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RIGHT MULTIPLIERS AND COMMUTATIVITY OF 3-PRIME NEAR-RINGS

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Abstract: In this paper we generalize some well-known results concerning a right multiplier satisfying certain differential identities on 3-prime near-rings.

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1. Introduction

Throughout this paper, $\mathcal{N}$ denotes a left near-ring with multiplicative center $Z(\mathcal{N})$. From [2], a near-ring $\mathcal{N}$ is called 3-prime if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. Recalling that $\mathcal{N}$ is said to be 2-torsion free, if whenever $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. A right near-ring $\mathcal{N}$ is called zero-symmetric, if $x0 = 0$ for all $x \in \mathcal{N}$, recall that right distributive yields $0x = 0$. Let α and β mapfrom $\mathcal{N}$ to $\mathcal{N}$. Let $x, y \in \mathcal{N}$, we write $[x, y]_{\alpha,\beta} = \alpha(x)\beta(y) - \beta(y)\alpha(x)$ and $(x \circ y)_{\alpha,\beta} = \alpha(x)\beta(y) + \beta(y)\alpha(x)$, in particular $[x, y]_{1,1} = [x, y]$ and $(x \circ y)_{1,1} = x \circ y$ in the usual sense. An additive mapping $F : \mathcal{N} \to \mathcal{N}$ is called a right (resp. left) multiplier (or centralizer), if $F(xy) = xF(y)$ (resp.

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$F(xy) = F(x)y$) holds for all $x, y \in \mathcal{N}$. F is called multiplier if it is both right as well as left multiplier. Several authors investigated the commutativity in prime and semi-prime rings admitting right (or left) multipliers, which satisfy appropriate algebraic conditions on suitable subset of the rings. For example, we refer the reader to [7], where more references can be found. Recently, the second author with M. Ashraf [4] proved that if a 3-prime near-ring $\mathcal{N}$ admits a left multiplier (resp. right multiplier) $F : \mathcal{N} \to \mathcal{N}$ satisfying any one of the following properties: (i) $F([x, y]) \in Z(\mathcal{N})$, (ii) $F(x \circ y) \in Z(\mathcal{N})$,, (iii) $F([x, y]) \pm (x \circ y) \in Z(\mathcal{N})$ and (iv) $F([x, y]) \pm x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring. In this note we intend to extend these results for 3-prime near-ring satisfying any one of the identities: (1) $[x, y]_{\alpha,\beta} \in Z(\mathcal{N})$, (2) $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, (3) $H([x, y]_{\alpha,\beta}) \in Z(\mathcal{N})$, (4) $H([x, y]_{\alpha,\beta}) \pm [x, y]_{\alpha,\beta} \in Z(\mathcal{N})$, (5) $H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$, (6)$H((x \circ y)_{\alpha,\beta}) \pm (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, (7) $H([x, y]_{\alpha,\beta}) \pm (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, (8) $H((x \circ y)_{\alpha,\beta}) \pm [x, y]_{\alpha,\beta} \in Z(\mathcal{N})$, holds for all $x, y \in \mathcal{N}$, where H and β nonzero right multipliers on $\mathcal{N}$ and α is an automorphism of $\mathcal{N}$.

2. Some preliminaries

In this paper it is assumed that $\mathcal{N}$ is 3-prime zero-symmetric near-ring and $\alpha : \mathcal{N} \to \mathcal{N}$ is an automorphism.

Lemma 1. ([3], Lemma 1.3 (iii)) *Let $\mathcal{N}$ be a 3-prime near-ring. If $z \in Z(\mathcal{N}) - \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

Lemma 2. ([3], Lemma 1.3 (i)) *Let $\mathcal{N}$ be a 3-prime near-ring. If x is an element of $\mathcal{N}$ such that $\mathcal{N}x = \{0\}$ (resp. $x\mathcal{N} = \{0\}$), then $x = 0$.*

Lemma 3. ([3], Lemma 1.5) *Let $\mathcal{N}$ is a 3-prime near-ring. If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 4. *Let $\mathcal{N}$ be a near-ring and H be a right multiplier of $\mathcal{N}$, then:*

(a) $H - I_{\mathcal{N}}$ *is a right multiplier.*

(b) $H + I_{\mathcal{N}}$ *is a right multiplier.*

(c) For each positive integer $n \geq 1$, H^n is a right multiplier.

Proof. (a) Let $F = H - I_{\mathcal{N}}$, then $F(xy) = H(xy) - xy = xH(y) - xy = x(H(y) - y) = xF(y)$ for all $x, y \in \mathcal{N}$. Thus $H - I_{\mathcal{N}}$ is a right multiplier.

(b) By using similar arguments, we get the required result.

(c) For $n = 1$, the result is true. Let $n \geq 1$ be a fixed positive integer. Suppose that H^n is a right multiplier, then $H^{n+1}(xy) = H(H^n(xy)) = H(xH^n(y)) = xH^{n+1}(y)$ for all $x, y \in \mathcal{N}$. Hence, H^{n+1} is a right multiplier. So by the induction hypothesis, we conclude that for each positive integer $n \geq 1$, H^n is a right multiplier.

Lemma 5. *Let $\mathcal{N}$ be a p-torsion free near-ring, where p is positive integer $p \geq 1$. If $\mathcal{N}$ admits a nonzero right multiplier H, then:*

(a) pH is a nonzero right multiplier;

(b) $-pH$ is a nonzero right multiplier.

Proof. (a) First we assume that H is a right multiplier. Let $G = pH$, then $G(xy) = pH(xy) = x(pH(y)) = xG(y)$ for all $x, y \in \mathcal{N}$. Suppose that $G = 0$, then $G(x) = pH(x) = 0$ for all $x \in \mathcal{N}$. Using the p-torsion freeness of $\mathcal{N}$, we get $H = 0$, which is a contradiction. Consequently, pH is a nonzero right multiplier.

(b) Using similar arguments as used in (a), we may obtain the required result.

Lemma 6. *Let $\mathcal{N}$ be a 3-prime near-ring. If H is a nonzero right multiplier on $\mathcal{N}$ such that $H(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Proof. Assume that $H(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Putting yx in place of x, we get $yH(x) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Since $H \neq 0$, by Lemma 1, we obtain $\mathcal{N} \subseteq Z(\mathcal{N})$, which forces that $\mathcal{N}$ is a commutative ring by Lemma 3.

3. Conditions on right multipliers

Motivated by the results in [3] our objective in the present paper is to generalize them.

Theorem 7. *If H and β are nonzero right multipliers on $\mathcal{N}$, then the following assertions are equivalent:*

(a) $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(b) $H([x,y]_{\alpha,\beta}) \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(c) $\mathcal{N}$ *is a commutative ring.*

Proof. It is obvious that $(c) \Rightarrow (a)$ and $(c) \Rightarrow (b)$.

$(a) \Rightarrow (c)$. We begin with the situation $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. Substituting $\alpha(x)y$ for y, we have $\alpha(x)[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. Using Lemma 1, we get $\alpha(x) \in Z(\mathcal{N})$ or $[x,y]_{\alpha,\beta} = [\alpha(x),\beta(y)] = 0$ for all $x,y \in \mathcal{N}$. Thus $[\alpha(x),\beta(y)] = 0$ for all $x,y \in \mathcal{N}$ it follows $\beta(\mathcal{N}) \subseteq Z(\mathcal{N})$. According to Lemma 6, we conclude that $\mathcal{N}$ is a commutative ring.

$(b) \Rightarrow (c)$. Assume that

$$H([x,y]_{\alpha,\beta}) \in Z(\mathcal{N}) \text{ for all } x,y \in \mathcal{N}. \tag{1}$$

Putting $\alpha(x)y$ instead of y in (1), we find that $\alpha(x)H([x,y]_{\alpha,\beta}) \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. By Lemma 1, we get $H([x,y]_{\alpha,\beta}) = 0$ or $\alpha(x) \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$, which gives $H([x,y]_{\alpha,\beta}) = 0$ for all $x,y \in \mathcal{N}$. Equivalently,

$$\beta(y)H(\alpha(x)) = \alpha(x)H \circ \beta(y) \text{ for all } x,y \in \mathcal{N}. \tag{2}$$

Replacing y by ty in (2), we arrive at

$$t\beta(y)H(\alpha(x)) = \alpha(x)tH \circ \beta(y) \text{ for all } x,y,t \in \mathcal{N}. \tag{3}$$

Putting $\alpha^{-1}([n,m]_{\alpha,\beta})$ in place of x in (3), we get $[n,m]_{\alpha,\beta}\mathcal{N}H \circ \beta(y) = \{0\}$ for all $y,n,m \in \mathcal{N}$. The 3-primeness of $\mathcal{N}$ implies

$$H \circ \beta = 0 \text{ or } [n,m]_{\alpha,\beta} = 0 \text{ for all } n,m \in \mathcal{N}.$$

If $H \circ \beta = 0$, then (3) becomes

$$\beta(y)H(x) = 0 \text{ for all } x,y \in \mathcal{N}. \tag{4}$$

Substituting tx for x in (4), we obtain $\beta(y)\mathcal{N}H(x) = 0$ for all $x,y \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we get $H = 0$ or $\beta = 0$, which gives a contradiction. Hence $[n,m]_{\alpha,\beta} = 0$ for all $n,m \in \mathcal{N}$. According to $(a) \Rightarrow (c)$, we conclude that $\mathcal{N}$ is a commutative ring.

Corollary 8. *If H and β are nonzero right multipliers on $\mathcal{N}$, then for each positive integer $n \geq 1$ such that $H^n \neq 0$ the following statements are equivalent:*

(i) $H^n([x,y]_{\alpha,\beta}) \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(ii) $\mathcal{N}$ *is a commutative ring.*

Corollary 9. *If* H *and* β *are nonzero right multipliers on* $\mathcal{N}$*, then for each positive integer* $n \geq 1$ *such that* $H^n \neq \pm I_{\mathcal{N}}$ *the following assertions are equivalent:*

(i) $H^n([x,y]_{\alpha,\beta}) + [x,y]_\beta \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{R}$*;*

(ii) $H^n([x,y]_{\alpha,\beta}) - [x,y]_\beta \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{R}$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

Corollary 10. ([4], Remark) *If* H *is a nonzero right multiplier on* $\mathcal{N}$*, then the following assertions are equivalent:*

(i) $H([x,y]) \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(ii) $\mathcal{N}$ *is a commutative ring.*

Theorem 11. *Let* $\mathcal{N}$ *be a 2-torsion free near-ring. If* H *and* β *are nonzero right multipliers on* $\mathcal{N}$*, then the following assertions are equivalent:*

(a) $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(b) $H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(c) $\mathcal{N}$ *is a commutative ring.*

Proof. It is obvious that (c) implies (a) and (b).

$(a) \Rightarrow (c)$. First we consider the case $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. Now we substitute $\alpha(x)y$ for y, we get $\alpha(x)(x \circ y)_\beta \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. So we have

$$(x \circ y)_{\alpha,\beta} = 0 \text{ or } \alpha(x) \in Z(\mathcal{N}) \text{ for all } x,y \in \mathcal{N}. \tag{5}$$

Suppose there exists $\alpha(x_0) \in Z(\mathcal{N}) - \{0\}$. Using our hypothesis, we get $\alpha(x_0)(\beta(t) + \beta(t)) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$. From Lemma 1, we arrive at

$$2\beta(t) \in Z(\mathcal{N}) \text{ for all } t \in \mathcal{N} \text{ or } x_0 = 0. \tag{6}$$

In view of (6), (5) becomes

$$(x \circ y)_{\alpha,\beta} = 0 \text{ for all } x,y \in \mathcal{N} \text{ or } 2\beta(t) \in Z(\mathcal{N}) \text{ for all } t \in \mathcal{N}. \tag{7}$$

If $2\beta(t) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$, we conclude that $\mathcal{N}$ is a commutative ring by Lemma 5 and Lemma 6.

If $(x \circ y)_{\alpha,\beta} = 0$ for all $x, y \in \mathcal{N}$, so $\alpha(x)\beta(y) = -\beta(y)\alpha(x)$ for all $x, y \in \mathcal{N}$. Putting xt in place of x in the above relation, we arrive at

$$\alpha(x)t\beta(y) = \alpha(x)(-\beta(y))(-t) \text{ for all } t, x, y \in \mathcal{N}. \tag{8}$$

Which gives $\mathcal{N}[-\beta(y), t] = \{0\}$ for all $y, t \in \mathcal{N}$. So by Lemma 1 and Lemma 2, we find that $-\beta(y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Since $-\beta$ is a nonzero right multiplier, Lemma 6 forces that $\mathcal{N}$ is a commutative ring.

$(b) \Rightarrow (c)$. Assume that

$$H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{9}$$

Case 1: If $H \circ \beta = 0$, then (9) becomes

$$\beta(y)H(\alpha(x)) \in Z(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{10}$$

Writing ty instead of y in (10), we get

$$t\beta(y)H(\alpha(x)) \in Z(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{11}$$

Using Lemma 1, we find that

$$\beta(y)H(\alpha(x)) = 0 \text{ or } t \in Z(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{12}$$

Suppose that $\beta(y)H(\alpha(x)) = 0$ for all $x, y \in \mathcal{N}$, by using the same technique as used previously, we get $H = 0$ or $\beta = 0$; a contradiction. So (12) becomes $\mathcal{N} \subseteq Z(\mathcal{N})$ which forces that $\mathcal{N}$ is a commutative rings by Lemma 6.

Case 2: If $H \circ \beta \neq 0$. Taking $\alpha(x)y$ instead of y in (9), we get $\alpha(x)H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$. So by Lemma 1, we have

$$H((x \circ y)_{\alpha,\beta}) = 0 \text{ or } \alpha(x) \in Z(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{13}$$

Suppose that there exists $x_0 \in \mathcal{N}$ such that $\alpha(x_0) \in Z(\mathcal{N}) - \{0\}$. From $H((x_0 \circ t)_{\alpha,\beta}) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$, it follows that $\alpha(x_0)(H \circ \beta(t) + H \circ \beta(t)) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$ and by Lemma 1, we obtain

$$x_0 = 0 \text{ or } 2H \circ \beta(t) \in Z(\mathcal{N}) \text{ for all } t \in \mathcal{N}. \tag{14}$$

Using (14), (13) yields

$$H((x \circ y)_{\alpha,\beta}) = 0 \text{ for all } x, y \in \mathcal{N} \text{ or } 2H \circ \beta(t) \in Z(\mathcal{N}) \text{ for all } t \in \mathcal{N}.$$

If $2H(\beta(t)) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$. Since $H \circ \beta$ is a nonzero right multiplier, by Lemma 5 and Lemma 6, we conclude that $\mathcal{N}$ is a commutative ring.

If $H((x \circ y)_{\alpha,\beta}) = 0$ for all $x, y \in \mathcal{N}$, then

$$-\beta(y)H(\alpha(x)) = \alpha(x)H \circ \beta(y) \text{ for all } x, y \in \mathcal{N}. \quad (15)$$

Substituting ty for y in (15), we obtain

$$-t\beta(y)H(\alpha(x)) = \alpha(x)tH \circ \beta(y) \text{ for all } x, y, t \in \mathcal{N}. \quad (16)$$

Writing $\alpha^{-1}((m \circ n)_{\alpha,\beta})$ instead of x in (16), we find that $(m \circ n)_{\alpha,\beta}\mathcal{N}H \circ \beta(y) = \{0\}$ for all $m, n, y \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$ and the fact that $H \circ \beta \neq 0$, we conclude that $(m \circ n)_{\alpha,\beta} = 0$ for all $m, n \in \mathcal{N}$. Using the same techniques as used previously, we conclude that $\mathcal{N}$ is a commutative ring.

Corollary 12. *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H and β are nonzero right multipliers on $\mathcal{N}$, then for each positive integer $n \geq 1$ such that $H^n \neq 0$ the following assertions are equivalent:*

(i) $H^n((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(ii) $\mathcal{N}$ *is a commutative ring.*

Corollary 13. *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H and β are nonzero right multipliers on $\mathcal{N}$, then for each positive integer $n \geq 1$ such that $H^n \neq \pm I_{\mathcal{N}}$ the following assertions are equivalent:*

(a) $H^n((x \circ y)_{\alpha,\beta}) + (x \circ y)_{\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{R}$;

(b) $H^n((x \circ y)_{\alpha,\beta}) - (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{R}$;

(c) $\mathcal{N}$ *is a commutative ring.*

Corollary 14. ([4], Remark) *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H is a nonzero right multiplier on $\mathcal{N}$, then the following assertions are equivalent:*

(a) $H(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(b) $\mathcal{N}$ *is a commutative ring.*

Theorem 15. *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H and β are nonzero right multipliers on $\mathcal{N}$, then the following assertions are equivalent:*

(a) $H([x, y]_{\alpha,\beta}) + (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(b) $H([x,y]_{\alpha,\beta}) - (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in \mathcal{N}$*;*

(c) $\mathcal{N}$ *is a commutative ring.*

Proof. It is clear that (c) implies (a) and (b).

$(a) \Rightarrow (c)$. Assume that $H = 0$, then $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$, so $\mathcal{N}$ is a commutative ring by Theorem 11.

Suppose that $H \neq 0$ and

$$H([x,y]_{\alpha,\beta}) + (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N}) \text{ for all } x,y \in \mathcal{N}. \tag{17}$$

For $\alpha(x) = \beta(y)$, (17) becomes

$$\beta(y)\left(\beta(y) + \beta(y)\right) = \left(\beta(y)\right)^2 + \left(\beta(y)\right)^2 \in Z(\mathcal{N}) \text{ for all } y \in \mathcal{N}. \tag{18}$$

Putting $\beta(y)y$ in place of y in (18) and using it, we arrive at

$$\left(\beta(y)\right)^2 \left(\left(\beta(y)\right)^2 + \left(\beta(y)\right)^2\right) \in Z(\mathcal{N}) \ y \in \mathcal{N}.$$

Using Lemma 1, we conclude that $\left(\beta(y)\right)^2 + \left(\beta(y)\right)^2 = 0$ or $\left(\beta(y)\right)^2 \in Z(\mathcal{N})$. Since $\mathcal{N}$ is 2-torsion free, the above expression gives $\left(\beta(y)\right)^2 \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Putting $\beta(y)y$ in place of y in (17), we get $2x\left(\beta(y)\right)^2 \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. From Lemma 1, it follows that

$$\left(\beta(y)\right)^2 = 0 \text{ or } 2x \in Z(\mathcal{N}) \text{ for all } x,y \in \mathcal{N}. \tag{19}$$

Assume that $\left(\beta(y)\right)^2 = 0$ for all $y \in \mathcal{N}$. Then

$$\beta(y)\left(\beta(y) + \beta(z)\right)^2 = 0 \text{ for all } y,z \in \mathcal{N}.$$

Which means that $\beta(y)\beta(z)\beta(y) = 0$ for all $y,z \in \mathcal{N}$. Replacing z by mz in the last equation we ge $\beta(y)m\beta(z)\beta(y) = 0$. This yields $\beta(y)\mathcal{N}\beta(z)\beta(y) = \{0\}$ for all $y,z \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we conclude that $\beta(z)\beta(y) = 0$ for all $y,z \in \mathcal{N}$. Taking ny in place of y in last relation, we get $\beta(z)\mathcal{N}\beta(y) = \{0\}$ for all $y,z \in \mathcal{N}$. 3-primness of $\mathcal{N}$ again forces that $\beta = 0$; a contradiction. Thus (19) becomes $2I_{\mathcal{N}}(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Since $I_{\mathcal{N}}$ is a nonzero right multiplier, then $\mathcal{N}$ is a commutative ring by Lemma 5 and Lemma 6.

$(b) \Rightarrow (c)$. Assume that $H([x,y]_{\beta}) - (x \circ y)_{\beta} \in Z(\mathcal{N})$ for all $x,y \in \mathcal{N}$. Arguing as in the proof of $(a) \Rightarrow (c)$, we arrive at $\mathcal{N}$ is a commutative ring.

Corollary 16. *Let* $\mathcal{N}$ *be a 2-torsion free near-ring. If* H *and* β *are nonzero right multipliers on* $\mathcal{N}$*, then for each positive integer* $n \geq 1$ *the following assertions are equivalent:*

(a) $H^n([x,y]_{\alpha,\beta}) + (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(b) $H^n([x,y]_{\alpha,\beta}) - (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(c) $\mathcal{N}$ *is a commutative ring.*

Corollary 17. ([4], Remark) *Let* $\mathcal{N}$ *be a 2-torsion free near-ring. If* H *is nonzero right multiplier on* $\mathcal{N}$*, then the following assertions are equivalent:*

(a) $H([x,y]) + (x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(b) $H([x,y]) - (x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(c) $\mathcal{N}$ *is a commutative ring.*

Theorem 18. *Let* $\mathcal{N}$ *be a 2-torsion free near-ring. If* H *and* β *are nonzero right multipliers on* $\mathcal{N}$ *such that* $H \circ \beta \neq 0$*, then the following assertions are equivalent:*

(a) $H((x \circ y)_{\alpha,\beta}) + [x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(b) $H((x \circ y)_{\alpha,\beta}) - [x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;

(c) $\mathcal{N}$ *is a commutative ring.*

Proof. It is obvious that (c) implies (a) and (b).

$(a) \Rightarrow (c)$. Suppose that $H = 0$, then $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ for all for all $x, y \in \mathcal{N}$, Hence $\mathcal{N}$ is a commutative ring by Theorem 7.

Suppose that $H \neq 0$ and

$$H((x \circ y)_{\alpha,\beta}) + [x,y]_{\alpha,\beta} \in Z(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{20}$$

Taking $\alpha(x) = \beta(y)$ in (20) we arrive at $2H((\beta(y))^2) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Substituting $\beta(y)y$ for y in the last expression and using it we obtain $(\beta(y))^2\,(2H((\beta(y))^2)) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. By using Lemma 1, we get

$$2H((\beta(y))^2) = 0 \text{ or } (\beta(y))^2 \in Z(\mathcal{N}) \text{ for all } y \in \mathcal{N}.$$

According to 2-torsion freeness of $\mathcal{N}$, we conclude that

$$H((\beta(y))^2) = 0 \text{ or } (\beta(y))^2 \in Z(\mathcal{N}) \text{ for all } y \in \mathcal{N}. \tag{21}$$

Suppose that there exists $y_0 \in \mathcal{N}$ such that $(\beta(y_0))^2 \in Z(\mathcal{N})$. Putting $\beta(y_0)y_0$ in place of y in (20), we arrive $(\beta(y_0))^2\,(2H(\alpha(x))) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. By using Lemma 1, we get

$$(\beta(y_0))^2 = 0 \text{ or } 2H(\alpha(x)) \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}. \tag{22}$$

In view of (22), (21) becomes

$$H((\beta(y))^2) = 0 \text{ for all } y \in \mathcal{N} \text{ or } 2H(x) \in Z(\mathcal{N}) \text{ for all } x \in \mathcal{N}. \tag{23}$$

Assume that $H((\beta(y))^2) = 0$ for all $y \in \mathcal{N}$. Then

$$H\left(\beta(y)\left(\beta(y)+\beta(z)\right)^2\right) = 0 \text{ for all } y, z \in \mathcal{N}.$$

Which gives $H(\beta(y))\beta(z)\beta(y) = 0$ for all $y, z \in \mathcal{N}$. Replacing z by nz in the last expression, we get $H(\beta(y))n\beta(z)\beta(y) = 0$ for all $n, y, z \in \mathcal{N}$ this can be written as $H(\beta(y))\mathcal{N}\beta(z)\beta(y) = \{0\}$ for all $y, z \in \mathcal{N}$. According to 3-primeness of $\mathcal{N}$, we get $H(\beta(y)) = 0$ or $\beta(z)\beta(y) = 0$ for all $y, z \in \mathcal{N}$. Which implies that $\beta(z)H(\beta(y)) = 0$ for all $y, z \in \mathcal{N}$. Taking my in place of y in last equation, we get $\beta(z)\mathcal{N}H(\beta(y)) = \{0\}$ for all $y, z \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we arrive at $\beta = 0$ or $H \circ \beta = 0$, which implies that $H \circ \beta = 0$; a contradiction. Hence (23) becomes $2H(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Consequently, $\mathcal{N}$ is a commutative ring by Lemma 5 and Lemma 6.

Corollary 19. *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H and β are nonzero right multipliers on $\mathcal{N}$ such that $H \circ \beta \neq 0$, then for each positive integer $n \geq 1$ the following assertions are equivalent:*

(a) $H^n((x \circ y)_{\alpha,\beta}) + [x, y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*;*

(b) $H^n((x \circ y)_{\alpha,\beta}) - [x, y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*;*

(c) $\mathcal{N}$ *is a commutative ring.*

Corollary 20. ([4], Remark) *Let $\mathcal{N}$ be a 2-torsion free near-ring. If H is a nonzero right multiplier on $\mathcal{N}$, then the following assertions are equivalent:*

(a) $H(x \circ y) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*;*

(b) $H(x \circ y) - [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$*;*

(c) $\mathcal{N}$ *is a commutative ring.*

The following example proves that the restriction of 3-primeness of $\mathcal{N}$ imposed on the hypothesis of the above theorems is not superfluous.

Example 1. Suppose that S is any abelian left near-ring, $n \geq 1$ be a fixed positive integer. Let

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}.$$

Then $\mathcal{N}$ is a right near-ring, which is not 3-prime. Let us define β, H and α : $\mathcal{N} \to \mathcal{N}$ as follows: $\beta \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$\alpha \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & x+y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that α is an automorphism of $\mathcal{N}$, and β, H are nonzero right multipliers of $\mathcal{N}$. Moreover, for each positive integer $n \geq 1$, $\mathcal{N}$ satisfies the conditions:

(*i*) $[u, v]_{\alpha,\beta} \in Z(\mathcal{N})$,
(*ii*) $(u \circ v)_{\alpha,\beta} \in Z(\mathcal{N})$,
(*iii*) $H^n([u, v]_{\alpha,\beta}) \in Z(\mathcal{N})$,
(*iv*) $H^n([u, v]_{\alpha,\beta}) \pm [u, v]_{\beta} \in Z(\mathcal{N})$,
(*v*) $H^n((u \circ v)_{\alpha,\beta}) \in Z(\mathcal{N})$,
(*vi*) $H^n((u \circ v)_{\alpha,\beta}) \pm (u \circ v)_{\alpha,\beta} \in Z(\mathcal{N})$,
(*vii*) $H^n([u, v]_{\alpha,\beta}) \pm (u \circ v)_{\alpha,\beta} \in Z(\mathcal{N})$,
(*viii*) $H^n((u \circ v)_{\alpha,\beta}) \pm [u, v]_{\alpha,\beta} \in Z(\mathcal{N})$,

for all $u, v \in \mathcal{N}$, but $\mathcal{N}$ is not commutative.

The following example proves that in Theorems 15 and 18 the hypothesis that $\mathcal{N}$ is 2-torsion free is crucial.

Example 2. Let $\mathcal{N} = M_2(Z_2)$. Then, $\mathcal{N}$ is a non-commutative prime ring, which is not 2-torsion free. Define β, H and $\alpha : \mathcal{N} \to \mathcal{N}$ by $H = \beta = \alpha = I_{\mathcal{N}}$. It is easy to verify that α is an automorphism, and β, H are nonzero right multipliers. Moreover, for each positive integer $n \geq 1$, $\mathcal{N}$ satisfies the conditions:

(i) $H^n([x,y]_{\alpha,\beta}) \pm (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$;

(ii) $H^n((x \circ y)_{\alpha,\beta}) \pm (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$.

The following example shows that for $n \geq 1$ the conditions:

(i) $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$,

(ii) $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$,

(iii) $H^n([x,y]_{\alpha,\beta}) \in Z(\mathcal{N})$,

(iv) $H^n([x,y]_{\alpha,\beta}) \pm [x,y]_{\alpha,\beta} \in Z(\mathcal{N})$,

(v) $H^n((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$,

(vi) $H^n((x \circ y)_{\alpha,\beta}) \pm (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$,

for all $x, y \in \mathcal{N}$ are crucial.

Example 3. Let $\mathcal{N} = M_2(Z)$. Define mappings β, H and $\alpha : \mathcal{N} \to \mathcal{N}$ by: $\alpha = I_{\mathcal{N}}$, $\beta \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & t \end{pmatrix}$ and $H \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix}$.

It easy to verify that $\mathcal{N}$ is a non-commutative prime ring, which is 2-torsion free, α is an automorphism, and H, β are right multipliers. Moreover, for $X = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, we have:

(i) $[X,Y]_{\alpha,\beta} \notin Z(\mathcal{N})$,

(ii) $(X \circ Y)_{\alpha,\beta} \notin Z(\mathcal{N})$,

(iii) $H^n([X,Y]_{\alpha,\beta}) \notin Z(\mathcal{N})$,

(iv) $H^n([X,Y]_{\alpha,\beta}) \pm [X,Y]_{\alpha,\beta} \notin Z(\mathcal{N})$,

(v) $H^n((X \circ Y)_{\alpha,\beta}) \notin Z(\mathcal{N})$,

(vi) $H^n((X \circ Y)_{\alpha,\beta}) \pm (X \circ Y)_{\alpha,\beta} \notin Z(\mathcal{N})$.

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