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ON BITSADZE-SAMARSKII TYPE ELLIPTIC DIFFERENTIAL PROBLEMS ON HYPERBOLIC PLANE

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Abstract: In the present article, we consider nonlocal boundary value problems (NBVP) of elliptic type on relatively compact domains in the hyperbolic plane. We establish the well- posedness of Neumann-Bitsadze-Samarskii type and also Dirichlet-Bitsadze-Samarskii type on such domains. Furthermore, we establish new coercivity inequalities for solutions of such elliptic NBVP on relatively compact domains in the hyperbolic plane with various Hölder norms.

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1. Introduction

The well-posedness of boundary value problems for partial differential equations is well known (see, e.g. [1],[2],[3]). Moreover, the coercivity inequalities of NBVP for partial differential equations of hyperbolic type in the Euclidean space has been studied widely (see, e.g. [4],[5],[6],[7][8],[9],[10],[11],[12],[13], [14],[15],[16],[17],[18] and the references therein).

In this paper, by considering differential equations on hyperbolic plane, we prove the well-posedness of NBVP on relatively compact domains in the hyperbolic plane with various Hölder norms. We also obtain new coercivity estimates for the solutions of such NBVP for elliptic equations compact domains in the hyperbolic plane with various Hölder norms.

2. Preliminary Results

In this section, we provide the basic definitions and facts about the Laplacian on Riemannian manifolds. For further information, we refer the reader to [19], [20] and the references therein.

A pair $(\mathcal{M}, g)$ is a said to be *a Riemannian manifold,* if $\mathcal{M}$ is a smooth manifold, and for each $x \in \mathcal{M}$ $\langle \cdot, \cdot \rangle_{g(x)} : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ is a non-degenerate symmetric positive definite bilinear form such that for all smooth vector fields $X, Y \in \Gamma_{C^\infty}(T\mathcal{M})$, $x \mapsto \langle X(x), Y(x) \rangle_{g(x)}$ is smooth. Let $\left\{ \left(\frac{\partial}{\partial x^1}\right)_x, \ldots, \left(\frac{\partial}{\partial x^n}\right)_x \right\}$ be the corresponding basis of tangent space $T_x\mathcal{M}$ in the local coordinates $(x_1, \ldots, x_n)$. Let g_{ij} and g^{ij} denote $\left\langle \left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x \right\rangle_{g(x)}$ and the entries of the inverse matrix of (g_{ij}), respectively. The *gradient* operator $\nabla_g : \mathscr{C}^\infty(\mathcal{M}) \to \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$ is defined by $\langle \nabla_g \varphi, X \rangle_g = d\varphi(X)$ for each $\varphi \in \mathscr{C}^\infty(\mathcal{M})$, $X \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$. Note that the gradient $\nabla_g \varphi$ is equal to $\sum_{i,j=1}^n g^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}$ in local coordinates $(x_1, \ldots, x_n)$. By the fact $d(\varphi + \psi) = d\varphi + d\psi$ for each $\varphi, \psi \in \mathscr{C}^1(\mathcal{M})$, we have $\nabla_g(\varphi + \psi) = \nabla_g \varphi + \nabla_g \psi$. Similarly, Leibniz property $d(\varphi \cdot \psi) = \varphi \cdot d\psi + \psi \cdot d\varphi$ yields $\nabla_g(\varphi \cdot \psi) = \varphi \cdot \nabla_g \psi + \psi \cdot \nabla_g \varphi$.

Suppose $\omega \in \Omega^n(\mathcal{M})$ is an $n-$form and X is a vector field on $\mathcal{M}$. Then, $\iota_X \omega \in \Omega^{n-1}(\mathcal{M})$ is the $(n-1)-$form defined by

$$\iota_X \omega (X_1, \ldots, X_{n-1}) = \omega (X, X_{1,} \ldots, X_{n-1}).$$

Here, $X_1, \ldots, X_{n-1}$ are vector fields on the Riemaniann manifold $\mathcal{M}$. By using $d(\iota_X \omega) \in \Omega^n(\mathcal{M})$, we have $d(\iota_X \omega) = \mathrm{div}_\omega(X) \omega$ for some number $\mathrm{div}_\omega(X)$.

The *divergence* operator $\operatorname{div}_g : \Gamma_{\mathscr{C}^\infty}(T\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M})$ is defined by $d(\iota_X \omega_g) = \operatorname{div}_g(X)\omega_g$ for all $X \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$. Here, $\omega_g \in \Omega^n(\mathcal{M})$ is the volume element obtained by Rimannian metric g. Clearly, in local coordinates $(x_1, \ldots, x_n)$, divergence is $\operatorname{div}_g(X) = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(b_i \sqrt{\det g}\right)$ for $X = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$. For $X, Y \in \Gamma_{C^\infty}(T\mathcal{M})$ and $\omega \in \Omega^n(\mathcal{M})$, $\iota_{X+Y}\omega = \iota_X\omega + \iota_Y\omega$. Thus, we have $\operatorname{div}_g(X+Y) = \operatorname{div}_g(X) + \operatorname{div}_g(Y)$, and also $\operatorname{div}_g(\varphi X) = \varphi \operatorname{div}_g X + \langle \nabla_g \varphi, X\rangle_g$ for $\varphi \in \mathscr{C}^\infty(M)$. $\Delta_g = -\operatorname{div}_g \circ \nabla_g$ is called the *Laplace-Beltrami* Δ_g on real-valued smooth functions $\mathscr{C}^\infty(\mathcal{M})$ on $(\mathcal{M}, g)$. Clearly, $\Delta_g(\varphi + \psi) = \Delta_g\varphi + \Delta_g\psi$ and $\Delta_g(\varphi \cdot \psi) = \psi\Delta_g\varphi + \varphi\Delta_g\psi - 2\langle \nabla_g\varphi, \nabla_g\psi\rangle_g$ for any $\varphi, \psi \in \mathscr{C}^\infty(\mathcal{M})$. By using local coordinates $(x_1, \ldots, x_n)$, we have $\Delta_g = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\frac{\partial}{\partial x_j}\right)$.

We consider

$$\mathbb{H}^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0, \mid x_1^2 + x_2^2 - x_3^2 = c - 1\right\},$$

the 2−dimensional hyperbolic plane in geodesic polar coordinates, more precisely, $\xi : (0,\infty) \times (0, 2\pi) \to \mathbb{H}^2$,

$$x_1 = \sinh(r)\cos\theta,\ x_2 = \sinh(r)\sin\theta,\ x_3 = \cosh(r), \tag{1}$$

where $0 < r < \infty$, $0 < \theta < 2\pi$. Then, we obtain $g_{\mathbb{H}^2} = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2(r) \end{bmatrix}$, $\sqrt{\det g_{\mathbb{H}^2}} = \sinh(r)$, $g_{\mathbb{H}^2}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sinh^2(r)} \end{bmatrix}$. The Laplace-Beltrami operator $\Delta_{\mathbb{H}^2}$ is equal to

$$\frac{-1}{\sinh(r)}\left\{\frac{\partial}{\partial r}\left(a_0(r,\theta)\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(a_1(r,\theta)\frac{\partial}{\partial \theta}\right)\right\}, \tag{2}$$

where $a_0 = \sinh(r)$ and $a_1 = \frac{1}{\sinh(r)}$.

Theorem 1 (Divergence Theorem)**.** *Let $\mathcal{M}$ be a Riemannian manifold with boundary $\partial\mathcal{M}$ and X be a C^1-vector field on $\mathcal{M}$. Then, $\int_{\mathcal{M}} \operatorname{div}_g(X)\, dV_g = \int_{\partial\mathcal{M}} \langle X, \nu\rangle_g\, d\sigma_g$, where div_g is the divergence operator on $(\mathcal{M}, g)$, dV_g is the natural volume element on $(\mathcal{M}, g)$, and ν is the unit vector normal to $\partial\mathcal{M}$.*

Theorem 2 (Stokes' Theorem)**.** *If $\mathcal{M}$ is oriented complete Riemannian $n-$manifold with boundary, $\alpha \in \Omega^{n-1}(\mathcal{M})$ with compact support, and $i : \partial\mathcal{M} \to \mathcal{M}$ is inclusion map, then $\int_{\partial\mathcal{M}} i^*\alpha = \int_{\mathcal{M}} d\alpha$.*

These results yield the following theorem:

Theorem 3 (Green's Theorem)**.** *Suppose $(\mathcal{M}, g)$ is an oriented complete Riemannian manifold with boundary $\partial\mathcal{M}$. Suppose also $\psi \in C^1\left(\overline{\mathcal{M}}\right)$ and $\varphi \in C^2\left(\overline{\mathcal{M}}\right)$. Then, $\int_{\mathcal{M}} \psi \cdot \Delta_{\mathcal{M}}\phi \, dV_g = \int_{\mathcal{M}} \langle \nabla_g \psi, \nabla_g \phi \rangle \, dV_g - \int_{\partial\mathcal{M}} \psi \cdot \frac{\partial \phi}{\partial \nu} d\sigma_g$. Here, ∇_g is the gradient operator on the Riemannian manifold $(\mathcal{M}, g)$.*

By using Green's Theorem, the following theorem holds:

Theorem 4. ([19, 20]) *Let $(\mathcal{M}, g)$ be a complete Riemannian manifold with boundary. Then,*
1. (Formal self-adjointness): $\langle \psi, \Delta_{\mathcal{M}}\phi \rangle_{L_2(\mathcal{M}, dV_g)} = \langle \phi, \Delta_{\mathcal{M}}\psi \rangle_{L_2(\mathcal{M}, dV_g)}$, 2. (Positivity): $\langle \Delta_{\mathcal{M}}\phi, \phi \rangle_{L_2(\mathcal{M}, dV_g)} \geq 0$, where $L_2(\mathcal{M}, dV_g)$ is Hilbert space $\{f : \mathcal{M} \to \mathbb{R}; \langle \phi, \phi \rangle_{L_2(\mathcal{M}, dV_g)} := \int_{\mathcal{M}} \phi^2(x) \, dV_g(x) < \infty\}$.

2.1. Neumann-Bitsadze-Samarskii Type NBVP on the Hyperbolic Plane

Let us consider the domain $\Omega = \xi((a_1, b_1) \times (a_2, b_2)) \subset \mathbb{H}^2$. Here, $\xi : (0, \infty) \times (0, 2\pi) \to \mathbb{H}^2$ denotes the geodesic polar parametrization (1), $(a_1, b_1) \subset (0, \infty)$, and $(a_2, b_2) \subset (0, 2\pi)$. We consider

$$\begin{cases} -u_{tt}(t, x) + \Delta_{\mathbb{H}^2} u(t, x) + \delta u(t, x) = f(t, x), \\ x \in \Omega, \ t \in (0, 1), \\ u_t(0, x) = 0, u_t(1, x) = \sum_{i=1}^{p} \beta_i u_t(\lambda_i, x), \ x \in \overline{\Omega}, \\ \sum_{i=1}^{p} |\beta_i| \leq 1, \ 0 \leq \lambda_1 < \cdots < \lambda_p < 1, \\ \frac{\partial u}{\partial \vec{n}}(t, x) \mid_{x \in \partial\Omega} = 0, \ 0 \leq t \leq 1, \end{cases} \tag{3}$$

where $\Delta_{\mathbb{H}^2}$ denotes the Laplace-Beltrami operator on the Riemannian manifold $(\mathbb{H}^2, g_{\mathbb{H}^2})$ and $\delta > 0$. We introduce the following theorem:

Theorem 5. *The solutions of problem (3) satisfy the coercivity inequality*

$$\|u_{tt}\|_{\mathscr{C}^\alpha(\mathscr{L}_2(\Omega, dV_g))} + \|u\|_{\mathscr{C}^\alpha\left(\mathscr{W}_2^2(\Omega, dV_g)\right)} \leq \frac{M(\delta, \lambda_p, a, a_1, b_1)}{\alpha(1-\alpha)} \|f\|_{\mathscr{C}^\alpha(\mathscr{L}_2(\Omega, dV_g))}.$$

Here, $M(\delta, \lambda_p, a, a_1, b_1)$ is independent of $f(t, x)$.

Consider problem (3) as the following Bitsadze-Samarskii type NBVP:

$$\begin{cases} -U_{tt}\left(t\right)+\mathbf{L}U\left(t\right)=F(t), \ \ 0\leq t\leq 1, \\ U_t(0)=0, \ U_t(1)=\sum_{i=1}^{p}\beta_i U_t(\lambda_i), \\ \sum_{i=1}^{p}\left|\beta_i\right|\leq 1, \ 0\leq\lambda_1<\cdots<\lambda_p<1 \end{cases}$$

in $\mathscr{L}_2(\Omega, dV_g)$ with the self-adjoint and positive definite operator $\mathbf{L}=\Delta_{\mathbb{H}^2}+\delta I$, where I is the identity operator.

The proof of Theorem 5 follows from the symmetry property of $\mathbf{L}$, Theorem 6 and Theorem 7 on the coercivity estimate for the solution of elliptic differential problem in $H=\mathscr{L}_2(\Omega, dV_g)$.

Theorem 6. ([17]) *Suppose A is a self-adjoint positive definite operator with dense domain $D(A)$ in a Hilbert space H, and $\varphi,\psi\in E_\alpha\left(D\left(A^{1/2}\right),H\right)$. Then, the elliptic type differential problem*

$$\begin{cases} -v_{tt}(t,x)+Av(t)=g(t), \ \ 0<t<1, \\ v_t(0)=\varphi, \ \ v_t(1)=\sum_{i=1}^{p}\beta_i v_t\left(\lambda_i\right)+\psi, \\ \sum_{i=1}^{p}\left|\beta_i\right|\leq 1, \ \ 0\leq\lambda_1<\cdots<\lambda_p<1 \end{cases} \tag{4}$$

is well-posed in Hölder space $\mathscr{C}^\alpha\left(H\right)$. For the solutions of problem (4), the coercivity estimate holds:

$$\left\|v_{tt}\right\|_{\mathscr{C}^\alpha(H)}+\left\|Av\right\|_{\mathscr{C}^\alpha(H)}$$

$$\leq M\left(\delta\right)\left[\left\|A^{1/2}\varphi\right\|_H+\left\|A^{1/2}\psi\right\|_H\right]+\frac{M\left(\delta,\lambda_p\right)}{\alpha\left(1-\alpha\right)}\left\|g\right\|_{\mathscr{C}^\alpha(H)}.$$

Theorem 7. *The solutions of the elliptic differential problem*

$$\begin{cases} \Delta_{\mathbb{H}^2}u(\xi(r,\theta))=\omega(\xi(r,\theta)), \ (r,\theta)\in(a_1,b_1)\times(a_2,b_2), \\ \frac{\partial u(\xi(r,\theta))}{\partial\vec{n}}=0, \ (r,\theta) \text{ in boundary of } [a_1,b_1]\times[a_2,b_2] \end{cases}$$

satisfy the coercivity estimate

$$\sum_{i=1}^{n}\left\|u_{\theta_i\theta_i}\right\|_{\mathscr{L}_2(\Omega,dV_g)}\leq M_1\left(a_1,b_1\right)\left|\left|w\right|\right|_{\mathscr{L}_2(\Omega,dV_g)}.$$

The proof of Theorem 7 relies on the following theorem:

Theorem 8. ([8]) *Consider the solutions of the elliptic differential problem*

$$\begin{cases} A^{\xi}u(\xi) = w(\xi), \ \ \xi \in (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n), \\ \frac{\partial u(\xi)}{\partial \vec{n}} = 0, \ \ \xi \ in \ \ boundary \, [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n]. \end{cases} \tag{5}$$

The solutions of equation (5) satisfy the coercivity inequality $\sum_{i=1}^{n} \|u_{\xi_i \xi_i}\|_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))} \leq M_2(a,\delta)||w||_{\mathscr{L}_2((\alpha_1,\beta_1)\times\cdots\times(\alpha_n,\beta_n))}$, *where* $A^{\xi} = \sum_{r=1}^{n} \frac{\partial}{\partial \xi_r}\left(a_r(\xi)\frac{\partial}{\partial \xi_r}\right) + \delta I, \ a_r(\xi) \geq a > 0,$ *and* $r = 1, \ldots, n$.

Proof of Theorem 7. The boundary of Ω is the image $\xi(r,\theta)$ of boundary of $[a_1, b_1] \times [a_2, b_2]$ and the interior of Ω is the image $\xi(r,\theta)$ of $(a_1, b_1) \times (a_2, b_2)$. If $u : \Omega \to \mathbb{R}$ is so that $\frac{\partial u}{\partial \nu}$ vanishes on the boundary of Ω, then $v = u \circ \xi : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ and $\frac{\partial v}{\partial \nu}$ vanishes on the boundary of the rectangle $[a_1, b_1] \times [a_2, b_2]$, where ν is the outward unit normal to the boundary. Clearly, $0 < m\,(a_1) \leq \sinh(r) \leq M\,(b_1)$, where $m\,(a_1) = \sinh(a_1)$ and $M\,(b_1) = \sinh(b_1)$. From Equation (2) and Theorem 8, it follows:

$$\begin{aligned}
&\int_{\Omega} \left|\Delta_{\mathbb{H}^2} u(x)\right|^2 dV_g(x) \\
&= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\left\{\frac{\partial}{\partial r}\left(\sinh(r)\frac{\partial u\circ\xi(r,\theta)}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sinh(r)}\frac{\partial u\circ\xi(r,\theta)}{\partial \theta}\right)\right\}^2}{\sinh(r)} d\theta dr \\
&\geq \frac{1}{M\,(a_1,b_1)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left\{\frac{\partial}{\partial r}\left(\sinh(r)\frac{\partial u\circ\xi(r,\theta)}{\partial r}\right)\right. \\
&\quad \left. + \frac{\partial}{\partial \theta}\left(\frac{1}{\sinh(r)}\frac{\partial u\circ\xi(r,\theta)}{\partial \theta}\right)\right\}^2 d\theta dr \\
&= \frac{1}{M\,(a_1,b_1)}\left\|A^{(r,\theta)} u\circ\xi\right\|^2_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} \\
&= \frac{1}{M\,(a_1,b_1)}\left\|A^{(r,\theta)} v\right\|^2_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} \\
&\geq \frac{1}{M\,(a_1,b_1)\cdot M_2^2(a,\delta)}\left(\|v_{rr}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} + \|v_{\theta\theta}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))}\right)^2.
\end{aligned}$$

Thus, we get

$$\left(\int_{\Omega} \left|\Delta_{\mathbb{H}^2} u(x)\right|^2 dV_g(x)\right)^{1/2}$$

$$\geq \frac{\left(\|v_{rr}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} + \|v_{\theta\theta}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))}\right)}{\sqrt{M\left(a_1,b_1\right)}M_2(a,\delta)}. \tag{6}$$

Note that

$$\|v_{rr}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} = \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |v_{rr}(r,\theta)|^2 \, drd\theta\right)^{1/2}$$

$$\geq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |v_{rr}(r,\theta)|^2 \, \frac{\sinh(r)}{M\left(a_1,b_1\right)} drd\theta\right)^{1/2}$$

$$= \frac{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |v_{rr}(r,\theta)|^2 \sinh(r) drd\theta\right)^{1/2}}{\sqrt{M\left(a_1,b_1\right)}}$$

$$= \frac{\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2} |(u\circ\xi)_{rr}(r,\theta)|^2 \sinh(r) drd\theta\right)^{1/2}}{\sqrt{M\left(a_1,b_1\right)}} = \frac{\|u_{rr}\|_{\mathscr{L}_2(\Omega,dV_g)}}{\sqrt{M\left(a_1,b_1\right)}}. \tag{7}$$

Similarly, we have

$$\|v_{\theta\theta}\|_{\mathscr{L}_2((a_1,b_1)\times(a_2,b_2))} \geq \frac{1}{\sqrt{M\left(a_1,b_1\right)}} \|u_{\theta\theta}\|_{\mathscr{L}_2(\Omega,dV_g)}. \tag{8}$$

Equations (6), (7), and (8) yield

$$\left(\int_{\Omega} |\Delta_{\mathbb{H}^2} u(x)|^2 \, dV_g(x)\right)^{1/2}$$

$$\geq \frac{1}{M\left(a_1,b_1\right)\cdot M_2(a,\delta)} \left(\|u_{rr}\|_{\mathscr{L}_2(\Omega,dV_g)} + \|u_{\theta\theta}\|_{\mathscr{L}_2(\Omega,dV_g)}\right).$$

This finishes proof of Theorem 7. □

2.2. Dirichlet-Bitsadze-Samarskii Type NBVP on the Hyperbolic Plane

We consider the mixed boundary value problem of Dirichlet-Bitsadze-Samarskii type

$$\begin{cases} -u_{tt}(t,x) + \Delta_{\mathbb{H}^2} u(t,x) + \delta u(t,x) = f(t,x), \\ x \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u(1,x) = \sum_{j=1}^{p} \alpha_j u(\lambda_j, x) + \psi(x), x \in \overline{\Omega}, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \quad \sum_{j=1}^{p} |\alpha_j| \leq 1, \ 0 \leq t \leq 1, \\ u(t,x) \mid_{x \in \partial\Omega} = 0, 0 \leq t \leq 1. \end{cases} \tag{9}$$

$\times(a_2, b_2)) \subset \mathbb{H}^2$ and $\xi : (0,\infty) \times (0, 2\pi) \to \mathbb{H}^2$ is the geodesic polar parametrization (1), $(a_1, b_1) \subset (0,\infty)$, $(a_2, b_2) \subset (0, 2\pi)$. $\Delta_{\mathbb{H}^2}$ is the Laplace-Beltrami operator on the Riemannian manifold $(\mathbb{H}^2, g_{\mathbb{H}^2})$. We prove the following theorem:

Theorem 9. *For the solutions of NBVP (9), the coercivity inequality holds:*

$$\|u_{tt}\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\Omega, dV_g))} + \|u\|_{\mathscr{C}_{01}^{\alpha}\left(\mathscr{W}_2^2(\Omega, dV_g)\right)} \leq M(\delta, a_1, b_1) \Big[\|\varphi\|_{\mathscr{W}_2^2(\Omega, dV_g)}$$
$$+ \|\psi\|_{\mathscr{W}_2^2(\Omega, dV_g)} \Big] + \frac{M(\delta, \lambda_1, \lambda_p, a_1, b_1)}{\alpha(1-\alpha)} \|f\|_{\mathscr{C}_{01}^{\alpha}(\mathscr{L}_2(\Omega, dV_g))}.$$

Here, $K(\delta, \lambda_1, \lambda_p)$ does not depend on $\varphi(x), \psi(x)$, and $f(t,x)$.

Consider problem (9) as the NBVP of Bitsadze-Samarskii type

$$\begin{cases} -U_{tt}(t) + \mathbf{L}U(t) = F(t), \quad t \in (0,1), \\ U(0) = \varphi, \quad U(1) = \sum_{j=1}^{p} \alpha_j U(\lambda_j) + \psi, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \quad \sum_{j=1}^{p} |\alpha_j| \leq 1 \end{cases} \tag{10}$$

in $\mathscr{L}_2(\Omega, dV_{g_{\mathbb{H}^2}})$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathbb{H}^2} + \delta I$. Here, $\|U\|_{\mathscr{L}_2(\Omega, dV_{g_{\mathbb{H}^2}})} = \left(\int_\Omega U^2(x) dV_g(x)\right)^{1/2}$, $dV_{g_{\mathbb{H}^2}}$ is natural volume element of $\mathbb{H}^2$ obtained from metric tensor $g_{\mathbb{H}^2}$, and I is the identity operator.

The proof of Theorem 9 relies on [18, Theorem 12] and the following theorem:

Theorem 10. ([16]) *Suppose A is a self-adjoint positive definite operator with dense $D(A) \subset H$ in a Hilbert space H and $\varphi, \psi \in D(A)$. Then, the* *following boundary value problem*

$$\begin{cases} -v_{tt}(t,x) + Av(t) = f(t), \ \ 0 < t < 1, \\ v(0) = \varphi, \ \ v(1) = \sum_{j=1}^{p} \alpha_j v(\lambda_j) + \psi, \\ 0 < \lambda_1 < \cdots < \lambda_p < 1, \ \ \sum_{j=1}^{p} |\alpha_j| \leq 1 \end{cases}$$

is well-posed in Hölder space $\mathscr{C}_{01}^{\alpha}(H)$. *Furthermore, for the solutions of problem, the coercivity estimate*

$$\|v_{tt}\|_{\mathscr{C}_{01}^{\alpha}(H)} + \|Av\|_{\mathscr{C}_{01}^{\alpha}(H)}$$

$$\leq M\left[\|A\varphi\|_H + \|A\psi\|_H\right] + \frac{M(\delta, \lambda_1, \lambda_p)}{\alpha(1-\alpha)} \|f\|_{\mathscr{C}_{01}^{\alpha}(H)}$$

is valid. Here $M(\delta, \lambda_1, \lambda_p)$ *does not depend on* $\varphi(x), \psi(x)$, *and* $f(t,x)$. $\mathscr{C}_{01}^{\alpha}(H)$ $(0 < \alpha < 1)$ *is the Banach space which is the completion of smooth functions* $v : [0,1] \to H$ *with the norm* $\|v\|_{\mathscr{C}_{01}^{\alpha}(H)} = \|v\|_{\mathscr{C}(H)} + \sup\limits_{0 \leq t < t+\tau \leq 1} \frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|v(t+\tau)-v(t)\|_H}{\tau^{\alpha}}$ *and* $\|v\|_{\mathscr{C}(H)} = \max\limits_{0 \leq t \leq 1} \|v(t)\|_H$.

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