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CALCULATING THERMAL COEFFICIENTS USING A HYBRID METHOD

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Abstract: In this paper we study the problem of determining two thermal parameters of a cylindrical metal sample. This is an inverse problem in heat conduction where boundary conditions are determined on the basis of temperature measurements taken at the selected internal points in the sample. A hybrid method is used to find the parameters based on the experimental data of the temperature of a metalic sample. Both the direct and inverse problems are described and numerical results are given.

AMS Subject Classification: 65P99
Key Words: inverse problem; parameters; temperature; cylindrical coordinates

1. Introduction

Parameters estimations of material samples and inverse problems are commonly used to derive physical models from experiments. To solve the inverse problem, one must first solve the direct problem, then solve the inverse problem for some coefficients and parameters. Solving such a problem therefore requires solving

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an optimization (minimization) problem. These problems have many applications in scientific areas such as heat transfer, geophysics, electromagnetic, astronomy, electrocardiography, elastic waves and acoustics. Some important references on inverse problems can be found in [1, 2, 3, 4, 5]. Theory and application of ill-posed problems and their solutions can be found in the book edited by Bakushinsky and Goncharsky [6].

More mathematically oriented references on inverse problems include [7, 8, 9, 10, 11, 12, 13]. Tomography, particularly in medical imaging and seismology, is a very large field. Some general references on electrocardiography, are [14, 15, 16, 19, 20].

The recent development of theory, methods, and applications of one-dimensional inverse problems of dynamic elasticity can be found in [21, 22, 23, 24].

In [25] a good overview of many computational aspects of the subject, with applications and related areas that provide an entry point to some of the current research in this area. There is a wide research literature in the area of parameter estimation in [23].

Wave propagation problems in environmental applications such as seismic analysis, acoustic and electromagnetic scattering are described in [21] for both forward and inverse problems.

In our thermodynamics model we will study the problem of determining two thermal coefficients from a mixed set of data using a hybrid method. This is an inverse problem where the experimental data need to coincide the numerical solution of the model problem.

2. The model problem

Our model problem consists on a solid right circular cylinder of radius R and height L sitting on a table (see Fig. 1). The steady state temperature is denoted by $u(r, z)$ where we have introduced cylindrical coordinates. The bottom of the cylinder is at $z = 0$, and the top at $z = L$. In what follows, there will be no angular dependence. The origin for r is at the center of the cylinder. A the top, a small circle of the cylinder of radius ρ centred at $r = 0$ is indicated, the significance of which will soon be clear. The bottom is assumed to be insulated so there,

$$\frac{\partial u}{\partial z}(r, 0) = 0.$$

We take the ambient temperature to be zero. k will denote the thermal

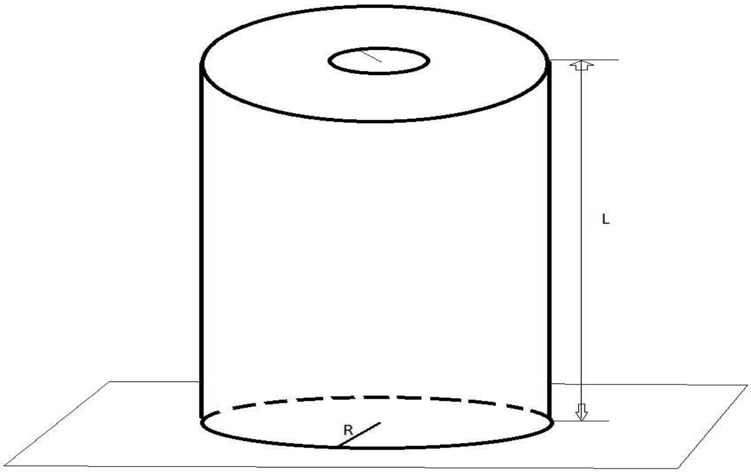

Figure 1: Geometry of the model problem.

conductivity and σ the heat transfer coefficient. We consider the model problem

$$\begin{aligned}\Delta u(r,z) &= \frac{\partial^2 u}{\partial r^2}(r,z)+\frac{1}{r}\frac{\partial u}{\partial r}(r,z)+\frac{\partial^2 u}{\partial z^2}(r,z)=0, \qquad (1)\\ 0 &< r<R,\quad 0<z<L,\\ \frac{\partial u}{\partial z}(r,0) &= 0,\quad 0\le r\le R,\\ \frac{\partial u}{\partial z}(R,z) &= -\gamma u(R,z),\quad \gamma=\frac{\sigma}{k},\quad 0\le z\le L, \qquad (2)\\ u(r,L) &= f(r)=\begin{cases} S, & 0\le r<\rho\\ T, & \rho\le r\le R\end{cases}.\end{aligned}$$

The problem is solved by means of the standard technique of separation of variables:

$$u(r,z)=\varphi(r)\psi(z).$$

We find that $\varphi(r)$ satisfies the equation

$$\begin{aligned}\varphi''(r)+\frac{1}{r}\varphi'(r)+\lambda^2\varphi(r) &= 0,\quad \lambda>0, \qquad (3)\\ \varphi'(R) &= -\gamma\varphi(R).\end{aligned}$$

The solution of (3) is well known as

$$\varphi(r)=J_0(\lambda r),$$

and λ is obtained by solving for μ

$$\mu J_1(\mu) = \gamma R J_0(\mu).$$

Here J_0 and J_1 are Bessel functions. We find

$$0 < \mu_1 < \mu_2 < .... < \mu_n < ...,,\ \mu_n \to \infty \text{ as } \ n \to \infty.$$

Now, $\lambda_n = \mu_n/R$. Then

$$\varphi_n(r) = J_0(\lambda_n r)$$

and

$$\psi_n(z) = \cosh(\lambda_n z).$$

The solution is

$$u(r,z) = \sum_{n=1}^{\infty} \alpha_n \, J_0(\lambda_n r) \, \cosh(\lambda_n z). \tag{4}$$

The α_n are determined from the boundary conditions

$$u(r,L) = f(r)$$

and for that we make use of the orthogonality relations

$$\int_0^R \varphi_m(r)\varphi_n(r) r dr = \begin{cases} 0, & m \neq n \\ \frac{R^2}{2} J_1^2(\lambda_n R)\left[\frac{\gamma^2}{\lambda_n^2}+1\right], & m = n \end{cases}.$$

Let

$$\upsilon_n = \frac{R^2}{2} J_1^2(\lambda_n R)\left[\frac{\gamma^2}{\lambda_n^2}+1\right].$$

Then

$$u(r,L) = f(r) = \sum_{n=1}^{\infty} \alpha_n \varphi_n(r) \cosh(\lambda_n L),$$

and

$$\alpha_n \upsilon_n \cosh(\lambda_n L) = \int_0^R f(r)\varphi_n(r) r dr.$$

So

$$\alpha_n = \frac{1}{\upsilon_n \cosh(\lambda_n L)}\left[S \int_0^{\rho} J_0(\lambda_n r) r dr + T \int_{\rho}^{R} J_0(\lambda_n r) r dr\right]. \tag{5}$$

We now construct a problem that will illustrate the hybrid method. Choose values for k, σ, S, T, all constants, positive, and take $S > T$. This allows us to construct the solution $u(r, z)$. Let

$$g(r) = ku_z(r, L), \quad 0 \le r \le \rho.$$

We assume at this point that $u(r, z)$ is not known but that we have measured values, say $h_1 = u(0, L/3)$ and $h_2 = u(0, 2L/3)$.

The values $T, g(r), h_1$ and h_2 are assumed known and we wish to determine k and σ. The ambient temperature is zero.

The problem as it stands is not amenable to an eigenvalue and eigenfunction approach because $u(r, L)$ is not given for all r but instead a mixed set of data is given at $z = L$. The first step is to solve for arbitrary k, σ Eqns. (1)-(2) to get values of $u(r, z)$ along $z = L$, that is $u(r, L)$. We then use these values to define $g(r)$ and solve numerically the model problem

$$\begin{aligned}
\Delta u(r,z) &= 0 \ \text{ in } \ 0 \le r < R, \ 0 < z < L, && (6)\\
\frac{\partial u}{\partial r}(R,z) &= -\gamma u(R,z), \ \gamma = \sigma/k, \ 0 \le z \le L, &&\\
u_z(r,0) &= 0, \quad 0 \le r \le R, &&\\
ku_z(r,L) &= g(r), \ 0 \le r < \rho, &&\\
u(r,L) &= T \ , \ \rho \le r \le R, && (7)
\end{aligned}$$

at the grid points. Let us call these values v_{ij} at $z = L$, v_{iJ}. This allows us to define a function $f(r)$ on $z = L$ and compare $u(0, L/3)$ with h_1 and $u(0, 2L/3)$ with h_2. The solution is done in an iteration process. We adjust the values for k and σ and repeat the process until that the computed values $u(0, L/3)$ and $u(0, 2L/3)$ coincide with the experimental values h_1 and h_2.

3. Numerical solution

To set up the finite difference method, we subdivide the interval $[0, L]$ into m intervals each of width l such that

$$z_j = jl, \quad j = 1, 2, ..., m-1, \quad l = L/m.$$

Let h be the radius-step size with $r_i = ih, \ i = 1, 2, ..., n, \ h = R/n$.

The differential equation (1) is discretized at (r_i, z_j) using the central difference in the z and r-directions, giving

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{1}{ih}\frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{l^2} = 0,$$
$$i = 1, 2, ..., n, \quad j = 0, 1, ..., m.$$

Rewrite to get

$$u_{i,j-1} + \lambda^2 u_{i-1,j}(1 - \tfrac{1}{2i}) - 2u_{i,j}(\lambda^2 + 1) + \lambda^2 u_{i+1,j}(1 + \tfrac{1}{2i}) + u_{i,j+1} = 0,$$
$$i = 1, 2, ..., n, \quad j = 0, 1, ..., m,$$

where $u_{i,j} = u(ih, jl)$ and $\lambda = l/h$. This is a five-point difference formula.

For the boundary conditions at $r = R$, and $z = 0$, we have

$$\begin{aligned} z &= 0, \quad u_{i,1} = u_{i,-1}, \\ u_{i,m+1} &= u_{i,m} + \frac{1}{k} g(ih),\ ih < \rho, \\ u_{i,m} &= T,\ \ ih > \rho, \\ u_{n+1,j} &= u_{n,j}(1 - \gamma h), \\ u_{1,j} &= u_{0,j}, \quad r = 0. \end{aligned}$$

4. Numerical results

In this section we will show some numerical results that determine the values of k and σ and therefore $u(x, t)$. For the infinite series in equations (4) we took 50 terms to guarantee the convergence of the series.

In this example we consider an unknown cylindrical metal material with height $L = 5$ cm, radius $R = 4$ cm. We choose $S = 100$, and $T = 80$. The results of the experiment are shown in Fig. 2 with the measured values of $h_1 = u(0, L/3) = 49.013$ and $h_2 = u(0, 2L/3) = 62.122$ which was done at Zayed University. We want to determine the values of k and σ of the sample. First we choose arbitrary values for $k = 4.0$ and $\sigma = 2.0$ that is $\gamma = 0.5$ and use 4 to get the values of the temperature at $u(r, L)$ which will be used to define $g(z)$. We then solve the model problem (6) numerically to obtain the values of the temperature at $u(0, L/3)$ and $u(0, 2L/3)$.

We compare these values with the experimental values h_1 and h_2. We continue the iteration process by using the new value for σ obtained from the condition

$$\frac{\partial u}{\partial r}(R, z) = -\gamma u(R, z), \quad \gamma = \sigma/k, \quad 0 \le z \le L,$$

until h_1 and h_2 coincide with $u(0, L/3)$ and $u(0, 2L/3)$. Table 1 shows the values of γ obtained during the iteration process.

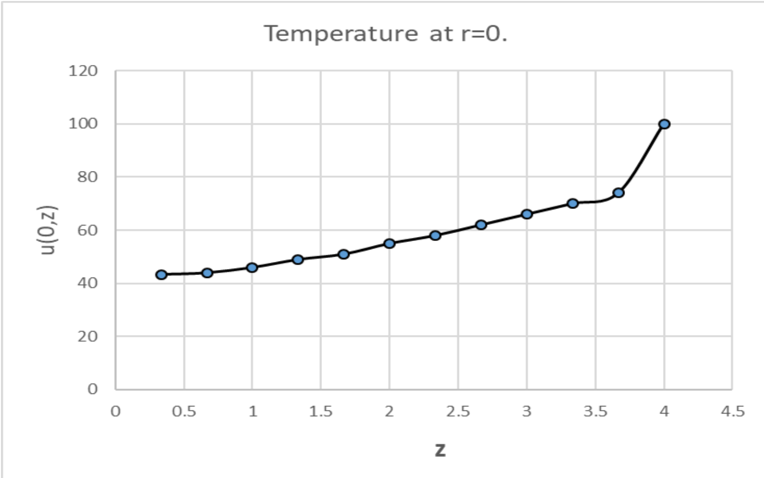

Figure 2: Experimental results.

Iterations	γ
Starting value	0.5
1	0.3380
2	0.2724
3	0.2529

Table 1

After 3 iterations we get convergence with $\gamma = 0.2529$, that gives $k = 0.6$ and $\sigma = 2.370$. These values are in good agreement with the actual parameters of the sample. Fig. 3 shows the numerical and experimental results along $r = 0$.

Nomenclature

σ = Heat transfer coefficient (W-s/Kg-K).
k = Thermal conductivity (W/cm-K).

5. Conclusion

This paper deals with the determination of two thermal coefficients of a metallic sample. Using a hybrid method and a set of experimental data, the parameters

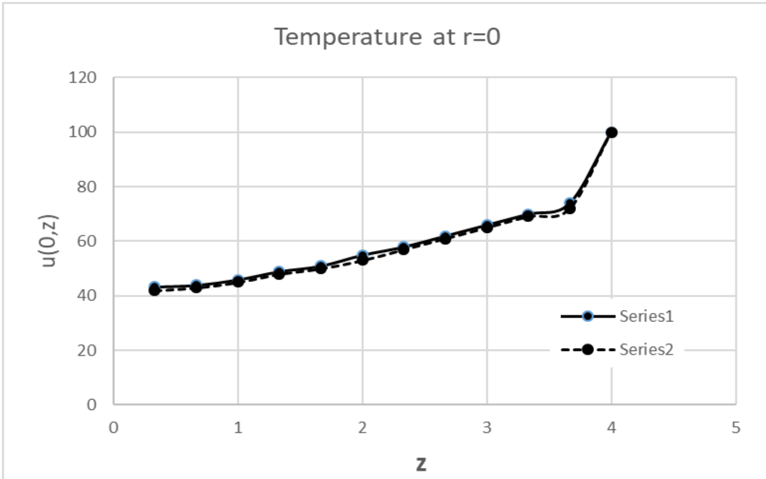

Figure 3: Experimental (series1) and numerical (series2) results of $u(x,t)$ at $r = 0$.

were obtained by solving an inverse problem. The present study shows that the values of the thermal coefficients of the sample obtained from the numerical results were in good agreement with experimental data. The model problem was presented and numerical results were given.

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References

[1] H.E. Kunze, D. La Torre, F. Mendivil, M.R. Galan and R. Zaki, Fractal-based methods and inverse problems for differential equations: current state-of-the-Art, *Mathematical Problems in Engineering*, Article ID 737694 (2014), 11 pages.

[2] Li Peng and al., Reconstruction of shredded paper documents by feature matching, *Mathematical Problems in Engineering*, Article ID 514748 (2014), 9 pages.

[3] G. Evensen, *Data Assimilation*, The Ensemble Kalman Filte, Springer (2006).

[4] M. Bertero and P. Boccacci, *Introduction to Inverse Problems in Imaging*, Institute of Physics, London (1998).

[5] A. Kharab, F. Howari and R.B. Guenther, Calculating three thermal coefficients from one data set, *Int. J. of Appl. Math.*, **32**, No 1 (2019), 91–100; doi: 10.12732/ijam.v32i1.9.

[6] A. Bakushinsky and A. Goncharsky, *Ill-Posed Problems: Theory and Applications*, Springer Science & Business Media, Mathematics and its Appl. (2012).

[7] D. Colton, R. Ewing and W. Rundell (Eds.), *Inverse Problems in Partial Differential Equations*, SIAM, Philadelphia (1990).

[8] G. Berryman, Analysis of approximate inverses in tomography. I. Resolution analysis, *Optimization and Engineering*, **1**, No 1 (2000), 87–115.

[9] G. De Carvalho and A.J. Silva Neto, An inverse analysis for polymers thermal properties estimation, *Proc. 3rd International Conference on Inverse Problems in Engineering: Theory and Practice*, Port Ludlow, WA (1999).

[10] H.W. Engl, A.K. Louis and W. Rundell (Editors), *Inverse Problems in Geophysical Applications*, SIAM, Philadelphia (1996).

[11] H.W. Engl and J.R. McLaughlin (Eds.), *Inverse Problems and Optimal Design in Industry*, B.G. Teubner, Stuttgart (1994).

[12] H.W. Engl and W. Rundell (Eds.), *Inverse Problems in Diffusion Processes*, SIAM, Philadelphia (1995).

[13] G. Gripenberg and S.O. Londen, *Volterra Iintegral and Functional Equations*, Cambridge University Press, Cambridge-New York (1990).

[14] F. Greensite, A new treatment of the inverse problem of multivariate analysis, *Inverse Problems*, **18** (2002).

[15] R.M. Gulrajani, P. Savard and F.A. Roberge, The inverse problem in electrocardiography: Solutions in terms of equivalent sources. *CRC Crit. Rev. Biomed. Eng.*, **16** (1988), 171–214.

[16] R.M. Gulrajani, F.A. Roberge and P. Savard, The Inverse problem of electrocardiography, In: P.W. Macfarlane and T.D. Lawrie Veitch (Eds.): *Comprehensive Electrocardiology.* Pergamon Press, Oxford, (1989), 237–288.

[17] B.M. Horacek, The forward and inverse problem of electrocardiography, In: *Biomedical and Life Physics, Biomedical and Life Physic*s, Vieweg-Verlag, Braunschweig (1996), 169–172.

[18] F. Greensite, The mathematical basis for imaging cardiac electrical function, *CRC Crit. Rev. Biomed. Eng.*, **22** (1994), 347–399.

[19] Y. Rudy, *The electrocardiogram and its relationship to excitation of the heart*, In: N. Sperelakis (Ed.), *Physiology and Pathophysiology of the Heart*, 3rd Ed., Kluwer Academic Publishers Group, Chapter 11 (1995), 201–239.

[20] S.I. Kabanikhin, *Inverse and Ill-Posed Problems: Theory and Applications of Inverse and Ill-posed Problems Series*, Walter de Gruyter, Berlin (2012).

[21] A.G. Megrabov, *Forward and Inverse Problems for Hyperboliic, Elliptic, and Mixed Type Equations*, Inverse and Ill-Posed Problems Series, VSP, Utrecht (2003).

[22] A.G. Ramm, *Inverse Problems: Mathematical and Analytical Techniques with Applications to Engineering*, Springer, New York, NY (2005).

[23] H.T. Banks and H.T. Tran, *Mathematical and Experimental Modeling of Physical and Biological Processes*, CRC Press, Boca Raton, FL (2009).

[24] I. Graham, U. Langer, J. Melenk and M. Sini, *Direct and Inverse Problems in Wave Propagation and Applications*, Radon Ser. on Computational and Applied Mathematics (2011).

[25] C.A. Richard, B. Borchers, C.H. Thurber, *Parameter Estimation and Inverse Problems*, Academic Press Inc., 2nd Ed. (2013).