can be expressed in terms of the generalized Laguerre polynomials

$$S_{2n,\nu}^{(1)}(z,t) = (4a^2 t)^n \Phi\left(-n, \mu; -\frac{z^2}{4a^2 t}\right)$$
$$= \frac{n!\Gamma(\mu)}{\Gamma(\mu + n)}(4a^2 t)^n L_n^{\mu-1}\left(-\frac{z^2}{4a^2 t}\right), \tag{5}$$

$$S_{2n,\nu}^{(2)}(z,t) = (4a^2 t)^n \left(\frac{z^2}{4a^2 t}\right)^{1-\mu} \Phi\left(1 - \mu - n, 2 - \mu; -\frac{z^2}{4a^2 t}\right)$$
$$= \frac{n!\Gamma(\mu)}{\Gamma(\mu + n)}(4a^2 t)^n \left(\frac{z^2}{4a^2 t}\right)^{1-\mu} L_n^{\mu-1}\left(-\frac{z^2}{4a^2 t}\right), \tag{6}$$

where $\mu = \dfrac{\nu + 1}{2}$. It should be noted that this formula is valid for $\mu > 0$ only.

Using the integral representation for the degenerate hypergeometric function

$$\Phi\left(-\frac{\beta}{2}, \mu, -z^2\right) = \frac{2\Gamma(\mu)}{\Gamma(\mu + \beta/2)} \exp(-z^2) z^{-\mu+1}$$
$$\times \int_0^\infty \exp(-x^2) x^{\mu+\beta} I_{\mu+1}(2zx) dx \tag{7}$$

and the asymptotic formula

$$\lim_{z\to\infty} \frac{e^{-z} I_\nu(z)}{\sqrt{2\pi z}} = 1,$$

it is possible to show that

$$\lim_{z\to\infty} \frac{1}{z^\beta} \Phi\left(-\frac{\beta}{2}, \mu; -z^2\right) = \frac{\Gamma(\mu)}{\Gamma\left(\mu + \dfrac{\beta}{2}\right)}. \tag{8}$$

In particular,

$$\lim_{z\to\infty} \frac{1}{z^\beta} \Phi\left(-\frac{\beta}{2}, 1; -z^2\right) = \frac{1}{\Gamma\left(1 + \dfrac{\beta}{2}\right)}. \tag{9}$$

For $\nu = 1$, both functions (1) coincide:

$$S_{\beta,1}^{(1)}(z,t) = S_{\beta,1}^{(2)}(z,t) = (2a\sqrt{t})^\beta \Phi\left(-\frac{\beta}{2}, 1; -\frac{z^2}{4a^2 t}\right).$$

In this case, the second linearly independent solution of the equation (3) is ([4])

$$\phi_2(x) = \Phi\left(-\frac{\beta}{2}, 1; x\right)\ln x + \sum_{k=1}^{\infty} M_k x^k, \tag{10}$$

where

$$M_k = \left(\begin{matrix} k \\ -\frac{\beta}{2} \end{matrix}\right) \frac{1}{k!} \sum_{m=0}^{k-1} \left(\frac{1}{m - \beta/2} + \frac{2}{m+1}\right).$$

This work is a continuation of the article [11] where we considered problems for generalized heat conduction, and here we consider the case with $\nu = 1$ for cylindrical equation.

2. Problem Definition

We consider the following problem of heat transfer in solid phase $0 < r < \alpha(t)$ and in liquid phase $\alpha(t) < r < \infty$ which can be modelled with cylindrical heat equation

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_1}{\partial r}\right), \quad 0 < r < \alpha(t), \ \ t > 0, \tag{11}$$

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_2}{\partial r}\right), \quad \alpha(t) < r < \infty, \ \ t > 0 \tag{12}$$

with initial conditions

$$\theta_1(0,0) = 0, \quad \alpha(0) = 0, \tag{13}$$

$$\theta_2(r,0) = f(r) \tag{14}$$

and boundary conditions

$$\theta_1(\alpha(t), t) = \theta_2(\alpha(t), t) = \theta_m. \tag{15}$$

Stefan's condition at free boundary is

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t), t)}{\partial r} = -\lambda_2 \frac{\partial \theta_2(\alpha(t), t)}{\partial r} + L\gamma \frac{d\alpha}{dt} \tag{16}$$

and the condition at infinity is

$$\theta_2(\infty, t) = 0. \tag{17}$$

3. Problem Solution

For $\beta = 2n$ we can represent solution of (11)-(17) as form of linear combinations of special functions

$$
\begin{aligned}
\theta_1(r,t) = & \sum_{n=0}^{\infty} A_n (4a_1^2 t)^n L_n \left(-\frac{r^2}{4a_1^2 t} \right) \\
& + \sum_{n=0}^{\infty} B_n (4a_1^2 t)^n \left[\Phi \left(-n, 1; -\frac{r^2}{4a_1^2 t} \right) \ln \left(\frac{r^2}{4a_1^2 t} \right) \right. \\
& \left. + \sum_{k=1}^{\infty} M_k \left(-\frac{r^2}{4a_1^2 t} \right)^k \right],
\end{aligned}
\tag{18}
$$

$$
\theta_2(r,t) = \sum_{n=0}^{\infty} C_n (4a_2 t)^n L_n \left(-\frac{r^2}{4a_2^2 t} \right),
\tag{19}
$$

where $M_k = \binom{k}{n} \frac{1}{k!} \sum_{m=0}^{k-1} \left(\frac{1}{m+n} + \frac{2}{m+1} \right)$ and $\theta_1(r,t)$, $\theta_2(r,t)$ are temperatures in solid and liquid zones, θ_m is melting temperature and A_n, B_n, C_n are unknowns which have to be determined. The equations (18) and (19) satisfy the problem (11)-(17). Consider that the function $f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{(n)!} r^n$ and free boundary $\alpha(t) = \sum_{n=1}^{\infty} \alpha_n t^{n/2}$ are given. From (14) we have

$$
\sum_{n=0}^{\infty} C_n \frac{r^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n
\tag{20}
$$

and comparing degree of r, we obtain definition for coefficient C_n as follows

$$
C_n = \frac{f^{(2n)}(0)}{(2n)!} n!, \quad n = 0, 1, 2, \dots .
\tag{21}
$$

The algorithm to find coefficients A_n and B_n is that from condition (16) we express coefficients A_n and by making substitution to (16) we can get coefficients B_n. At first, we take m-th derivative of (14) and (15) when $\tau = 0$, where $\tau = \sqrt{t}$:

$$
\left. \frac{\partial^m \theta_1(\alpha(\tau), \tau)}{\partial \tau^m} \right|_{\tau=0} = \left. \frac{\partial^m \theta_2(\alpha(\tau), \tau)}{\partial \tau^m} \right|_{\tau=0} = \frac{\partial^m \theta_m}{\partial \tau^m},
\tag{22}
$$

$$-\lambda_1 \frac{\partial^m}{\partial \tau^m}\left[\frac{\partial \theta_1(\alpha(\tau),\tau)}{\partial r}\right]\Bigg|_{\tau=0}$$

$$= -\lambda_2 \frac{\partial^m}{\partial \tau^m}\left[\frac{\partial \theta_2(\alpha(\tau),\tau)}{\partial r}\right]\Bigg|_{\tau=0} + L\gamma \frac{d^m \alpha}{d\tau^m}. \tag{23}$$

We use the Leibniz rule for (22) and (23), then we have

$$\frac{\partial^m\left[(4a_1^2)^n \tau^2 L_n(-\delta(\tau))\right]}{\partial \tau^m}\Bigg|_{\tau=0} = \frac{(4a_1^2)^n m!}{(m-2n)!}\left[L_n(-\delta(\tau))\right]^{(m-2n)}\Bigg|_{\tau=0}$$

and

$$\frac{\partial^m\left[(4a_1^2)^n \tau^n \left(\Phi[-n,1;-\delta(\tau)]\ln(\delta(\tau)) + \sum\limits_{k=1}^{\infty} M_k(-\delta(\tau))^k\right)\right]}{\partial \tau^m}\Bigg|_{\tau=0}$$

$$= \frac{(4a_1^2)m!}{(m-2n)!}\left[\Phi[-n,1;-\delta(\tau)]\ln(\delta(\tau)) + \sum\limits_{k=1}^{\infty} M_k(-\delta(\tau))^k\right]^{(m-2n)}\Bigg|_{\tau=0}$$

$$= \frac{(4a_1^2)m!}{(m-2n)!}\sum\limits_{l=0}^{m-2n}\binom{m-2n}{l}\left[\Phi[-n,1;-\delta(\tau)]\right]^{(l)}\left[\ln(\delta(\tau))\right]^{(m-2n-l)}$$

$$+ \sum\limits_{k=1}^{\infty} M_k\left[(-\delta(\tau))^k\right]^{(m-2n)}\Bigg|_{\tau=0},$$

where

$$\delta(\tau) = \frac{1}{4a_1^2}(\alpha_1 + \alpha_2\tau + \alpha_3\tau^2 + ...)^2 = \frac{1}{4a_1^2}\left(\sum\limits_{n=1}^{\infty}\alpha_n\tau^{n-1}\right)^2, \quad i=1,2.$$

Then we use Faa-di Bruno formula for taking derivative from composite function and we get the following results

$$\frac{\partial^{m-2n}\left[L_n(-\delta(\tau))\right]}{\partial \tau^{m-2n}}\Bigg|_{\tau=0}$$

$$= \sum\limits_{l=0}^{m-2n}\left[L_n(-\delta_1)\right]^{(l)}\sum\limits_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}, \tag{24}$$

$$\frac{\partial^{m-2n}\left[\Phi(-n,1;-\delta(\tau))\right]}{\partial \tau^{m-2n}}\Bigg|_{\tau=0}$$

$$= \sum\limits_{l=0}^{m-2n}\left[\Phi(-n,1;-\delta_1)\right]^{(l)}\sum\limits_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}, \tag{25}$$

$$\frac{\partial^{m-2n-l}\left[\ln(\delta(\tau))\right]}{\partial\tau^{m-2n-l}}\Bigg|_{\tau=0}$$

$$=\sum_{p=0}^{m-2n-l}\left[\ln(\delta_1)\right]^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2!b_3!...b_{m-2n-l-p+2}!}, \tag{26}$$

$$\frac{\partial^{m-2n}\left[(-\delta(\tau))^k\right]}{\partial\tau^{m-2n}}\Bigg|_{\tau=0}$$

$$=\sum_{l=0}^{m-2n}(-\delta_1)^{k-l}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}, \tag{27}$$

where $\delta_1=\dfrac{\alpha_1^2}{4a_i^2}$, $\delta_2=\dfrac{\alpha_2^2}{4a_i^2},...,\delta_{m-2n-l-p+2}=\dfrac{\alpha_{m-2n-l-p+2}^2}{4a_i^2}$, $i=1,2$ and $b_1,b_2,b_3,...$ satisfy the following equations

$$b_2+b_3+...+b_{m-2n-l-p+2}=m,$$

$$2b_2+3b_3+...+(m-2n-l-p+2)b_{m-2n-l-p+2}=m-2n.$$

In particular, when $m=0$ and $\tau=0$ we have the following initial coefficients

$$C_0=\theta_m,\quad B_0=-\frac{L\gamma\alpha_1}{\lambda_1\left[\dfrac{4}{\alpha_0}+4a_1\sum\limits_{k=1}^{\infty}M_k k\left(-\dfrac{\alpha_1}{2a_1}\right)^{2k-1}\right]},$$

$$A_0=\theta_m+\frac{L\gamma\alpha_1\left[\ln\left(\dfrac{\alpha_1^2}{4a_1^2}\right)+\sum\limits_{k=1}^{\infty}M_k\left(-\dfrac{\alpha_1}{2a_1}\right)^{2k}\right]}{\lambda_1\left[\dfrac{4}{\alpha_1}+4a_1\sum\limits_{k=1}^{\infty}M_k k\left(-\dfrac{\alpha_1}{2a_1}\right)^{2k-1}\right]}. \tag{28}$$

By using formulas (24)-(27) we have the following recurrent expressions for conditions (15) and (16)

$$\sum_{n=0}^{m}A_n(4a_1^2)^n\frac{m!}{(m-2n)!}\sum_{l=0}^{m-2n}\left[L_n(-\delta_1)\right]^{(l)}$$

$$\times\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}$$

$$+ \sum_{n=0}^{m} B_n (4a_1^2)^n \frac{m!}{(m-2n)!} \sum_{l=0}^{m-2n} \binom{m-2n}{l} \left[\Phi(-n,1;-\delta_1) \right]^{(l)}$$

$$\times \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2! b_3! ... b_{m-2n-l+2}!}$$

$$\times \sum_{p=0}^{m-2n-l} \left[\ln(\delta_1) \right]^{(p)} \sum_{b_i} \frac{(m-2n-l)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2! b_3! ... b_{m-2n-l-p+2}!}$$

$$+ \sum_{k=1}^{\infty} M_k \sum_{l=0}^{m-2n} \delta_1^{k-l} \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2! b_3! ... b_{m-2n-l+2}!} = 0$$

(29)

and

$$-\lambda_1 \Bigg[\sum_{n=0}^{m} A_n (4a_1^2)^n \frac{m!}{(m-2n)!} \sum_{l=0}^{m-2n} \left[\frac{\partial}{\partial r} L_n(-\delta_1) \right]^{(l)}$$

$$\times \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2 b_3! ... b_{m-2n-l+2}!}$$

$$+ \sum_{n=0}^{m} B_n (4a_1^2)^2 \frac{m!}{(m-2n)!} \sum_{l=0}^{m-2n} \binom{m-2n}{l} \left[\frac{\partial}{\partial r} \Phi(-n,1;-\delta_1) \right]^{(l)}$$

$$\times \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2 b_3! ... b_{m-2n-l+2}!}$$

$$\times \sum_{p=0}^{m-2n-l} \left[\ln(\delta_1) \right]^{(p)} \sum_{b_i} \frac{(m-2n-l)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2 b_3! ... b_{m-2n-l-p+2}!}$$

$$+ \sum_{l=0}^{m-2n} \binom{m-2n}{l} \left[\Phi(-n,1;-\delta_1) \right]^{(l)} \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2 b_3! ... b_{m-2n-l+2}!}$$

$$\times \sum_{p=0}^{m-2n-l} (-1)^p \frac{1}{\delta_1^{p+1}} \sum_{b_i} \frac{(m-2n-l)! \delta_2^{b_2} \delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2 b_3!...b_{m-2n-l-p+2}!}$$

$$+ \sum_{k=1}^{\infty} M_k \sum_{l=0}^{m-2n} \binom{m-2n}{l} \delta_1^{k-l} \sum_{b_i} \frac{(m-2n)! \delta_2^{b_2} \delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2 b_3!...b_{m-2n-l+2}!}$$

$$\times \sum_{p=0}^{m-2n-l} \beta_1^{(p)} \sum_{b_i} \frac{(m-2n-l)! \delta_2^{b_2} \delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2 b_3!...b_{m-2n-l-p+2}!} \Bigg] \tag{30}$$

$$= -\lambda_2 \Bigg[\sum_{n=0}^{m} C_n (4a_2^2) n \frac{m!}{(m-2n)!} \sum_{l=0}^{m-2n} \left[\frac{\partial}{\partial r} L_n(-\delta_1) \right]^{(l)}$$

$$\times \sum_{b_i} \frac{(m-2n)! \beta_2^{b_2} \beta_3^{b_3}...\beta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2 b_3!...b_{m-2n-l+2}!} \Bigg] + L\gamma m! \alpha_m,$$

where $\beta(\tau) = \frac{1}{2a_1^2} \left(\sum_{n=1}^{\infty} \alpha_n \tau^{n-1} \right)^2$. From recurrent expression (29) we express coefficient A_n. Making substitutions to (30), we can determine B_n as free boundary is given and coefficient C_n can be found from (21).

4. Convergence of Series

The convergence of series (18)-(19) can be proved as follows. Let $\alpha(t_0) = \eta_0$ for any $t = t_0$. Then the series (18) can be written as

$$\theta_1(r, t_0) = \sum_{n=0}^{\infty} A_n (4a_1^2 t_0) L_n \left(-\frac{r_0^2}{4a_1^2 t_0} \right) + \sum_{n=0}^{\infty} B_n (4a_1^2 t_0)^n$$

$$\times \left[\Phi\left(-n, 1; -\frac{r_0^2}{4a_1^2 t_0} \right) \ln\left(\frac{r_0^2}{4a_1^2 t_0} \right) + \sum_{k=1}^{\infty} M_k \left(-\frac{r_0^2}{4a_1^2 t_0} \right)^k \right]. \tag{31}$$

The series (18) and (19) must be convergence because
$\theta_1(\alpha(t), t) = \theta_2(\alpha(t), t) = \theta_m$. Then there exists some constant D_1 independent of n and for the first term of (31) we have

$$|A_n| < D_1 / (4a_1^2 t_0)^n L_n \left(-\frac{\eta_0^2}{4a_1^2 t_0} \right). \tag{32}$$

Since A_n bounded, then multiply both sides of (32) by

$(4a_1^2 t)^n L_n\left(-\dfrac{(\alpha(t))^2}{4a_1^2 t}\right)$ we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} A_n (4a_1^2 t)^n L_n\left(-\frac{(\alpha(t))^2}{4a_1^2 t}\right) \\
&< D_1 \sum_{n=0}^{\infty} \frac{(4a_1^2 t)^n L_n\left(-\dfrac{(\alpha(t))^2}{4a_1^2 t}\right)}{(4a_1^2 t_0)^n L_n\left(-\dfrac{\eta^2}{4a_1^2 t_0}\right)} < D_1 \sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n.
\end{aligned} \tag{33}
$$

For the second term of (33) we consider that there exists some constant D_2 and we obtain

$$
\begin{aligned}
|B_n| > D_2 / &\left\{ (4a_1^2 t_0)^n \left[\left(\Phi - n, 1; -\frac{r_0^2}{4a_1^2 t_0}\right) \ln\left(\frac{r_0^2}{4a_1^2 t_0}\right) \right. \right. \\
&\left. \left. + \sum_{k=1}^{\infty} M_k \left(-\frac{r_0^2}{4a_1^2 t_0}\right)^k \right] \right\}.
\end{aligned} \tag{34}
$$

As B_n is bounded, multiplying both sides of (34) by

$$
(4a_1^2 t)^n \left[\Phi\left(-n, 1; -\frac{(\alpha(t))^2}{4a_1^2 t}\right) \ln\left(\frac{(\alpha(t))^2}{4a_1^2 t}\right) + \sum_{k=1}^{\infty} M_k \left(-\frac{(\alpha(t))^2}{4a_1^2 t}\right)^k \right],
$$

we get

$$
\begin{aligned}
&B_n (4a_1^2 t)^n \left[\Phi\left(-n, 1; -\frac{(\alpha(t))^2}{4a_1^2 t}\right) \ln\left(\frac{(\alpha(t))^2}{4a_1^2 t}\right) + \sum_{k=1}^{\infty} M_k \left(-\frac{(\alpha(t))^2}{4a_1^2 t}\right)^k \right] \\
&< D_2 \frac{(4a_1^2 t)^n \Phi\left[\left(-n, 1; -\dfrac{(\alpha(t))^2}{4a_1^2 t}\right) \ln\left(\dfrac{(\alpha(t))^2}{4a_1^2 t}\right) + \sum_{k=1}^{\infty} M_k \left(-\dfrac{(\alpha(t))^2}{4a_1^2 t}\right)^k \right]}{(4a_1^2 t_0)^n \left[\Phi\left(-n, 1; -\dfrac{\eta^2}{4a_1^2 t_0}\right) \ln\left(\dfrac{\eta^2}{4a_1^2 t_0}\right) + \sum_{k=1}^{\infty} M_k \left(-\dfrac{\eta^2}{4a_1^2 t_0}\right)^k \right]} \\
&< D_2 \sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n.
\end{aligned} \tag{35}
$$

These geometric series and θ_1 converge for all $r < \mu_0$ and the same is the θ_2 convergence for all $r > \mu_0$ and $t < t_0$. The convergence for equation (19) and $\alpha(t)$ can be determined analogously.

Conclusion

The undetermined coefficients of heat in liquid and solid phases are found from (14) by comparing degree of cylindrical material radius in expression (20) and from recurrent formulas (29) and (30) for the conditions (15) and (16) which obtained by using Leibniz Rule and Faa di Bruno formulas.

Acknowledgment

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