ON THE SOLVABILITY OF HIGHER-ORDER OPERATOR-DIFFERENTIAL EQUATIONS IN A WEIGHTED SOBOLEV SPACE

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Abstract: In a weighted Sobolev space on the whole real axis, we obtain the sufficient conditions for the well-posed and unique solvability of m, n order operator-differential equations. These conditions were formulated only by the operator coefficients of the considered equation. According to the values of m, n the operator-differential equation has complicated and multiple characteristics. In addition, by using the main part of the equation, the norms of the operators of intermediate derivative were estimated. We deduce the relationship between the exponent of the weight and the lower bound of the spectrum of the operator of the main part of the equation. As an applied result of this paper, we formulated a problem for higher-order partial differential equations and we provided an alternative method for obtaining the regular solvability of operator pencil.

1. Introduction

The theory of operator-differential equations in Banach or Hilbert space played a key role in searching both of the ordinary and the partial differential operators (see [25]). In this paper, the investigated equation expressed an interest in applications, for instance, the dynamic problems of arches as well as rings and

modeling the stability of the plates from plastic (see [21], [25]). The solvability of initial boundary value problems for higher order operator differential equations has been researched by many authors as A.A. Gasymov, V.I. Gorbachuk, M.L. Gorbachuk, S.Ya. Yakubov, V.N. Pilipchuk and their followers (see [6], [9], [19], [20], [22], [24]). Nowadays, a large number of papers concerning the study of solvability of the operator differential equations in Hilbert or Banach spaces has been published. The principal part of the investigated equation has mixed multiple-complicated characteristics with m, n order $(m, n \geq 1)$. It should be noted that for specific values of m, n, the solvability problem for second $(m = 0, n = 2)$, third $(m = 1, n = 2)$, fourth $(m = 1, n = 3)$ and fifth $(m = 5, n = 0)-$ order operator differential equations have been studied elsewhere (see [1]-[7], [16]). This study differs from the study in work [5]. The main part of the equation in work [5] contains $\frac{du(t)}{dt}$ and $\frac{d^n u(t)}{dt^n}$ terms, $(t \in [0, +\infty))$ with only multiple characteristics while in this study the solvability problem for operator-differential equations include $\frac{d^m u(t)}{dt^m}$ and $\frac{d^n u(t)}{dt^n}$ terms, $(t \in (-\infty, +\infty))$ with both complicated and multiple characteristics. The m, n-order differential equations are very difficult to solve because they must be solved in more complete form as the main part of the equation contains $\frac{d^m u(t)}{dt^m}$ and $\frac{d^n u(t)}{dt^n}$ terms. In the whole real axis and in a weighted Sobolev space, the general higher-order operator-differential equations with complicated and multiple characteristic have not been studied yet. The interest of this paper is to provide a general case of the solvability for operator-differential equations in a weighted Sobolev space that cover many applications in the future.

In a separable Hilbert space H, we have the following operator-differential equations:

$$\left(\frac{d}{dt} - A\right)^m \left(\frac{d}{dt} + A\right)^n u(t) + \sum_{j=0}^{m+n} A_{m+n-j} u^{(j)}(t) = f(t),$$

$$t \in R = (-\infty, +\infty), \tag{1}$$

where A is a self-adjoint positive-definite operator $(A = A^* \geq \sigma_0 E, \sigma_0 > 0)$, σ_0 is the lower bound of spectrum $(\sigma_0 \in \sigma(A))$, E is the unit operator, and A_j, $j = \overline{0, m+n}$ are generally linear unbounded operators. All derivatives are understood in the sense of distributions theory. We consider $f(t) \in L_{2,\alpha} =$

$L_{2,\alpha}(R;H)$, $u(t) \in W_{2,\alpha}^{n+m}(R;H)$, and $\alpha \in R$, where

$$L_{2,\alpha} = \left\{ f(t) : \|f(t)\|_{L_{2,\alpha}} = \left(\int_{-\infty}^{+\infty} \|f(t)\|_H^2 e^{-\alpha t} dt \right)^{\frac{1}{2}} < +\infty \right\},$$

$$W_{2,\alpha}^{n+m}(R;H) = \left\{ u(t) : \frac{d^{n+m}u(t)}{dt^{n+m}} \in L_{2,\alpha}, A^{n+m}u(t) \in L_{2,\alpha} \right\},$$

$$\|u\|_{W_{2,\alpha}^{n+m}(R;H)}$$

$$= \left(\int_{-\infty}^{+\infty} \left(\|A^{n+m}(u)\|_{L_{2,\alpha}}^2 + \left\| \frac{d^{n+m}(u)}{dt^{n+m}} \right\|_{L_{2,\alpha}}^2 e^{-\alpha t} dt \right) \right)^{\frac{1}{2}} < +\infty.$$

At $\alpha = 0$, for simplification we denote the space $L_{2,0}(R;H)$ by $L_2(R;H)$ and the space $W_{2,0}^{n+m}(R;H)$ by $W_2^{n+m}(R;H)$ (see [15], [18]).

Definition 1. If for any $f(t) \in L_{2,\alpha}(R;H)$ there exists a vector function $u(t) \in W_{2,\alpha}^{n+m}(R;H)$ that satisfies (1) almost everywhere in R, and the inequality

$$\|u\|_{W_{2,\alpha}^{n+m}(R;H)} \leq \mathrm{const}\, \|f\|_{L_{2,\alpha}}$$

is true, then $u(t)$ is called a regular solution of equation (1) and equation (1) is called regularly solvable (see [1], [2], [12], [16]).

We denote

$$P_0 u(t) = \left(\frac{d}{dt} - A \right)^m \left(\frac{d}{dt} + A \right)^n u(t), \qquad (2)$$

$$P_1 u(t) = \sum_{j=0}^{m+n} A_{m+n-j} u^{(j)}(t). \qquad (3)$$

Then equation (1) can be written in the form

$$Pu(t) \equiv P_0 u(t) + P_1 u(t) = f(t),\ t \in R.$$

2. Main results

Theorem 2. Let $|\alpha| < 2\sigma_0$. Then the operator P_0 is an isomorphism from $W_{2,\alpha}^{m+n}(R;H)$ to $L_{2,\alpha}(R;H)$ (see [2]-[6], [11]).

Proof. Equation (2) can be written in the form:

$$P_0\left(\frac{d}{dt};A\right)u\left(t\right)=f(t),\tag{4}$$

where $f\left(t\right)\in L_{2,\alpha}\left(R;H\right),u(t)\in W_{2,\alpha}^{n+m}(R;H).$

Let $u(t)=v(t)e^{\frac{\alpha}{2}t}$, then equation (2) takes the form

$$P_{0,\alpha}\left(\frac{d}{dt}+\frac{\alpha}{2};A\right)v\left(t\right)=g(t),\tag{5}$$

where $v(t)\in W_2^{n+m}(R;H)$, $g(t)=f(t)e^{\frac{-\alpha}{2}t}\in L_2\left(R;H\right).$ Since the mapping $v(t)\to u(t)e^{-\frac{\alpha}{2}t}$ is an isomorphism between the spaces $W_2^{n+m}(R;H)$ and $W_{2,\alpha}^{n+m}(R;H)$, It is sufficient to prove that $P_{0,\alpha}:W_2^{m+n}(R;H)\to L_2(R;H)$ is an isomorphism, where

$$P_{0,\alpha}v(t)\equiv\left(\frac{d}{dt}+\frac{\alpha}{2}-A\right)^m\left(\frac{d}{dt}+\frac{\alpha}{2}+A\right)^n v(t)=g(t).\tag{6}$$

So we must find the solution of (6) in the form

$$v(t)=\int_{-\infty}^{+\infty}G(t-s)g(s)ds\equiv P_{0,\alpha}^{-1}g.$$

By using Fourier transform for (6), we obtain

$$\left(i\zeta E+\frac{\alpha}{2}E-A\right)^m\left(i\zeta E+\frac{\alpha}{2}E+A\right)^n\widetilde{v}\left(\zeta\right)=\widetilde{g}\left(\zeta\right),\ \ \zeta\in R,\tag{7}$$

where $\widetilde{v}\left(\zeta\right)$ and $\widetilde{g}\left(\zeta\right)$ are Fourier transforms for the functions $v(t)$ and $g(t)$, respectively. For $|\alpha|<2\sigma_0$, the operator pencil

$$\left(i\zeta E+\frac{\alpha}{2}E-A\right)^m\left(i\zeta E+\frac{\alpha}{2}E+A\right)^n\widetilde{v}\left(\zeta\right)=\widetilde{g}\left(\zeta\right)$$

is invertible and moreover,

$$\widetilde{v}\left(\zeta\right)=\left(i\zeta E+\frac{\alpha}{2}E-A\right)^{-m}\left(i\zeta E+\frac{\alpha}{2}E+A\right)^{-n}\widetilde{g}\left(\zeta\right),\tag{8}$$

hence

$$v(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(i\zeta E + \frac{\alpha}{2}E - A\right)^{-m} \left(i\zeta E + \frac{\alpha}{2}E + A\right)^{-n}$$

$$\times \widetilde{g}(\zeta) e^{i\zeta t} d\zeta, \quad t \in R$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\zeta E + \frac{\alpha}{2}E - A)^{-m}(i\zeta E + \frac{\alpha}{2}E + A)^{-n}$$

$$\times \left(\int_{-\infty}^{+\infty} g(s)e^{-i\zeta s}ds\right) e^{i\zeta t} d\zeta$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{0}^{+\infty} (i\zeta E + \frac{\alpha}{2}E - A)^{-m}(i\zeta E + \frac{\alpha}{2}E + A)^{-n} e^{i\zeta(t-s)}$$

$$\times d\zeta g(s)ds = \int_{-\infty}^{+\infty} G(t-s)g(s)ds,$$

then

$$G(t-s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\zeta E + \frac{\alpha}{2}E - A)^{-m}(i\zeta E + \frac{\alpha}{2}E + A)^{-n} e^{i\zeta(t-s)}d\zeta.$$

By taking $i\zeta = w$, then

$$G(t-s) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} (\omega E + \frac{\alpha}{2}E - A)^{-m}(\omega E + \frac{\alpha}{2}E + A)^{-n} e^{w(t-s)}d\omega.$$

If $\mu \in \sigma(A)$, then

$$G(t-s) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\omega(t-s)}}{(\omega + \frac{\alpha}{2} - \mu)^m (\omega + \frac{\alpha}{2} + \mu)^n}d\omega.$$

Using the Cauchy integral:
If $t > s$, we get

$$G(t-s) = Re\, s_{\omega=-\mu-\frac{\alpha}{2}} \frac{e^{\omega(t-s)}}{(\omega + \frac{\alpha}{2} - \mu)^m (\omega + \frac{\alpha}{2} + \mu)^n}$$

$$= \frac{1}{(n-1)!} \lim_{\omega\to-\mu-\frac{\alpha}{2}} \frac{d^{n-1}}{dw^{n-1}} \left[\frac{e^{\omega(t-s)}}{(\omega + \frac{\alpha}{2} - \mu)^m} \right]$$

$$= \frac{1}{(n-1)!} \sum_{k=0}^{2} \frac{2^{-(n+m)+1}(n-k)(n-1)![\mu(t-s)]^k}{2^{(-k)}k!\mu^{(n+m)-1}} e^{-(\mu+\frac{\alpha}{2})(t-s)}$$

$$+ \frac{1}{(n-1)!} \sum_{k=3}^{n-1} \frac{2^{-(n+m)+1}(n-1)![\mu(t-s)]^k}{2^{(-k)}k!\mu^{(n+m)-1}} e^{-(\mu+\frac{\alpha}{2})(t-s)}$$

$$= \sum_{k=0}^{2} \frac{2^{-(n+m)+1}(n-k)[\mu(t-s)]^k}{2^{(-k)}k!} e^{-(\mu+\frac{\alpha}{2})(t-s)} \mu^{-(n+m)+1}$$

$$+ \sum_{k=3}^{n-1} \frac{2^{-(n+m)+1}[\mu(t-s)]^k}{2^{(-k)}k!} e^{-(\mu+\frac{\alpha}{2})(t-s)} \mu^{-(n+m)+1},$$

similarly for $t < s$, we have

$$G(t-s) = Re\, s_{\omega=\mu-\frac{\alpha}{2}} \frac{e^{\omega(t-s)}}{(\omega + \frac{\alpha}{2} - \mu)^m (\omega + \frac{\alpha}{2} + \mu)^n}$$

$$= \frac{1}{(m-1)!} \lim_{\omega\to\mu-\frac{\alpha}{2}} \frac{d^{m-1}}{dw^{m-1}} \left[\frac{e^{\omega(t-s)}}{(\omega + \frac{\alpha}{2} + \mu)^n} \right]$$

$$= \frac{1}{(m-1)!} \sum_{k=0}^{2} \frac{2^{-(n+m)+1}(m-k)(m-1)![\mu(t-s)]^k}{2^{(-k)}k!\mu^{(n+m)-1}} e^{(\mu-\frac{\alpha}{2})(t-s)}$$

$$+ \frac{1}{(m-1)!} \sum_{k=3}^{m-1} \frac{2^{-(n+m)+1}(m-1)![\mu(t-s)]^k}{2^{(-k)}k!\mu^{(n+m)-1}} e^{(\mu-\frac{\alpha}{2})(t-s)}$$

$$= \sum_{k=0}^{2} \frac{2^{-(n+m)+1}(m-k)[\mu(t-s)]^k}{2^{(-k)}k!} e^{(\mu-\frac{\alpha}{2})(t-s)} \mu^{-(n+m)+1}$$

$$+ \sum_{k=3}^{m-1} \frac{2^{-(n+m)+1}[\mu(t-s)]^k}{2^{(-k)}k!} e^{(\mu-\frac{\alpha}{2})(t-s)} \mu^{-(n+m)+1}.$$

From the spectral expansion of operator A, $(\mu \in \sigma(A)$, we get

$$
G(t-s) = \begin{cases}
\sum_{k=0}^{2} Q_{m,n}[A(t-s)]^k e^{-(A+\frac{\alpha}{2}E)(t-s)} A^{-(n+m)+1} \\
+ \sum_{k=3}^{n-1} R_{m,n}[A(t-s)]^k e^{-(A+\frac{\alpha}{2}E)(t-s)} A^{-(n+m)+1}, \ t > s \\
\\
\sum_{k=0}^{2} S_{m,n}[(A-\frac{\alpha}{2}E)(t-s)]^k e^{(A-\frac{\alpha}{2}E)(t-s)} A^{-(n+m)+1} \\
+ \sum_{k=3}^{m-1} T_{m,n}[A(t-s)]^k e^{(A-\frac{\alpha}{2}E)(t-s)} A^{-(n+m)+1}, \ t < s,
\end{cases}
$$

(9)

where

$$
Q_{m,n} = \frac{2^{-(n+m)+1}(n-k)}{2^{(-k)}k!}, \quad R_{m,n} = \frac{2^{-(n+m)+1}}{2^{(-k)}k!},
$$

$$
S_{m,n} \frac{2^{-(n+m)+1}(m-k)}{2^{(-k)}k!}, \quad T_{m,n} \frac{2^{-(n+m)+1}}{2^{(-k)}k!}, \quad m,n \geq 2.
$$

The solution $v(t)$ satisfies equation (6) almost everywhere.

According to (8), now we show that $v(t) \in W_2^{m+n}(R;H)$. By using Parseval's equality, we obtain:

$$
\|v\|_{W_2^{m+n}(R;H)}^2 = \left\|\frac{d^{m+n}v}{dt^{m+n}}\right\|_{L_2(R;H)}^2 + \|A^{m+n}v\|_{L_2(R;H)}^2
$$

$$
= \left\|(i\zeta)^{m+n}\widetilde{v}(\zeta)\right\|_{L_2(R;H)}^2 + \left\|A^{m+n}\widetilde{v}(\zeta)\right\|_{L_2(R;H)}^2
$$

$$
= \left\|(i\zeta)^{m+n}\left(i\zeta E + \frac{\alpha}{2}E - A\right)^{-m}\left(i\zeta E + \frac{\alpha}{2}E + A\right)^{-n}\widetilde{g}(\zeta)\right\|_{L_2(R;H)}^2
$$

$$
+ \left\|A^{m+n}\left(i\zeta E + \frac{\alpha}{2}E - A\right)^{-m}\left(i\zeta E + \frac{\alpha}{2}E + A\right)^{-n}\widetilde{g}(\zeta)\right\|_{L_2(R;H)}^2
$$

(10)

$$
\leq \sup_{\zeta \in R}\left\|i\zeta^{m+n}\left(i\zeta \mathrm{E} + \frac{\alpha}{2}E - A\right)^{-m}\left(i\zeta \mathrm{E} + \frac{\alpha}{2}E + A\right)^{-n}\right\|_{H \to H}^2
$$

$$
\times \left\|\widetilde{g}(\zeta)\right\|_{L_2(R;H)}^2
$$

$$
+ \sup_{\zeta \in R}\left\|A^{m+n}\left(i\zeta \mathrm{E} + \frac{\alpha}{2}E - A\right)^{-m}\left(i\zeta \mathrm{E} + \frac{\alpha}{2}E + A\right)^{-n}\right\|_{H \to H}^2
$$

$$
\times \left\|\widetilde{g}(\zeta)\right\|_{L_2(R;H)}^2.
$$

As $\sigma(A)$ is a spectrum of the operator A, then:

$$\sup_{\zeta \in R} \left\| (i\zeta)^{m+n} \left(i\zeta \mathrm{E} + \tfrac{\alpha}{2}E - A \right)^{-m} \left(i\zeta \mathrm{E} + \tfrac{\alpha}{2}E + A \right)^{-n} \right\|_{H \to H}$$

$$\leq \sup_{\zeta \in R} \sup_{\sigma \in \sigma(A)} \left| (i\zeta)^{m+n} \left(i\zeta + \tfrac{\alpha}{2} - \sigma \right)^{-m} \left(i\zeta + \tfrac{\alpha}{2} + \sigma \right)^{-n} \right| \tag{11}$$

$$= \sup_{\zeta \in R} \frac{|\zeta|^{m+n}}{\left(\zeta^2 + (\sigma + \tfrac{\alpha}{2})^2 \right)^{\frac{m+n}{2}}} \leq 1,$$

$$\sup_{\zeta \in R} \left\| A^{m+n} \left(i\zeta \mathrm{E} + \tfrac{\alpha}{2}E - A \right)^{-m} \left(i\zeta \mathrm{E} + \tfrac{\alpha}{2}E + A \right)^{-n} \right\|_{H \to H}$$

$$\leq \sup_{\zeta \in R} \sup_{\sigma \in \sigma(A)} \left| \sigma^{m+n} \left(i\zeta + \tfrac{\alpha}{2} - \sigma \right)^{-m} \left(i\zeta + \tfrac{\alpha}{2} + \sigma \right)^{-n} \right| \tag{12}$$

$$= \sup_{\sigma \in \sigma(A)} \frac{\sigma^{m+n}}{\left(\zeta^2 + (\sigma + \tfrac{\alpha}{2})^2 \right)^{\frac{m+n}{2}}} \leq 1.$$

From (12) and (11) into (10) we obtain:

$$\|v\|_{W_2^{m+n}(R;H)}^2 \leq 2 \left\| \widetilde{g}(\zeta) \right\|_{L_2(R;H)}^2 = 2 \|g(t)\|_{L_2(R;H)}^2, \tag{13}$$

where $\widetilde{v}(\zeta)$ and $\widetilde{g}(\zeta)$ are the Fourier transforms of the functions $v(t)$ and $g(t)$, respectively. Then $v(t) \in W_2^{m+n}(R;H)$. Similarly we can prove the boundedness of the operator $P_{0,\alpha}$, hence

$$\|P_{0,\alpha}v\|_{L_2(R;H)}^2 \leq const \, \|v\|_{W_2^{m+n}(R;H)}^2.$$

Using the Banach theorem on the inverse operator (see [10], [13]), there exists a bounded inverse operator $P_{0,\alpha}^{-1}$: $L_2(R;H)$ $\to W_2^{m+n}(R;H)$. Then the theorem is proved. $\square$

Remark. The operators of intermediate derivatives:

$$A^j \frac{d^{n+m-j}u}{dt^{n+m-j}} : W_{2,\alpha}^{n+m}(R;H) \to L_{2,\alpha}(R;H), \quad j = \overline{0, m+n},$$

are continuous.

Now we estimate the norms of intermediate derivative operators participating in the main part of the equation (1) for finding the exact conditions on regular solvability of the investigated equation, expressed only by its operator coefficients.

From Theorem 2, we have that the norms $\|P_0 u\|_{L_{2,\alpha}(R;H)}$ and $\|u\|_{W_{2,\alpha}^{m+n}(R;H)}$ are equivalent in the space $W_{2,\alpha}^{m+n}(R;H)$, then we can estimate the norms of intermediate derivatives operators by the norm $\|P_0 u\|_{L_{2,\alpha}(R;H)}$ (see [14]).

Theorem 3. *The operator $P_1 : W_{2,\alpha}^{n+m}(R; H) \to L_{2,\alpha}(R; H)$ is continuous, provided that the operators $A_j A^{-j}$, $j = \overline{0, m+n}$, are bounded in H (see [2]-[4]).*

Proof. Since $u(t) \in W_{2,\alpha}^{n+m}(R; H)$ then from the theorem on intermediate derivatives (see [14]), we have

$$\|P_1 u\|_{L_{2,\alpha}} \leq \sum_{j=0}^{n+m} \|A_j A^{-j}\|_{H \to H} \left\| A^j \frac{d^{n+m-j} u}{dt^{n+m-j}} \right\|_{L_{2,\alpha}}$$

$$\leq \text{const} \|u\|_{W_{2,\alpha}^{n+m}(R;H)}.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It follows by Theorem 2 and Theorem 3 that the following lemma is true:

Lemma 4. *Consider that the operators $A_j A^{-j}, j = \overline{0, n+m}$, are bounded on H. Then the operator P, in case $A_j \neq 0$ acting from the space $W_{2,\alpha}^{n+m}(R; H)$ to $L_{2,\alpha}$ be bounded (see [15], [16]).*

Theorem 5. *Let $|\alpha| < 2\sigma_0$. Then for any $u(t) \in W_{2,\alpha}^{m+n}(R; H)$, the following inequalities hold:*

$$\left\| A^{m+n-j} \frac{d^j u(t)}{dt^j} \right\|_{L_{2,\alpha}} \leq b_j \|P_0 u\|_{L_{2,\alpha}}, \tag{14}$$

where

$$
b_j = \begin{cases}
\frac{1}{(m+n)^{\frac{m+n}{2}}} j^{\frac{j}{2}} (m+n-j)^{\frac{m+n-j}{2}}, & j = \overline{1, m+n-1}, \, m \neq n, \\
\left(\frac{m+n-j}{m+n}\right)^{\frac{m+n-j}{2(m+n)}} \left(\frac{j}{m+n}\right)^{\frac{j}{2(m+n)}}, & j = \overline{1, m+n-1}, \, m = n, \\
1, & j = 0, m+n.
\end{cases}
$$

Proof. Apply the Fourier transformation on equation (6) (see [1]-[3]), we get

$$\left\| A^{m+n-j} (i\zeta)^j \left(i\zeta \mathrm{E} + \frac{\alpha}{2} E - A \right)^{-m} \left(i\zeta \mathrm{E} + \frac{\alpha}{2} E + A \right)^{-n} \widetilde{g}(\zeta) \right\|_{L_2(R;H)}$$

$$\leq \sup_{\zeta \in R} \left\| A^{m+n-j} (i\zeta)^j \left(i\zeta \mathrm{E} + \frac{\alpha}{2} E - A \right)^{-m} \left(i\zeta \mathrm{E} + \frac{\alpha}{2} E + A \right)^{-n} \right\|_{H \to H} \tag{15}$$

$$\times \left\| \widetilde{g}(\zeta) \right\|_{L_2(R;H)}.$$

For $\zeta \in R$, we estimate the following norms:

$$\left\| A^{m+n-j} (i\zeta)^j \left(i\zeta\mathrm{E} + \frac{\alpha}{2}E - A \right)^{-m} \left(i\zeta\mathrm{E} + \frac{\alpha}{2}E + A \right)^{-n} \right\|_{H \to H}$$

$$\leq \sup_{\sigma \in \sigma(A)} \left| \sigma^{m+n-j} (i\zeta)^j \left(i\zeta + \frac{\alpha}{2} - \sigma \right)^{-m} \left(i\zeta + \frac{\alpha}{2} + \sigma \right)^{-n} \right|$$

$$= \sup_{\sigma \in \sigma(A)} \left| \sigma^{-j} (i\zeta)^j \left(i\frac{\zeta}{\sigma} - (-\frac{\alpha}{2\sigma} + 1) \right)^{-m} \left(i\frac{\zeta}{\sigma} + (\frac{\alpha}{2\sigma} + 1) \right)^{-n} \right| \qquad (16)$$

$$\leq \sup_{\mu = \frac{\zeta^2}{\sigma^2} \geq 0} \frac{\mu^{j/2}}{(\mu + 1)^{(m+n)/2}} = b_j.$$

Finally, from (15) we have

$$\left\| A^{m+n} (i\zeta)^j (i\mathrm{E} - A)^{-m} (i\zeta\mathrm{E} + A)^{-n} \widetilde{f}(\zeta) \right\|_{L_{2,\alpha}} \leq b_j \left\| \widetilde{f}(\zeta) \right\|_{L_{2,\alpha}}. \qquad (17)$$

Now, we introduce the following specific cases at certain values of m and n:

Case (i) $m = 3$, $n = 0$, then we have an initial-boundary value problem of a third order operator-differential equation with multiple characteristics with $b_1 = b_2 = \frac{2}{3\sqrt{3}}$ (see [23]).

Case (ii) $m = 0$, $n = 3$, then we have an initial-boundary value problem of a fourth order operator-differential equation with multiple characteristics with $b_1 = b_2 = \frac{2}{3\sqrt{3}}$ (see [17]).

Case (iii) $m = 3$, $n = 1$, then we have an initial-boundary value problem of a fourth order operator-differential equation with multiple characteristics with $b_1 = b_3 = \frac{3\sqrt{3}}{16}$, $b_2 = \frac{1}{4}$.

Case (iv) $m = 1$, $n = 3$, then we have an initial-boundary value problem of a fourth order operator-differential equation with complicated characteristics with $b_1 = b_3 = \frac{3\sqrt{3}}{16}$, $b_2 = \frac{1}{4}$ (see [7]). $\qquad\Box$

Now in $H_{2(m+n)}$ we consider the following pencil operator

$$Q_j(\lambda, \beta, A) = \begin{cases} \left((i\lambda)^2 E + A^2 \right)^{m+n} - \beta(i\lambda)^{2j} A^{2(m+n)-2j}, & m \neq n, \\ \lambda^{2(m+n)} E + A^{2(m+n)} - \beta(i\lambda)^{2j} A^{2(m+n)-2j}, & m = n, \end{cases} \qquad (18)$$

β is a real parameter. From (18) we have

$$B_{m+n,j} = \begin{cases} (m+n)^{m+n} j^{-j} \left(\frac{1}{m+n-j}\right)^{m+n-j}, & m \neq n, j = \overline{1, m+n-1}, \\ \left(\frac{m+n}{m+n-j}\right)^{\frac{m+n-j}{(m+n)}} \left(\frac{m+n}{j}\right)^{\frac{j}{(m+n)}}, & m = n, j = \overline{1, m+n-1}, \\ 1, & j = 0, m+n. \end{cases} \tag{19}$$

Theorem 6. For $\beta \in [0, B_{m+n,j}), j = \overline{0, m+n}$, the operator pencils $Q_j(\lambda, \beta, A)$ can be represented in the form

$$Q_j(\lambda, \beta, A) = Q_j^-(\lambda, \beta, A)Q_j^+(-\lambda, \beta, A), \ j = \overline{0, m+n}, \tag{20}$$

where

$$Q_j^-(\lambda, \beta, A) = \sum_{i=0}^{m+n} \alpha_{i,j} \lambda^i A^{m+n-i},$$

$$Q_j^+(-\lambda, \beta, A) = \sum_{i=0}^{m+n} \alpha_{i,j}(-\lambda)^i A^{m+n-i},$$

and satisfy the following system of equations:

$$Q_j(\lambda, \beta, A) = \left(\sum_{i=0}^{m+n} \alpha_{i,j} \lambda^i A^{m+n-i}\right)\left(\sum_{i=0}^{m+n} \alpha_{i,j}(-\lambda)^i A^{m+n-i}\right). \tag{21}$$

Proof. For $\beta \in [0, B_{m+n,j}), j = \overline{0, m+n}$, the polynomial

$$Q_j(\lambda, \beta) = \begin{cases} ((i\lambda)^2 + 1)^{m+n} - \beta(i\lambda)^{2j}, & m \neq n, \\ \lambda^{2(m+n)} + 1 - \beta(i\lambda)^{2j}, & m = n, \end{cases}$$

has not purely imaginary roots, its roots are simple and symmetrically situated relatively to the real axis and the origin. So it can be represented in the form

$$Q_j(\lambda, \beta) = Q_j^-(\lambda, \beta) Q_j^+(-\lambda, \beta),$$

$$Q_j^-(\lambda, \beta) = \prod_{i=1}^{m+n} (\lambda - \omega_{i,j}(\beta)),$$

where $\omega_{i,j}(\beta)$ are the roots of $Q_j^-(\lambda, \beta)$, $\operatorname{Re}\omega_{i,j}(\beta) < 0$,
hence

$$Q_j^-(\lambda, \beta) = \sum_{i=1}^{m+n} \alpha_{i,j}(\beta)\lambda^i, \ \alpha_{i,j} > 0 \text{ are real coefficients.}$$

Let E_σ be the spectral decomposition of A.

For the case $m \neq n$:

$$Q_j(\lambda, \beta, A) = \left((i\lambda)^2 E + A^2\right)^{m+n} - \beta(i\lambda)^{2j} A^{2(m+n)-2j}$$

$$= \int_{\sigma^0}^{\infty} \left((i\lambda)^2 + \sigma^2\right)^{m+n} - \beta(i\lambda)^{2j} \sigma^{2(m+n)-2j} dE_\sigma$$

$$= \int_{\sigma^0}^{\infty} \sigma^{2(m+n)} \left[\left(\frac{(i\lambda)^2}{\sigma^2} + 1\right)^{m+n} - \beta\frac{(i\lambda)^{2j}}{\sigma^{2j}}\right] dE_\sigma$$

$$= \int_{\sigma^0}^{\infty} \sigma^{2(m+n)} Q_j\left(\frac{\lambda}{\sigma}, \beta\right) dE_\sigma$$

$$= \int_{\sigma^0}^{\infty} \sigma^{2(m+n)} Q_j^-\left(\frac{\lambda}{\sigma}, \beta\right) Q_j^+\left(-\frac{\lambda}{\sigma}, \beta\right) dE_\sigma$$

$$= \int_{\sigma^0}^{\infty} \sigma^{(m+n)} Q_j^-\left(\frac{\lambda}{\sigma}, \beta\right) dE_\sigma \int_{\sigma^0}^{\infty} \sigma^{(m+n)} Q_j^+\left(-\frac{\lambda}{\sigma}, \beta\right) dE_\sigma$$

$$= \left(\sum_{i=0}^{m+n} \alpha_{i,j}(\beta)\lambda^i A^{m+n-i}\right) \left(\sum_{i=0}^{m+n} \alpha_{i,j}(\beta)(-\lambda)^i A^{m+n-i}\right).$$

By taking

$$Q_j^-(\lambda, \beta, A) = \sum_{i=o}^{m+n} \alpha_{i,j}(\beta)\lambda^i A^{m+n-i},$$

we get

$$Q_j(\lambda, \beta, A) = Q_j^-(\lambda, \beta, A) Q_j^+(-\lambda, \beta, A).$$

Similarly for the case $m = n$.

For example,

(i) If $m = 1, n = 3$ satisfy the following system of equations:

(1) for $j = 0$

$$\begin{cases} \alpha_{0,0}(\beta) = \sqrt{1 - \beta}, \\ 2\alpha_{2,0}(\beta) - \alpha_{3,0}^2(\beta) + 4 = 0, \\ 2\alpha_{0,0}(\beta)\alpha_{4,0}(\beta) + \alpha_{2,0}^2(\beta) - 2\alpha_{1,0}(\beta)\alpha_{3,0}(\beta) - 6 = 0, \\ 2\alpha_{0,0}(\beta)\alpha_{2,0}(\beta) - \alpha_{1,0}^2(\beta) + 4 = 0; \end{cases}$$

(2) for $j = 1$

$$\begin{cases} 2\alpha_{2,1}(\beta) - \alpha_{3,1}^2(\beta) + 4 = 0, \\ 2\alpha_{0,1}(\beta)\alpha_{4,1}(\beta) + \alpha_{2,1}^2(\beta) - 2\alpha_{1,1}(\beta)\alpha_{3,1}(\beta) - 6 = 0, \\ 2\alpha_{0,1}(\beta)\alpha_{2,1}(\beta) - \alpha_{1,1}^2(\beta) + 4 = \beta; \end{cases}$$

(3) for $j = 2$

$$\begin{cases} 2\alpha_{2,2}(\beta) - \alpha_{3,2}^2(\beta) + 4 = 0, \\ 2\alpha_{0,2}(\beta)\alpha_{4,2}(\beta) + \alpha_{2,2}^2(\beta) - 2\alpha_{1,2}(\beta)\alpha_{3,2}(\beta) - 6 + \beta = 0, \\ 2\alpha_{0,2}(\beta)\alpha_{2,2}(\beta) - \alpha_{1,2}^2(\beta) + 4 = 0; \end{cases}$$

(4) for $j = 3$

$$\begin{cases} 2\alpha_{2,3}(\beta) - \alpha_{3,3}^2(\beta) + 4 - \beta = 0, \\ 2\alpha_{0,3}(\beta)\alpha_{4,3}(\beta) + \alpha_{2,3}^2(\beta) - 2\alpha_{1,3}(\beta)\alpha_{3,3}(\beta) - 6 = 0, \\ 2\alpha_{0,3}(\beta)\alpha_{2,3}(\beta) - \alpha_{1,3}^2(\beta) + 4 = 0; \end{cases}$$

(5) for $j = 4$

$$\begin{cases} \alpha_{4,4}(\beta) = \sqrt{1 + \beta}, \\ 2\alpha_{2,4}(\beta) - \alpha_{3,4}^2(\beta) + 4 = 0, \\ 2\alpha_{0,4}(\beta)\alpha_{4,4}(\beta) + \alpha_{2,4}^2(\beta) - 2\alpha_{1,4}(\beta)\alpha_{3,4}(\beta) - 6 = 0, \\ 2\alpha_{0,4}(\beta)\alpha_{2,4}(\beta) - \alpha_{1,4}^2(\beta) + 4 = 0; \end{cases}$$

(ii) If $m = 2, n = 2$ satisfy the following system of equations:
(1) for $j = 0$

$$\begin{cases} \alpha_{0,0}(\beta) = \sqrt{1 - \beta}, \\ \alpha_{1,0}^2(\beta) - 2\alpha_{0,0}(\beta)\alpha_{2,0}(\beta) = 0, \\ \alpha_{2,0}^2(\beta) - 2\alpha_{1,0}(\beta)\alpha_{3,0}(\beta) + 2\alpha_{0,0}(\beta) = 0, \\ \alpha_{3,0}^2(\beta) - 2\alpha_{2,0}(\beta) = 0; \end{cases}$$

(2) for $j = 1$

$$\begin{cases} \alpha_{3,1}^2(\beta)^2 - 2\alpha_{2,1}(\beta) = 0, \\ \alpha_{1,1}^2(\beta) - 2\alpha_{2,1}(\beta) + \beta = 0, \\ \alpha_{2,1}^2(\beta) - 2\alpha_{1,1}(\beta)\alpha_{3,1}(\beta) + 2 = 0; \end{cases}$$

(3) for $j = 2$

$$\begin{cases} \alpha_{1,2}^2(\beta) - 2\alpha_{2,2}(\beta) = 0, \\ \alpha_{3,2}^2(\beta) - 2\alpha_{2,2}(\beta) = 0, \\ \alpha_{2,2}^2(\beta) - 2\alpha_{1,2}(\beta)\alpha_{3,2}(\beta) + 2 = -\beta; \end{cases}$$

(4) for $j = 3$

$$\begin{cases} \alpha_{3,3}(\beta) - 2\alpha_{2,3}(\beta) = -\beta, \\ \alpha_{1,3}^2(\beta) - 2\alpha_{2,3}(\beta) = 0, \\ \alpha_{2,3}^2(\beta) - 2\alpha_{1,3}(\beta)\alpha_{3,3}(\beta) + 2 = 0; \end{cases}$$

(5) for $j = 4$

$$
\begin{cases}
\alpha_{4,4}(\beta) = \sqrt{1-\beta}, \\
\alpha_{1,4}^2(\beta) - 2\alpha_{0,4}(\beta)\alpha_{2,4}(\beta) = 0, \\
\alpha_{2,4}^2(\beta) - 2\alpha_{1,4}(\beta)\alpha_{3,4}(\beta) + 2\alpha_{4,4}(\beta) = 0, \\
\alpha_{3,4}^2(\beta) - 2\alpha_{2,4}(\beta)\alpha_{4,4}(\beta) = 0.
\end{cases}
,
$$

$\square$

Lemma 7. *Let* $\beta \in [0, B_{m+n,j})$, *then for any* $u \in W_{2,\alpha}^{m+n}(R;H)$

$$
\left\| Q_j^- \left(\frac{d}{dt}; \beta; A \right) u \right\|_{L_2(R;H)}^2
$$

$$
= \| P_0 u \|_{L_{2,\alpha}}^{2,\alpha} - \beta \left\| A^{m+n-j} \frac{d^j}{dt^j} u \right\|_{L_{2,\alpha}}^2 , \quad j = \overline{0, m+n}. \tag{22}
$$

From Theorem 5 and the theorem of intermediate derivatives (see [18]) the norms $\| u \|_{W_{2,\alpha}^{m+n}(R;H)}$ and $\| P_0 u \|_{L_2(R;H)}$ are equivalent in the space $W_{2,\alpha}^{m+n}(R;H)$. Therefore, the numbers

$$
N_j = \sup_{0 \neq u \in W_{2,\alpha}^{m+n}(R;H)} \frac{\left\| A^{m+n-j} \frac{d^j}{dt^j} u \right\|_{L_{2,\alpha}}}{\| P_0 u \|_{L_{2,\alpha}}}, \quad j = \overline{0, m+n},
$$

are finite numbers. To find exact values of these numbers, we provide the following lemma:

Theorem 8. *The numbers* N_j *are determined as follows:*

$$
N_j = b_j, \; j = \overline{0, m+n}.
$$

Proof. As (22) goes to the limit as $\beta \to B_{m+n,j}$, it is clear that for any vector function $u(t) \in W_{2,\alpha}^{m+n}(R;H)$:

$$
\| P_0 u \|_{L_{2,\alpha}}^2 \geq B_{m+n,j} \left\| A^{m+n-j} \frac{d^j}{dt^j} u \right\|_{L_{2,\alpha}}^2 , \quad j = \overline{0, m+n}. \tag{23}
$$

So, $N_j \leq B_{m+n,j}, j = \overline{0, m+n}$. Moreover, we must show that $N_j = B_{m+n,j}, j = \overline{0, m+n}$ is also hold. To do this, it is sufficient for any $\delta > 0$, there exist a vector function $u_\delta(t) \in W_{2,\alpha}^{m+n}(R;H)$ such that the functional

$$
\chi(u_\delta(t)) \equiv \| P_0 u_\delta \|_{L_{2,\alpha}}^2 - (\delta + B_{m+n,j}) \left\| A^{m+n-j} \frac{d^j}{dt^j} u_\delta \right\|_{L_{2,\alpha}}^2 < 0. \tag{24}
$$

Let the vector $\nu \in D(A^{m+n})$ such that $\|\nu\| = 1, r(t) \in W_{2,\alpha}^{m+n}(R; H)$ be scalar function. Using Parseval's equality, we obtain

$$\chi(r(t)\nu) = \|P_0(r(t)\nu)\|_{L_{2,\alpha}}^2 - (\delta + B_{m+n,j}) \left\| A^{m+n-j} \frac{d^j r(t)}{dt^j} \nu \right\|_{L_{2,\alpha}}^2$$

$$= \int_{-\infty}^{+\infty} \Big((P_0(-i\zeta; A)\nu, P_0(-i\zeta; A)\nu) |\tilde{r}(\zeta)|^2$$

$$- \zeta^{2j}(\delta + B_{m+n,j})(A^{m+n-j}\nu, A^{m+n-j}\nu) |\tilde{r}(\zeta)|^2 \Big) d\zeta \qquad (25)$$

$$= \int_{-\infty}^{+\infty} \Big(P_0(-i\zeta; A)P_0(-i\zeta; A)\nu$$

$$- \zeta^{2j}(\delta + B_{m+n,j})A^{2(m+n-j)}\nu, \nu) |\tilde{r}(\zeta)|^2 \Big) d\zeta$$

$$= \int_{-\infty}^{+\infty} (P_j(i\zeta; \delta + B_{m+n,j}; A)\nu, \nu) |\tilde{r}(\zeta)|^2 d\zeta,$$

where $\tilde{r}(\zeta)$ is the Fourier transform of $r(t)$. Then it is necessary to show that $(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu, \nu)$ for a given vector ν has negative values in some interval (ϵ^-, ϵ^+). For $\sigma_0 > 0$ is an eigenvalue of A, and ζ is its corresponding eigenvector, then

$$(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu, \nu) = (P_j(i\zeta; \delta + B_{m+n,j}; \sigma_0)\nu, \nu).$$

From the properties of the polynomial $P_j(i\zeta; \beta; \sigma_0)$, it is negative for $\beta = \delta + B_{m+n,j}$ for sufficiently small $\delta > 0$. If $\sigma_0 \in \sigma(A)$ is not an eigenvalue, then σ_0 is close to an eigenvalue, i.e. there exist a vector ν_δ such that $\|\nu_\delta\| = 1$ and

$$(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu_\delta, \nu_\delta) = (P_j(i\zeta; \delta + B_{m+n,j}; \sigma_0)\nu_\delta, \nu_\delta) + 0(\delta)$$

$$\text{as} \quad \delta \to 0.$$

In which, the smallest value of $(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu, \nu))$ is negative for sufficiently small δ and some ν_δ. Then there exist an interval (ϵ^-, ϵ^+) such that

$$(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu, \nu)) < \delta, \quad \zeta \in (\epsilon^-, \epsilon^+).$$

Further, we consider the $m + n$ times differentiable function $\tilde{r}(\zeta), \zeta \in (\epsilon^-, \epsilon^+)$, then from the negativity of $(P_j(i\zeta; \delta + B_{m+n,j}; A)\nu_\delta, \nu_\delta))$ in the interval (ϵ^-, ϵ^+) and from (25), we obtain

$$\chi(r(t)\nu_\delta) = \int_{\epsilon^-}^{\epsilon^+} (P_j(i\zeta; \delta + B_{m+n,j}; A)\nu_\delta, \nu_\delta) |\tilde{r}(\zeta)|^2 d\zeta < 0.$$

Consequently, $N_j = b_j, j = \overline{0, m + n}$. $\qquad \square$

Theorem 9. Let $A = A^* \geq \sigma_0 E \ (\sigma_0 > 0), |\alpha| < 2\sigma_0$, the operators $A_j A^{-j}, \ j = \overline{0, m+n}$ are bounded in H and holds the inequality

$$\sum_{j=0}^{m+n} b_j \left\| A_{m+n-j} A^{-(m+n-j)} \right\|_{H \to H} < 1, \tag{26}$$

in which the numbers $b_j, \ j = \overline{0, m+n}$, are determined in Theorem 5, then the equation (1) is regularly solvable.

Proof. From Theorem 2, the operator P_0 has a bounded inverse operator P_0^{-1} acting from $L_{2,\alpha}(R; H)$ to $W_{2,\alpha}^{m+n}(R; H)$. Then equation (1) can be written as

$$\left(E + P_1 P_0^{-1} \right) z(t) = f(t),$$

where $P_0 u(t) = z(t)$. To prove the existence of a solution, we must show that the norm

$$\left\| p_1 p_0^{-1} \right\|_{L_{2,\alpha}(R;H) \to L_{2,\alpha}(R;H)} < 1.$$

By Theorem 2, we have

$$\left\| P_1 P_0^{-1} z \right\|_{L_{2,\alpha}} = \left\| P_1 u \right\|_{L_{2,\alpha}} \leq \sum_{j=0}^{m+n} \left\| A_j \frac{d^{m+n-j} u}{dt^{m+n-j}} \right\|_{L_{2,\alpha}}$$

$$\leq \sum_{j=0}^{m+n} \left\| A_j A^{-j} \right\|_{H \to H} \left\| A^j \frac{d^{m+n-j} u}{dt^{m+n-j}} \right\|_{L_{2,\alpha}}$$

$$\leq \sum_{j=0}^{m+n} b_j \left\| A_j A^{-j} \right\|_{H \to H} \left\| P_0 u \right\|_{L_{2,\alpha}}$$

$$= \sum_{j=0}^{m+n} b_j \left\| A_j A^{-j} \right\|_{H \to H} \left\| z \right\|_{L_{2,\alpha}}.$$

Consequently,

$$\left\| P_1 P_0^{-1} \right\|_{L_{2,\alpha} \to L_{2,\alpha}} \leq \sum_{i=0}^{m+n} b_j \left\| A_j A^{-j} \right\|_{H \to H} < 1.$$

Providing that the operator $E + P_1 P_0^{-1}$ is invertible in $L_2(R; H)$, hence $u(t)$ can be determined by $u(t) = P_0^{-1} \left(E + P_1 P_0^{-1} \right)^{-1} f(t)$. Moreover,

$$\| u \|_{W_{2,\alpha}^{m+n}} \leq \left\| p_0^{-1} \right\|_{L_{2,\alpha} \to W_{2,\alpha}^{m+n}(R;H)}$$

$$\times \left\| \left(\left(E + P_1 P_0^{-1} \right) \right)^{-1} \right\|_{L_2(R;H) \to L_{2,\alpha}} \| f \|_{L_{2,\alpha}(R;H)} \tag{27}$$

$$\leq \text{const} \, \| f \|_{L_{2,\alpha}}.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Consider a polynomial operator pencil (see [8]) of a $m+n$ order in H is

$$P(\lambda) = \prod_{k=1}^{m} (\lambda + \mu_k A) \prod_{k=1}^{n} (\lambda + \mu_k A) + \sum_{j=1}^{m+n-1} A_{m+n-j} \lambda^{(j)}, \tag{28}$$

$$P_0(\lambda) = \prod_{k=1}^{m}(\lambda + \mu_k A)\prod_{k=1}^{n}(\lambda + \mu_k A),$$

$$P_1(\lambda) = \sum_{j=1}^{m+n-1} A_{m+n-j}\lambda^{(j)}.$$

According to Theorem 3, it suffices to establish the following theorem (see [5], [21], [20]).

Theorem 10. *Suppose that the operators $A_j A^{-j}$ are bounded operators in H , $j = \overline{0, m+n-1}$ and the inequality*

$$\sum_{j=0}^{m+n-1} b_j \|A_j A^{-j}\|_{H\to H} < 1,$$

holds true, where $b_j, j = \overline{0, m+n}$ are calculated in Theorem 5, then the resolvent of the pencil (28) exists on the imagery axis and the inequalities

$$\sum_{r=0}^{m+n-1} \|\lambda^{m+n-r}A^r P^{-1}(\lambda)\| \le const, \tag{29}$$

$$\|A^\alpha P^{-1}(\lambda)\| \le const|\lambda|^{\alpha-(m+n)}, \ \ 0 < \alpha < m+n, \ \lambda \ne 0 \tag{30}$$

hold.

Theorem 11. *From Theorem 10, for sufficiently small ϕ (greater than zero) on the sectors*

$$\Gamma_{\frac{\pi}{2}\pm\phi} = \{\lambda : \lambda = re^{i(\frac{\pi}{2}\pm\phi)}, r > 0\},$$

$$\Gamma_{\frac{-\pi}{2}\pm\phi} = \{\lambda : \lambda = re^{-i(\frac{\pi}{2}\pm\phi)}, r > 0\},$$

the operator pencil (28) is invertible and the estimates of the form (29) and (30) hold.

3. Example

As a result of the solvability of the differential equation (1), we introduce the following problem of partial differential equation as an applied example on the strip $R \times [0; \pi]$:

$$\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^n u(t, x)$$

$$+ \sum_{j=0}^{n+m} r_{n+m-j}(x) \frac{\partial^{2(n+m)-j} u(t,x)}{\partial t^j \partial x^{2(n+m-j)}} = f(t,x), \qquad (31)$$

$$\frac{\partial^{2k} u(t,0)}{\partial x^{2k}} = \frac{\partial^{2k} u(t,\pi)}{\partial x^{2k}} = 0, \ k = \overline{0, n+m}, \qquad (32)$$

where $r_{n+m-j}(x), j = \overline{0, n+m}$ are bounded functions on $[0,\pi]$, $f(t,x) \in L_2(R; L_2[0;\pi])$. We note that problem (31), (32) is a special case and can be reduced to the operator-differential equation (1) in which:

$$A_j = r_{n+m-j}(x) \frac{\partial^{2(n+m-j)}}{\partial x^{2(n+m-j)}}, \ \ j = \overline{0, n+m}.$$

The operator A is defined on $H = L_2[0,\pi]$ by $Au = -\frac{d^2 u}{dx^2}$, and the conditions $u\,|_{x=0} = u\,|_{x=\pi} = 0$. Applying Theorem 10, where

$$\sum_{j=0}^{m+n} b_j \sup_{x \in [0,\pi]} |r_j(x)| < 1,$$

then problem (31)-(32) has a unique solution

$$u(x,t) \in W_{t,x,2}^{n+m,2(n+m)}(R; L_2[0;\pi]).$$

4. Conclusion

In a weighted Sobolev space for all $t \in R$, we calculated the exact conditions of regular solvability of equation (1), expressed only by its operator coefficients. We deduced the relationship between the exponent of the weight $(e^{\frac{-\alpha t}{2}})$ and the lower bound (σ_0) of the spectrum of the operator of the main part of the equation. We estimated the norms of intermediate derivative operators participating in the principle part of the given equation. In the perturbed part of equation (1), the norms of the linear operators $A_j(t), j = \overline{0, m+n}$, were estimated. An alternative method on the regular solvability of operator pencil was investigated. As an applied result, we formulated a problem of $2(m+n)$ order partial differential equations.

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