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ON A BOUNDARY VALUE PROBLEM FOR SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH INVOLUTION

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Abstract: A linear boundary value problem for a system of integro-differential equations with involution is studied by the parameterization method. Sufficient conditions for the existence of a unique solution to the problem are established in terms of coefficients. An algorithm for finding the solution to the problem under consideration is proposed.

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1. Introduction

On the interval $[0, T]$ we consider the following two-point boundary value problem for the system of integro-differential equations with involution:

$$\frac{dx(t)}{dt} + \operatorname{diag}(a_1, a_2, \ldots, a_n) \frac{dx(\alpha(t))}{dt} = \int_0^T K(t,s)x(s)ds + f(t), \quad t \in [0,T], \tag{1}$$

$$Bx(0) + Cx(T) = d, \qquad d \in R^n, \tag{2}$$

where $K(t,s)$ is a continuous on $[0,T]\times[0,T]$ matrix and $f(t)$ is an n-dimensional

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vector-function continuous on $[0,T]$.

Here $\alpha(t)$ is a changing orientation homeomorphism $\alpha(t):[0,T]\to[0,T]$ such that $\alpha^2(t)=\alpha(\alpha(t))=t$. Such a homeomorphism is called a Carleman shift or deviation of involution. Its properties were studied by G. Litvinchuk [1], N. Karapetyants and S. Samko [2] and other (see [3], [4], [5], [6], [7], [8], [9], [10]). As an example of a deviation with involution on $[0,T]$ we can take the homeomorphism $\alpha(t)=T-t$.

The study of integro-differential equations has been considered in many works (see [11], [12], [13]).

By a solution to problem (1),(2) we mean a vector-function $x(t)$ that is continuous on $[0,T]$ and continuously differentiable on $(0,T)$, satisfies the system of integro-differential equations with involution (1) and the boundary condition (2).

In the present paper, problem (1),(2) is studied by the parameterization method [14]. On the basis of this method, we establish necessary and sufficient conditions for the unique solvability of the problem in question and propose an algorithm for finding its solution.

Let us consider equation (1) for $t=\alpha(t)$:

$$\begin{aligned}&\tfrac{dx(\alpha(t))}{dt}+\operatorname{diag}(a_1,a_2,\dots,a_n)\tfrac{dx(t)}{dt}\\&=\int_0^T K(\alpha(t),s)\,x(s)ds+f(\alpha(t)),\quad t\in[0,T].\end{aligned}$$

From the system

$$\begin{aligned}&\tfrac{dx(t)}{dt}+\operatorname{diag}(a_1,a_2,\dots,a_n)\tfrac{dx(\alpha(t))}{dt}\\&=\int_0^T K(t,s)x(s)ds+f(t),\ t\in[0,T],\end{aligned}$$

$$\begin{aligned}&\tfrac{dx(\alpha(t)}{dt}+\operatorname{diag}(a_1,a_2,\dots,a_n)\tfrac{dx(t)}{dt}\\&=\int_0^T K(\alpha(t),s)x(s)ds+f(\alpha(t)),\ t\in[0,T],\end{aligned}$$

we obtain

$$\begin{aligned}&\operatorname{diag}\left(1-a_1^2,1-a_2^2,\dots,1-a_n^2\right)\tfrac{dx(t)}{dt}\\&=\int_0^T\left[K(t,s)-\operatorname{diag}(a_1,a_2,\dots,a_n)K(\alpha(t),s)\right]x(s)ds\\&+\left[f(t)-\operatorname{diag}(a_1,a_2,\dots,a_n)f(\alpha(t))\right].\end{aligned}$$

Under assumption that the matrix $\mathrm{diag}(1-a_1^2, 1-a_2^2, \ldots, 1-a_n^2)$ is invertible, we can rewrite problem (1),(2) in the form

$$\frac{dx(t)}{dt} = \int\limits_0^T K_1(t,s)x(s)ds + f_1(t),\ t \in [0,T], \tag{3}$$

$$Bx(0) + Cx(T) = d, \tag{4}$$

where

$$K_1(t,s) = \mathrm{diag}(1/(1-a_1^2), 1/(1-a_2^2), \ldots, 1/(1-a_n^2))$$
$$\times[K(t,s) - \mathrm{diag}(a_1, a_2, \ldots, a_n)K(\alpha(t), s)]$$

and

$$f_1(t) = \mathrm{diag}(1/(1-a_1^2), 1/(1-a_2^2), \ldots, 1/(1-a_n^2))$$
$$\times[f(t) - \mathrm{diag}(a_1, a_2, \ldots, a_n)f(\alpha(t))].$$

2. Method of Investigation

Let us divide $[0,T]$ into N equal parts with a step size h: $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$. We denote by $x_r(t)$ the restriction of the function $x(t)$ to the r-th subinterval, i.e., $x_r(t) = x(t)$ for $t \in [(r-1)h, rh)$. The original problem is then transformed to an equivalent multipoint boundary value problem

$$\frac{dx_r(t)}{dt} = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(t,s)x_j(s)ds + f_1(t),$$
$$t \in [(r-1)h, rh),\ r = \overline{1,N}, \tag{5}$$

$$Bx_1(0) + C \lim_{t\to T-0} x_N(t) = d, \tag{6}$$

$$\lim_{t\to sh-0} x_s(t) = x_{s+1}(sh),\ s = \overline{1,N-1}. \tag{7}$$

Here (7) are conditions for the continuity of $x(t)$ at the interior partition points $t = sh$, $s = \overline{1,N-1}$.

If $x(t)$ is a solution to problem (3),(4), then the system of its restrictions $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))'$ is a solution to multipoint problem (5)-(7). And vice versa, if a system of vector-functions $\widetilde{x}[t] = (\widetilde{x}_1(t), \widetilde{x}_2(t), \ldots, \widetilde{x}_N(t))'$ is a solution to problem (5)-(7), then the function $\widetilde{x}(t)$ defined by the equations

$\widetilde{x}(t) = \widetilde{x}_r(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$, and $\widetilde{x}(T) = \lim\limits_{t\to T-0} \widetilde{x}_N(t)$, is a solution to original problem (3),(4).

Let us now introduce parameters λ_r, $r = \overline{1,N}$, that are equal to the values of the functions $x_r(t)$ at the points $t = (r-1)h$. Now, on each subinterval $[(r-1)h, rh)$, $r = \overline{1,N}$, we make the substitution $x_r(t) = u_r(t) + \lambda_r$. The problem (5)-(7) is then reduced to an equivalent multipoint problem with parameters:

$$\frac{du_r(t)}{dt} = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(t,s)[u_j(s) + \lambda_j]ds + f_1(t), \tag{8}$$

$$u_r[(r-1)h] = 0, \;\; t \in [(r-1)h, rh), \;\; r = \overline{1,N}, \tag{9}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t\to T-0} u_N(t) = d, \tag{10}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1}, \;\; s = \overline{1, N-1}. \tag{11}$$

Problems (5)-(7) and (8)-(11) are equivalent in the following sense. If a system of functions $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))'$ is a solution to problem (5)-(7), then the pair $(\lambda, u[t])$ with

$$\lambda = (x_1(0), x_2(h), \ldots, x_N[(N-1)h])'$$

and

$$u[t] = (x_1(t) - x_1(0), x_2(t) - x_2(h), \ldots, x_N(t) - x_N[(N-1)h])'$$

is a solution to problem (8)-(11). Vice versa, if a pair $(\widetilde{\lambda}, \widetilde{u}[t])$ is a solution to (8)-(11), then the system of functions

$$\widetilde{x}(t) = (\widetilde{\lambda}_1 + \widetilde{u}_1(t), \widetilde{\lambda}_2 + \widetilde{u}_2(t), \ldots, \widetilde{\lambda}_N + \widetilde{u}_N(t))'$$

is a solution to problem (5)-(7).

The presence of initial conditions (9) allows us, for fixed λ_r, to determine $u_r(t)$, $r = \overline{1,N}$, from the system of integral equations

$$\begin{aligned} u_r(t) = &\int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)[u_j(s) + \lambda_j]ds d\tau \\ &+ \int\limits_{(r-1)h}^{t} f_1(\tau)d\tau, \quad t \in [(r-1)h, rh). \end{aligned} \tag{12}$$

By substituting the expressions for $\lim\limits_{t\to T-0} u_N(t)$ and $\lim\limits_{t\to sh-0} u_s(t)$, $s=\overline{1,N}$, obtained from (12), into conditions (10),(11), and multiplying both sides of (10) by $h>0$, we get the system of linear algebraic equations in unknown parameters λ_r, $r=\overline{1,N}$:

$$\begin{aligned} &hB\lambda_1 + hC\lambda_N + hC\int\limits_{(N-1)h}^{Nh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)\lambda_j ds d\tau \\ &= hd - hC\int\limits_{(N-1)h}^{Nh} f_1(\tau)d\tau \\ &-hC\int\limits_{(N-1)h}^{Nh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau, \end{aligned} \tag{13}$$

$$\begin{aligned} &\lambda_s + \int\limits_{(s-1)h}^{sh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)\lambda_j dsd\tau - \lambda_{s+1} \\ &= -\int\limits_{(s-1)h}^{sh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau \\ &-\int\limits_{(s-1)h}^{sh} f_1(\tau)d\tau, \quad s=\overline{1,N-1}. \end{aligned} \tag{14}$$

Let us denote by $Q(h)$ the $(nN\times nN)$-matrix corresponding to the left-hand side of system (13),(14). This system thus can be represented in the form

$$Q_h(\lambda) = -F(h) - G(u,h), \qquad \lambda\in R^{nN}, \tag{15}$$

where

$$F(h) = \Big(-hd + hC\int\limits_{(N-1)h}^{Nh} f_1(\tau)d\tau, \int\limits_0^h f_1(\tau)d\tau, \ldots, \int\limits_{(N-2)h}^{(N-1)h} f_1(\tau)d\tau\Big)',$$

$$\begin{aligned} G(u,h) = \Big(&hC\int\limits_{(N-1)h}^{Nh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau, \\ &\int\limits_0^h\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau, \\ &\ldots, \int\limits_{(N-2)h}^{(N-1)h}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau\Big)'. \end{aligned}$$

Thus, to find a solution to problem (8)-(11), for the pair $(\lambda, u[t])$, we have the system of equations (12), (15). We will find this solution as the limit of the sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$ according to the following algorithm.

Step 0. (a) Assuming that the matrix $Q(h)$ is invertible, from the equation $Q_h(\lambda) = -F(h)$ we find the initial approximation to the parameter $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)})'^{mN}$:

$$\lambda^{(0)} = -[Q(h)]^{-1}F(h).$$

(b) Substituting $\lambda_r^{(0)}$, $r = \overline{1,N}$, into the right-hand side of the system of integro-differential equations (8) and solving special Cauchy problems with initial conditions (9), we obtain

$$u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), \dots, u_N^{(0)}(t))'.$$

Step 1. (a) Substituting $u_r^{(0)}(t)$, $r = \overline{1,N}$, into the right-hand side of (15), from the equation $Q_h(\lambda) = -F(h) - G(u^{(0)}, h)$ we determine $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_N^{(1)})'$.

(b) Substituting $\lambda_r^{(1)}$, $r = \overline{1,N}$, into the right-hand side of (8) and solving special Cauchy problems (8),(9), we obtain

$$u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), \dots, u_N^{(1)}(t))',$$

and so forth. Proceeding by the algorithm, in the k-th step we find the pair $(\lambda^{(k)}, u^{(k)}[t])$, $k = 0, 1, 2, \dots$. We introduce spaces $C([0,T],\ h,\ R^{nN})$ of systems of functions

$$x[t] = (x_1(t), x_2(t), \dots, x_N(t))',$$

where the functions $x_r(t)$ are continuous on $[(r-1)h, rh)$ and have a finite left-side limit $\lim\limits_{t \to rh-0} u_r(t), r = \overline{1,N}$, with norm

$$\|x[\cdot]\|_2 = \max_{r=\overline{1,N}} \sup_{t \in [(r-1)h, rh)} (x_1(t), x_2(t), \dots, x_N(t))'.$$

As we said, the unknown functions $u[t] = (u_1(t), u_2(t), \dots, u_N(t))'$ are determined by solving special Cauchy problems (8),(9) for systems of integro-differential equations. But, unlike the Cauchy problems for ordinary differential equations, special Cauchy problems for integro-differential equations are not always solvable.

3. Results of Investigation

The following theorem provides a sufficient condition for the unique solvability of the special Cauchy problem (8),(9) for fixed values of parameters.

Theorem 1. *Let the step size of the partition $h = T/N$ satisfy the inequality*

$$\delta(h) = \beta T h < 1, \tag{16}$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} \|K_1(t,s)\|$.

Then the special Cauchy problem (8),(9) has a unique solution.

Proof. As is known, the Cauchy problem (8),(9) is equivalent to a system of integral equations

$$\begin{aligned} u_r(t) = &\int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j(s)dsd\tau \\ &+ \int\limits_{(r-1)h}^{t} \tilde{F}(\tau)d\tau, \ \ t \in [(r-1)h, rh), \ \ r = \overline{1,N}, \end{aligned} \tag{17}$$

where $\tilde{F}(\tau) = \sum\limits_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)dsd\tau \cdot \lambda_j + f_1(\tau)$.

We find the solution of integral equations (16) by the method of successive approximations. As a zero approximation we take the family of functions $u_r(t) = 0$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$ and $u^{(k)}[t] = (u_1^{(k)}(t), u_2^{(k)}(t), \dots, u_N^{(k)}(t))'$ from the systems of integral equations

$$\begin{aligned} u_r^{(k)}(t) = &\int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)u_j^{(k-1)}(s)dsd\tau \\ &+ \int\limits_{(r-1)h}^{t} \tilde{F}(\tau)d\tau, \ \ t \in [(r-1)h, rh), \ \ r = \overline{1,N}. \end{aligned}$$

Denote by $\Delta_r^{(k)}(t)$ the difference of $u^{(k)}(t) - u^{(k-1)}(t)$, then

$$\begin{aligned} &\Delta u_r^{(k)}(t) = \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau,s)\Delta u_j^{(k-1)}(s)dsd\tau, \\ &t \in [(r-1)h, rh), \ \ r = \overline{1,N}, \ k = 1, 2, \dots . \end{aligned}$$

As

$$\sup_{t\in[(r-1)h,rh)} \|\Delta u_r^{(k)}(t)\| = \beta T h \max_{j=\overline{1,N}} \sup_{t\in[(j-1)h,jh)} \|\Delta u_j^{(k-1)}(t)\|,$$

hence

$$\|\Delta u^{(k)}[\cdot]\|_2 = \beta T h \|\Delta u^{(k-1)}[\cdot]\|_2, \ k = 1, 2, \dots . \tag{18}$$

It follows from (18) and (16) that the system of the function $u^{(k)}[t]$ at $k \to \infty$ converges to $u^*[t]$ by the norm of the space $C([0,T],\ h,\ R^{nN})$. Theorem 1 is proved. $\square$

Sufficient conditions for the feasibility and convergence of the proposed algorithm, as well as the existence of a unique solution to the problem (1),(2), provided that the

$$\mathrm{diag}(1-a_1^2, 1-\ a_2^2,\ \dots, 1-\ a_n^2)$$

matrix is not degenerate, are established.

Theorem 2. *Let the condition of Theorem 1 be satisfied, the matrix $Q(h)$ be invertible, and the following inequalities hold:*

$$\|[Q(h)]^{-1}\| \le \gamma(h),$$

$$q(h) = \frac{\delta(h)}{1-\delta(h)}\gamma(h)\max(1, h\|C\|) < 1. \tag{19}$$

Then problem (1),(2) has a unique solution.

Proof. The invertibility of the $Q(h)$ matrix implies the existence of $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)}) \in R^{nN}$ and

$$\begin{aligned}\|\lambda^{(0)}\| &= \max_{r=\overline{1,N}} \|\lambda_r^{(0)}\| = \|[Q(h)]^{-1}F(h)\| \\ &\le \gamma(h)(1+h\|C\|)\max(\|f\|, \|d\|)h.\end{aligned}$$

Since the conditions of Theorem 1 are met, then the special Cauchy problem has a unique solution $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), \dots, u_N^{(0)}(t))'$. Substituting the obtained systems of the function $u^{(0)}(t)$ in the right-hand side of equation (15), we determine the values of $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_N^{(1)})$. Then

$$\begin{aligned}\|\lambda^{(1)} - \lambda^{(0)}\| &= \max_{r=\overline{1,N}} \|\lambda_r^{(0)}\| = \|\ [Q(h)]^{-1}\ G(u^{(0)}, h)\| \\ &\le \gamma(h)(1, h\|C\|)\beta Th\|u^{(0)}[\cdot]\|_2.\end{aligned}$$

Continuing the process, at the $k-$th step we find a sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$, where $\lambda^{(k)} \in R^{nN}$, $u^{(k)}[t] \in C([0,T],\ h,\ R^{nN})$. Estimate the difference of solutions

$$\begin{aligned}&\|u^{(k)}[\cdot] - u^{(k-1)}[\cdot]\|_2 \\ &\le \beta Th[\|u^{(k)}[\cdot] - u^{(k-1)}[\cdot]\|_2 + \|\lambda^{(k)} - \lambda^{(k-1)}\|].\end{aligned} \tag{20}$$

Because of (16),

$$\|u^{(k)}[\cdot] - u^{(k-1)}[\cdot]\|_2 \leq \frac{\delta(h)}{1-\delta(h)} \|\lambda^{(k)} - \lambda^{(k-1)}\|. \tag{21}$$

From the system (15) it follows

$$\begin{aligned} &\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \|[Q(h)]^{-1}\| \|G(u^{(k)}, h) - G(u^{(k-1)}, h)\| \\ &\leq \gamma(h)(1, h\|C\|)\beta T h \|u^{(k)}[\cdot] - u^{(k-1)}[\cdot]\|_2. \end{aligned} \tag{22}$$

Substituting (21) in the right-hand side of (22) we get

$$\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq q(h)\|\lambda^{(k)} - \lambda^{(k-1)}\|, \ k = 1, 2, \dots . \tag{23}$$

As $q(h) < 1$, the inequalities (21), (23) imply the convergence of the sequence $\lambda^{(k)}$ to λ^* and the convergence of the sequence of functions $u^{(k)}[t]$ with respect to the norm of the space $C([0,T],\ h,\ R^{nN})$ to a function $u^*[t]$.

As the pair $(\lambda^*, u^*[t])$ is the solution of the problem (8)–(11), then by virtue of the conditions (16), (19), the function $x^*(t)$ defined by the equalities $x^*(t) = u_r^*(t) + \lambda_r^*,\ \ t \in [(r-1)h, rh),\ \ r = \overline{1, N}$, $x^*(T) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$ will be the solution of the problem (3), (4). And if the $\mathrm{diag}(1 - a_1^2, 1 - \ a_2^2,\ \dots, 1 - \ a_n^2)$ matrix is not degenerate, $x^*(t)$ will be the solution of the problem (1), (2).

Theorem 2 is proved. □

In [16], the sufficient and necessary conditions were established for the unique solvability of the linear boundary value problem for a system of integro-differential equations

$$\frac{dx}{dt} = \int_0^T K(t,s)x(s)ds + f(t), \qquad t \in [0,T], \tag{24}$$

$$Bx(0) + Cx(T) = d, \qquad d \in R^n. \tag{25}$$

Theorem 3. (see [16]) *Problem (16), (17) is uniquely solvable if and only if there exists $h \in (0, h_0] : Nh = T$ such that the matrix $Q_*(h)$ is invertible.*

Let us now state a corollary of this theorem regarding problem (1), (2).

Corollary 4. *Problem (1), (2) is uniquely solvable if and only if there exists $h \in (0, h_0] : Nh = T$ such that the matrix $Q_*(h)$ is invertible.*

Here h_0 is determined by the condition $q(h_0) = \frac{T}{1+a}\beta h_0 < 1$.

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