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ON HYPERBOLIC-PARABOLIC PROBLEMS WITH INVOLUTION AND NEUMANN BOUNDARY CONDITION

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Abstract: We study a nonlocal boundary value problem and a space-wise dependent source identification problem for one-dimensional hyperbolic-parabolic equation with involution and Neumann boundary condition. The stability estimates for the solutions of these two problems are established. The first order of accuracy stable difference schemes are constructed for the approximate solutions of the problems under consideration. Numerical results for two test problems are provided.

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1. Introduction

The theory and applications of local and nonlocal problems for mixed type partial differential equations have been investigated by many scientists (see, e.g., [12], [16], [18] and the references given therein). In particular, the theory of nonlocal boundary value problems for hyperbolic-parabolic equations and the numerical methods for their approximate solutions have been the subject of recent research (see, e.g., [3], [4] and the references therein).

Partial differential equations with unknown source terms are used in mathe-

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matical modeling of real-life systems in different fields of science and technology. In the study of so-called direct problems, the solution of a differential equation is realized by means of (non)local initial and/or boundary conditions, while in inverse problems the equation itself is also unknown. The determination of both the governing equation and its solution requires imposing some conditions additional to those in the corresponding direct problem. The theory of inverse source identification problems for partial differential equations has been developed in great detail over many decades (see, e.g., [14], [15] and the references therein). In recent years, the first attempts have been made to study the source identification problems for hyperbolic-parabolic equations and the corresponding difference schemes for their approximate solutions (see [2], [9], [10], [11]). In this ongoing research, a particular attention is given recently to the source identification problem for hyperbolic-parabolic equation with involution and Dirichlet boundary condition (see [1]). Note that partial differential equations with involution have been recently investigated in the context of direct problems (see [5], [6], [7], [13]). However, the theory of inverse source identification problems for partial differential equations with involution has not been well developed yet.

The present paper is devoted to the study of source identification problems for hyperbolic-parabolic differential and difference equations with involution and Neumann boundary condition. The study of inverse source identification problems is usually based on the reduction of these problems to corresponding direct problems with nonlocal conditions, and therefore, we consider additionally a nonlocal boundary value problems for hyperbolic-parabolic differential and difference equations with involution and Neumann boundary condition. The stability of these problems is established. Numerical results are presented.

2. Stability of Differential Equations

First, we consider the following nonlocal boundary value problem:

$$\begin{cases} u_{tt}(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x \\ \quad + \delta u(t,x) = f(t,x), \ -\ell < x < \ell, \ 0 < t < 1, \\ u_t(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x \\ \quad + \delta u(t,x) = g(t,x), \ -\ell < x < \ell, \ -1 < t < 0, \\ u_x(t,-\ell) = u_x(t,\ell) = 0, \ -1 \le t \le 1, \\ u(0^+,x) = u(0^-,x), u_t(0^+,x) = u_t(0^-,x), \ -\ell \le x \le \ell, \\ u(-1,x) = \sum\limits_{j=1}^{P} \alpha_j u(\lambda_j,x) + \varphi(x), \ -\ell \le x \le \ell, \end{cases} \tag{1}$$

for one-dimensional hyperbolic-parabolic differential equation with involution and Neumann boundary condition. Throughout this paper, we assume that the following conditions hold:

$$\overline{a} \ge a(x) = a(-x) \ge \underline{a} > 0, \ x \in (-\ell,\ell), \ \underline{a} - \overline{a}|\beta| \ge 0, \tag{2}$$

$$\sum_{j=1}^{P} |\alpha_j| \le 1, \ -1 < \lambda_1 \le \cdots \le \lambda_P \le 1. \tag{3}$$

Under compatibility conditions, nonlocal boundary value problem (1) has a unique smooth solution $u(t,x)$ for the given smooth functions $a(x)$, $\varphi(x)$, $f(t,x)$, $g(t,x)$ and positive constant δ.

Here and in the rest of this paper, let the Sobolev space $W_2^2[-\ell,\ell]$ be defined as the set of all functions $v(x)$ defined on $[-\ell,\ell]$ such that both $v(x)$ and $v''(x)$ are locally integrable in $L_2[-\ell,\ell]$, equipped with the norm

$$\|v(x)\|_{W_2^2[-\ell,\ell]} = \left(\int_{-\ell}^{\ell} |v(x)|^2\,dx\right)^{1/2} + \left(\int_{-\ell}^{\ell} \left|v''(x)\right|^2\,dx\right)^{1/2}.$$

Throughout this paper, we denote by M the positive constants which are not expected to be evaluated. We write $M(\delta)$ to emphasize that the constant M depends only on δ.

Theorem 1. *Suppose that $\varphi \in W_2^2[-\ell,\ell]$. Let function $f(t,x)$ be continuously differentiable in t on $[0,1] \times [-\ell,\ell]$ and function $g(t,x)$ be continuously* *differentiable in t on $[-1,0]\times[-\ell,\ell]$. Then, the solution of the nonlocal boundary value problem (1) satisfies the following stability estimates*

$$\|u\|_{C([-1,1],L_2[-\ell,\ell])} \le M_1(\delta)\Big[\,\|\varphi\|_{L_2[-\ell,\ell]}+\|f\|_{C([0,1],L_2[-\ell,\ell])}+\|g\|_{C([-1,0],L_2[-\ell,\ell])}\,\Big],$$

$$\|u\|_{C^{(2)}([0,1],L_2[-\ell,\ell])}+\|u\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])}+\|u\|_{C([-1,1],W_2^2[-\ell,\ell])} \le M_2(\delta)\Big[\,\|\varphi\|_{W_2^2[-\ell,\ell]}+\|f\|_{C^{(1)}([0,1],L_2[-\ell,\ell])}+\|g\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])}\,\Big].$$

Proof. Problem (1) can be written as the abstract nonlocal boundary value problem

$$\begin{cases} u''(t)+Au(t)=f(t),\ 0<t<1,\\ u'(t)+Au(t)=g(t),\ \ -1<t<0,\\ u(0^+)=u(0^-),\ u'(0^+)=u'(0^-),\\ u(-1)=\sum\limits_{j=1}^{P}\alpha_j u(\lambda_j)+\varphi \end{cases} \tag{4}$$

in a Hilbert space $L_2[-\ell,\ell]$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^x u(x)=-\big(a(x)u_x(x)\big)_x-\beta\big(a(-x)u_x(-x)\big)_x+\delta u(x) \tag{5}$$

with the domain $D(A^x)=\Big\{u\in W_2^2[-\ell,\ell]\ \Big|\ u'(-\ell)=u'(\ell)=0\Big\}$. Here, $f(t)=f(t,x)$ and $g(t)=g(t,x)$ are given abstract functions, $u(t)=u(t,x)$ is unknown function. The proof of Theorem 1 is based on the theorem on stability of nonlocal abstract problem (4) (see [3]), the self-adjointness and positive definiteness of the space operator A^x defined by formula (5) (see [6]). □

Second, we consider the following space-wise dependent source identification problem

$$\begin{cases} u_{tt}(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x \\ \quad + \delta u(t,x) = p(x) + f(t,x), \quad -\ell < x < \ell, \ 0 < t < 1, \\ u_t(t,x) - \big(a(x)u_x(t,x)\big)_x - \beta\big(a(-x)u_x(t,-x)\big)_x \\ \quad + \delta u(t,x) = p(x) + g(t,x), \quad -\ell < x < \ell, -1 < t < 0, \\ u(0^+,x) = u(0^-,x), \ u_t(0^+,x) = u_t(0^-,x), \quad -\ell \le x \le \ell, \\ u_x(t,-\ell) = u_x(t,\ell) = 0, \quad -1 \le t \le 1, \\ u(-1,x) = \varphi(x), \ u(1,x) = \psi(x), \quad -\ell \le x \le \ell \end{cases} \tag{6}$$

for one-dimensional hyperbolic-parabolic differential equation with involution and Neumann boundary condition. Under assumption (2) and compatibility conditions, problem (6) has a unique smooth solution $\big(u(t,x),p(x)\big)$ for the given smooth functions $a(x)$, $\varphi(x)$, $\psi(x)$, $f(t,x)$, $g(t,x)$ and constant $\delta > 0$.

Theorem 2. *Suppose that $\varphi,\psi \in W_2^2[-\ell,\ell]$. Let function $f(t,x)$ be continuously differentiable in t on $[0,1]\times[-\ell,\ell]$ and function $g(t,x)$ be continuously differentiable in t on $[-1,0]\times[-\ell,\ell]$. Then, the solution of the identification problem (6) satisfies the stability estimates*

$$\|u\|_{C([-1,1],L_2[-\ell,\ell])} + \big\|(A^x)^{-1}p\big\|_{L_2[-\ell,\ell]} \le M_3(\delta)\Big[\|\varphi\|_{L_2[-\ell,\ell]} + \|\psi\|_{L_2[-\ell,\ell]} + \|f\|_{C([0,1],L_2[-\ell,\ell])} + \|g\|_{C([-1,0],L_2[-\ell,\ell])}\Big],$$

$$\|u\|_{C^{(2)}([0,1],L_2[-\ell,\ell])} + \|u\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])} + \|u\|_{C([-1,1],W_2^2[-\ell,\ell])} + \|p\|_{L_2[-\ell,\ell]} \le M_4(\delta)\Big[\|\varphi\|_{W_2^2[-\ell,\ell]} + \|\psi\|_{W_2^2[-\ell,\ell]} + \|f\|_{C^{(1)}([0,1],L_2[-\ell,\ell])} + \|g\|_{C^{(1)}([-1,0],L_2[-\ell,\ell])}\Big].$$

Proof. Problem (6) can be written in the following abstract form

$$\begin{cases} u''(t) + Au(t) = p + f(t), \ 0 < t < 1, \\ u'(t) + Au(t) = p + g(t), \quad -1 < t < 0, \\ u(0^+) = u(0^-), \ u'(0^+) = u'(0^-), \\ u(-1) = \varphi, \ u(1) = \psi \end{cases} \tag{7}$$

in a Hilbert space $L_2[-\ell,\ell]$ with the space operator $A = A^x$ defined by the formula (5). Here, $f(t) = f(t,x)$ and $g(t) = g(t,x)$ are given abstract functions,

$u(t) = u(t,x)$ is unknown function and $p = p(x)$ is the unknown element of $L_2[-\ell,\ell]$. The proof of Theorem 2 is based on the theorem on stability of the identification problem (7) (see [2]), the self-adjointness and positive definiteness of the space operator A^x defined by formula (5) (see [6]). □

3. Stability of Difference Schemes

The development and implementation of stable numerical methods for solving the problems at hand are crucial for practical reasons since the analytical solutions are not available most of the time.

In this section, we construct and analyse the first order of accuracy stable difference schemes for the approximate solutions of the nonlocal boundary value problem (1) and the space-wise dependent source identification problem (6). The discretization of these problems is carried out in two steps. In the first step, the spatial discretization is conducted. We define the grid space

$$[-\ell,\ell]_h = \left\{x = x_n \;\middle|\; x_n = nh,\ -M \le n \le M,\ Mh = \ell\right\}.$$

We introduce the Hilbert space $L_{2h} = L_2([-\ell,\ell]_h)$ of the grid functions $\varphi^h(x) = \{\varphi^n\}_{-M}^{M}$ defined on $[-\ell,\ell]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[-\ell,\ell]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2}.$$

To the differential operator A^x defined by the formula (5), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \left\{-\left(a(x)\varphi_{\bar{x}}^n\right)_x - \beta\left(a(-x)\varphi_{\bar{x}}^{-n}\right)_x + \delta\varphi^n\right\}_{-M+1}^{M-1}, \tag{8}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^n\}_{-M}^{M}$ and satisfying the conditions $\varphi_{-M} = \varphi_{-M+1}$, $\varphi_M = \varphi_{M-1}$. Here

$$\varphi_{\bar{x}}^n = \frac{\varphi^n - \varphi^{n-1}}{h},\quad -M+1 \le n \le M,$$

$$\varphi_x^n = \frac{\varphi^{n+1} - \varphi^n}{h},\quad -M \le n \le M-1.$$

It is known that under the assumption (2) the difference operator A_h^x, defined by (8), is a self-adjoint positive definite operator in L_{2h}. Using A_h^x, the first discretization step for problems (1) and (6) results in the nonlocal boundary value problem

$$\begin{cases} u_{tt}^h(t,x) + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1, \\ u_t^h(t,x) + A_h^x u^h(t,x) = g^h(t,x),\ \ -1 < t < 0, \\ u^h(0^+,x) = u^h(0^-,x),\ u_t^h(0^+,x) = u_t^h(0^-,x), \\ u^h(-1,x) = \sum\limits_{j=1}^{P} \alpha_j u^h(\lambda_j,x) + \varphi^h(x) \end{cases} \tag{9}$$

and the identification problem

$$\begin{cases} u_{tt}^h(t,x) + A_h^x u^h(t,x) = p^h(x) + f^h(t,x),\ 0 < t < 1, \\ u_t^h(t,x) + A_h^x u^h(t,x) = p^h(x) + g^h(t,x),\ \ -1 < t < 0, \\ u^h(0^+,x) = u^h(0^-,x),\ u_t^h(0^+,x) = u_t^h(0^-,x), \\ u^h(-1,x) = \varphi^h(x),\ u^h(1,x) = \psi^h(x), \end{cases} \tag{10}$$

respectively. Here and in the rest of this section, $x \in [-\ell,\ell]_h$.

Let $\tau = \frac{1}{N}$ and $t_k = k\tau$, $-N \le k \le N$. In the second discretization step, we replace the problems (9) and (10) with the following first order of accuracy difference schemes

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(t_k,x),\ 1 \le k \le N-1, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h u_k^h(x) = g_k^h(x), \\ g_k^h(x) = g^h(t_k,x), -N+1 \le k \le 0, \\ u_1^h(x) - u_0^h(x) = u_0^h(x) - u_{-1}^h(x), \\ u_{-N}^h(x) = \sum\limits_{j=1}^{P} \alpha_j u_{\left[\frac{\lambda_j}{\tau}\right]}^h(x) + \varphi^h(x), \end{cases} \tag{11}$$

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h(x) = p^h(x) + f_k^h(x), \\ f_k^h(x) = f^h(t_k, x),\ 1 \le k \le N-1, \\ \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p^h(x) + g_k^h(x), \\ g_k^h(x) = g^h(t_k, x),\ \ -N+1 \le k \le 0, \\ u_1^h(x) - u_0^h(x) = u_0^h(x) - u_{-1}^h(x), \\ u_{-N}^h(x) = \varphi^h(x),\ u_N^h(x) = \psi^h(x), \end{cases} \tag{12}$$

respectively.

Theorem 3. *Let τ and h be sufficiently small numbers. For the solution $\left\{u_k^h(x)\right\}_{-N}^N$ of difference problem (11) the following stability estimates hold*

$$\max_{-N\le k\le N} \|u_k\|_{L_{2h}} \le M_5(\delta)\Big[\ \|\varphi^h\|_{L_{2h}} + \max_{-N+1\le k\le 0}\|g_k^h\|_{L_{2h}} + \max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}\ \Big],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}} + \max_{-N+1\le k\le 0}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}} + \max_{-N\le k\le N}\|u_k^h\|_{W_{2h}^2} \le M_6(\delta)\Big[\ \|\varphi^h\|_{W_{2h}^2} + \|g_0^h\|_{L_{2h}} + \max_{-N+2\le k\le 0}\left\|\frac{g_k^h-g_{k-1}^h}{\tau}\right\|_{L_{2h}} + \|f_1^h\|_{L_{2h}} + \max_{2\le k\le N-1}\left\|\frac{f_k^h-f_{k-1}^h}{\tau}\right\|_{L_{2h}}\ \Big].$$

Proof. Difference scheme (11) can be written as the following abstract difference scheme

$$\begin{cases} \frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2} + A_h u_{k+1}^h = f_k^h,\ 1 \le k \le N-1, \\ \frac{u_k^h-u_{k-1}^h}{\tau} + A_h u_k^h = g_k^h,\ \ -N+1 \le k \le 0, \\ u_1^h - u_0^h = u_0^h - u_{-1}^h,\ u_{-N}^h = \sum\limits_{j=1}^P \alpha_j u_{\left[\frac{\lambda_j}{\tau}\right]}^h + \varphi^h \end{cases} \tag{13}$$

in a Hilbert space L_{2h} with operator $A_h = A_h^x$ defined by formula (8). Here, $f_k^h = f_k^h(x)$ and $g_k^h = g_k^h(x)$ are given abstract functions, $u_k^h = u_k^h(x)$ is unknown mesh function. The proof of Theorem 3 is based on the stability of the difference scheme (13) (see [4]), the self-adjointness and positive definiteness of the space operator A_h in L_{2h} (see [8]). □

Theorem 4. *Let τ and h be sufficiently small numbers. For the solution $\left\{\left\{u_k^h(x)\right\}_{-N}^N, p^h(x)\right\}$ of problem (12) the following stability estimates hold*

$$\max_{-N\le k\le N}\|u_k\|_{L_{2h}}+\left\|(A_h^x)^{-1}p^h\right\|_{L_{2h}}\le M_7(\delta)\Big[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}$$
$$+\max_{-N+1\le k\le 0}\left\|g_k^h\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\Big],$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}+\left\|p^h\right\|_{L_{2h}}$$
$$+\max_{-N\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}\le M_8(\delta)\Big[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|g_0^h\right\|_{L_{2h}}$$
$$+\max_{-N+2\le k\le 0}\left\|\frac{g_k^h-g_{k-1}^h}{\tau}\right\|_{L_{2h}}+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\frac{f_k^h-f_{k-1}^h}{\tau}\right\|_{L_{2h}}\Big].$$

Proof. Difference scheme (12) can be written in the following abstract form

$$\begin{cases}\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+A_h u_{k+1}^h=p^h+f_k^h,\ 1\le k\le N-1,\\ \frac{u_k^h-u_{k-1}^h}{\tau}+A_h u_k^h=p^h+g_k^h,\ -N+1\le k\le 0,\\ u_1^h-u_0^h=u_0^h-u_{-1}^h,\ u_{-N}^h=\varphi^h,\ u_N^h=\psi^h\end{cases}\tag{14}$$

in a Hilbert space L_{2h} with operator $A_h=A_h^x$ defined by formula (8). Here, $f_k^h=f_k^h(x)$ and $g_k^h=g_k^h(x)$ are given abstract functions, $u_k^h=u_k^h(x)$ is unknown mesh function and $p^h=p^h(x)$ is the unknown mesh element of L_{2h}. The proof of Theorem 4 is based on the stability of the difference scheme (14) (see [11]), the self-adjointness and positive definiteness of the space operator A_h in L_{2h} (see [8]). □

4. Numerical Examples

In this section, we illustrate how the constructed first order of accuracy difference schemes can be applied for two test problems. The numerical algorithms for implementing these difference schemes are described in [1] and based on a procedure of modified Gauss elimination method [17]. Through the error analysis, we show the convergence of the first order of accuracy difference schemes.

First, we consider the following nonlocal problem

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) - 0.5\big(u_x(t,-x)\big)_x + u(t,x) \\ \quad = f(t,x),\ x \in (-\pi,\pi),\ t \in (0,1), \\ u_t(t,x) - u_{xx}(t,x) - 0.5\big(u_x(t,-x)\big)_x + u(t,x) \\ \quad = g(t,x),\ x \in (-\pi,\pi),\ t \in (-1,0), \\ u(0^+,x) = u(0^-,x),\ u_t(0^+,x) = u_t(0^-,x),\ x \in [-\pi,\pi], \\ u(-1,x) = u(1,x),\ x \in [-\pi,\pi], \\ u_x(t,-\pi) = u_x(t,\pi) = 0,\ t \in [-1,1] \end{cases} \tag{15}$$

for one-dimensional hyperbolic-parabolic equation with involution and Neumann boundary condition, where

$$f(t,x) = 1.5\cos t \cos x,\ x \in (-\pi,\pi),\ t \in (0,1),$$
$$g(t,x) = (2.5\cos t - \sin t)\cos x,\ x \in (-\pi,\pi),\ t \in (-1,0).$$

The exact solution of problem (15) is given by

$$u(t,x) = \cos t \cos x,\quad -\pi \le x \le \pi,\quad -1 \le t \le 1.$$

Let $\tau = \frac{1}{N}$ and $h = \frac{\pi}{M}$. We define the set of grid points as following

$$\big\{(t_k,x_n)\ \big|\ t_k = k\tau,\quad -N \le k \le N,\ x_n = nh,\quad -M \le n \le M\big\}.$$

For the numerical solution of problem (15), we construct the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} - \frac{u_{-n+1}^{k+1}-2u_{-n}^{k+1}+u_{-n-1}^{k+1}}{2h^2} + u_n^{k+1} \\ \quad = f(t_k,x_n),\ 1 \le k \le N-1, -M+1 \le n \le M-1, \\ \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} - \frac{u_{-n+1}^k-2u_{-n}^k+u_{-n-1}^k}{2h^2} + u_n^k \\ \quad = g(t_k,x_n), -N+1 \le k \le 0,\quad -M+1 \le n \le M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1},\ u_n^{-N} = u_n^N,\quad -M \le n \le M, \\ u_{-M}^k = u_{-M+1}^k,\ u_M^k = u_{M-1}^k,\quad -N \le k \le N. \end{cases}$$

The numerical solutions of this difference scheme are computed for different values of M and N. We measure the error between the exact and numerical solutions by

$$\|E_u\|_\infty = \max_{\substack{-N\le k\le N\\ -M\le n\le M}} \left|u(t_k,x_n) - u_n^k\right|,$$

where $u(t_k,x_n)$ is the exact value of $u(t,x)$ at (t_k,x_n) and u_n^k represents the corresponding numerical solution. Table 1 shows the errors between the exact solution of the problem (15) and the numerical solutions of the first order of accuracy difference scheme.

Table 1: The errors of the numerical solutions of the first order of accuracy difference scheme for the problem (15).

	$\|E_u\|_\infty$	Order
$N = M = 20$	1.020×10^{-1}	-
$N = M = 40$	4.975×10^{-2}	1.036
$N = M = 80$	2.457×10^{-2}	1.018
$N = M = 160$	1.221×10^{-2}	1.009
$N = M = 320$	6.084×10^{-3}	1.005

Second, we consider the following source identification problem

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) - 0.5\big(u_x(t,-x)\big)_x + u(t,x) \\ \quad = p(x) + f(t,x),\ x\in(-\pi,\pi),\ t\in(0,1), \\ u_t(t,x) - u_{xx}(t,x) - 0.5\big(u_x(t,-x)\big)_x + u(t,x) \\ \quad = p(x) + g(t,x),\ x\in(-\pi,\pi),\ t\in(-1,0), \\ u(0^+,x) = u(0^-,x),\ u_t(0^+,x) = u_t(0^-,x),\ x\in[-\pi,\pi], \\ u(-1,x) = \varphi(x),\ u(1,x) = \psi(x),\ x\in[-\pi,\pi], \\ u_x(t,-\pi) = u_x(t,\pi) = 0,\ t\in[-1,1] \end{cases} \tag{16}$$

for one-dimensional hyperbolic-parabolic equation with involution and Neumann boundary condition, where

$$f(t,x) = (1.5\cos t - 1)\cos x,\ x\in(-\pi,\pi),\ t\in(0,1),$$

$$g(t,x) = (2.5\cos t - \sin t - 1)\cos x,\ x\in(-\pi,\pi),\ t\in(-1,0),$$

$$\varphi(x) = \cos 1\cos x,\ \psi(x) = \cos 1\cos x,\ x\in[-\pi,\pi].$$

The exact solution of problem (16) is the pair of functions

$$\big(u(t,x),p(x)\big)=\big(\cos t\cos x,\cos x\big),\ \ -\pi\le x\le\pi,\ \ -1\le t\le 1.$$

For the numerical solution of source identification problem (16), we construct the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}-\frac{u_{-n+1}^{k+1}-2u_{-n}^{k+1}+u_{-n-1}^{k+1}}{2h^2}+u_n^{k+1} \\ \quad =p_n+f(t_k,x_n),\ 1\le k\le N-1,-M+1\le n\le M-1, \\ \frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\frac{u_{-n+1}^k-2u_{-n}^k+u_{-n-1}^k}{2h^2}+u_n^k \\ \quad =p_n+g(t_k,x_n),-N+1\le k\le 0,-M+1\le n\le M-1, \\ u_n^1-u_n^0=u_n^0-u_n^{-1},\ u_n^{-N}=\varphi(x_n),\ u_n^N=\psi(x_n),-M\le n\le M, \\ u_{-M}^k=u_{-M+1}^k,\ u_M^k=u_{M-1}^k,-N\le k\le N, \end{cases}$$

where u_n^k and p_n denote the numerical approximations of $u(t,x)$ at $(t,x)=(t_k,x_n)$ and $p(x)$ at $x=x_n$, respectively. The numerical solutions of this difference scheme are computed for different values of M and N. Table 2 shows the errors between the exact solution of the problem (16) and the numerical solutions of the first order of accuracy scheme. We observe that the scheme has the first order convergence as it is expected to be.

Table 2: The errors of the numerical solutions of the first order of accuracy difference scheme for the problem (16).

	$\|E_p\|_\infty$	Order	$\|E_u\|_\infty$	Order
$N=M=20$	9.787×10^{-2}	-	5.229×10^{-2}	-
$N=M=40$	4.927×10^{-2}	0.990	2.591×10^{-2}	1.013
$N=M=80$	2.472×10^{-2}	0.995	1.289×10^{-2}	1.007
$N=M=160$	1.238×10^{-2}	0.997	6.428×10^{-3}	1.004
$N=M=320$	6.198×10^{-3}	0.999	3.210×10^{-3}	1.002

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