International Journal of Applied Mathematics

Volume 34 No. 1 2021, 93-108
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v34i1.4

A NONLOCAL $p(x)\&q(x)$ ELLIPTIC TRANSMISSION PROBLEM WITH DEPENDENCE ON THE GRADIENT

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Abstract: In this work, we consider a nonlocal $p\&q$ elliptic transmission problem involving nonstandard growth conditions with dependence on the gradient. Under suitable conditions, we prove the existence of weak solutions by means of an abstract result of the monotone operator theory.

AMS Subject Classification: 35D05, 35J60, 35J70
Key Words: $p(x)$-Laplacian; variable exponent Sobolev space; transmission problem

1. Introduction

We are concerned with the existence of weak solutions to the following system

Received: July 20, 2020 © 2021 Academic Publications

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of nonlinear elliptic equations

$$
\begin{aligned}
&-M_1(u_1)\ \operatorname{div}(a(|\nabla u_1|^{p(x)})|\nabla u_1|^{p(x)-2}\nabla u_1) = f(x,u_1,\nabla u_1)|u_1|^{t(x)}_{s(x)} \\
&\hspace{10cm} \text{in} \quad \Omega_1, \\
&-M_2(u_2)\operatorname{div}(a(|\nabla u_2|^{p(x)})|\nabla u_2|^{p(x)}dx)\|\nabla u_2|^{p(x)-2} = |u_2|^{\beta(x)-2}u_2 \\
&\hspace{10cm} \text{in } \Omega_2, \\
&\qquad \frac{\partial u_1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1, \qquad u_1 = u_2 \qquad \text{on} \quad \Gamma_2, \\
&M_1(u_1)a(|\nabla u_1|^{p(x)})|\nabla u_1|^{p(x)-2}\ \frac{\partial u_1}{\partial \nu} \\
&\qquad\qquad = M_2(u_2)a(|\nabla u_2|^{p(x)})|\nabla u_2|^{p(x)-2}\ \frac{\partial u_2}{\partial \nu} \qquad \text{on } \Gamma_2, \\
&M_2(u_2)a(|\nabla u_2|^{p(x)})|\nabla u_2|^{p(x)-2}\ \frac{\partial u_2}{\partial \nu} + |u_2|^{\alpha(x)-2}u_2 = 0 \qquad \text{on } \Gamma_3,
\end{aligned}
\tag{1}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^n, n = 2,3$, such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$; the boundary $\Gamma = \partial\Omega$ is assumed to be splitted in three disjoint parts: Γ_1, Γ_2 which represents the common boundary between Ω_1 and Ω_2, and Γ_3; the functions M_1, M_2 and a satisfy the following assumptions:

(M_0) $M_i : W^{1,\gamma(x)}(\Omega_i) \longrightarrow (0,+\infty)$ are continuous and bounded on any bounded subset of $W^{1,\gamma(x)}(\Omega_i)$ such that there are constants $m_{0i}, m_{1i} > 0,\ i = 1,2$ such that $m_{0i} \le M_i(u) \le m_{1i}$,

(A_0) $a : [0,+\infty[\longrightarrow \mathbb{R}$ is a C^1-function such that

$$
a_0 + H(a_3)a_2|t|^{\frac{q(x)-p(x)}{p(x)}} \le a(t) \le a_1 + a_3|t|^{\frac{q(x)-p(x)}{p(x)}} \text{ for all } t \ge 0,
$$

$a_0, a_1 > 0,\ a_2, a_3 \ge 0$ are positive numbers, $p,\ \alpha,\ \beta,\ \gamma,\ s,\ t$ are continuous functions on $\overline{\Omega}$, q is some continuous function with $N > q(x) > p(x)$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a suitable Carathеódory function, $H(\tau) = 1$ if $\tau > 0$ and $H(\tau) = 0$ if $\tau = 0$. We confine ourselves to the case where $M_1 = M_2 = M$ with $m_{0i} = m_0,\ m_{1i} = m_1,\ i = 1,2$ for simplicity. Notice that the results of this work remain valid for $M_1 \ne M_2$.

We write $\gamma(x) = (1 - H(a_3))p(x) + H(a_3)q(x)$.

The study of nonlinear boundary value problems involving variable exponents has been received considerable attention in the las decades. This is motivated by the developments in elastic mechanics, electrorheological fluids and image restoration, see [1, 9, 21]. We refer the readers to [11, 12, 13] for the study of $p(x)$-Laplacian equations and the corresponding variational problems.

Transmission problems arise in several applications in physics and biology, see [3, 6, 7, 18]. Recently, in [17] the authors have studied the existence of ground-state solutions for a class of Kirchhoff-type transmission problem. The purpose of this work is to study the existence of solutions to the problem (1) in the Sobolev spaces with variable exponents. We observe that our problem cannot settled in the variational framework because of the functions M_i and f. Indeed, these functions create serious technical difficulties and make us force to apply different tools, such as the monotone operator theory. Also, the nonlinearity on the boundary (Newton boundary condition), which has a polynomial behaviour, causes difficulty in proving coercivity of the problem. In that context, we use an abstra ct result of [10] for monotone maps, to obtain the existence of weak solutions.

Let us point that the condition (A_0) is an extension of the condition given in [15] and that problem (1) is a generalization of the system proposed in [8] to describe the bioheat transfer for the bare human foot.

This paper is organized as follows. In Section 2 we present some necessary preliminary knowledge on variable exponent Sobolev spaces. Section 3 is devoted to the proof of the main result.

2. Preliminaries

We recall the definitions of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$, and $W^{1,p(x)}(\Omega)$. In that context, we refer to [13] for the fundamental properties of these spaces.

Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{p(x),\Omega} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \leq 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x),\Omega} = |u|_{p(x),\Omega} + |\nabla u|_{p(x),\Omega}.$$

The Sobolev space $W^{1,p(x)}(\Omega) \cap W^{1,\gamma(x)}(\Omega)$ is endowed with the norm

$$\|u\|_{1,\gamma(x),\Omega} = \|u\|_{1,p(x),\Omega} + H(a_3)\|u\|_{1,q(x),\Omega}.$$

Since $\gamma(x) \geq p(x)$ we have $W^{1,p(x)}(\Omega) \cap W^{1,\gamma(x)}(\Omega) = W^{1,\gamma(x)}(\Omega)$.

Let us define the Banach space $X = W^{1,\gamma(x)}(\Omega_1) \times W^{1,\gamma(x)}(\Omega_2)$ equipped with the norm

$$\|u\|_X = \|u_1\|_{1,\gamma(x),\Omega_1} + \|u_2\|_{1,\gamma(x),\Omega_2}, \quad \forall u = (u_1, u_2) \in X,$$

where $\|u_i\|_{1,\gamma(x),\Omega_i}$ is the norm of u_i in $W^{1,\gamma(x)}(\Omega_i)$, $i = 1, 2$. By $|u|_X$ we denote the seminorm in X,

$$|u|_X = \sum_{i=1}^{2} \left(|\nabla u_i|_{p(x),\Omega_i} + H(a_3)|\nabla u_i|_{q(x),\Omega_i}\right).$$

It is obvious that

$$|\nabla u_i|_{p(x),\Omega_i} + H(a_3)|\nabla u_i|_{q(x),\Omega_i} \leq |u|_X \leq \|u\|_X, \ \forall u = (u_1, u_2) \in X.$$

Given $(u_1^*, u_2^*) \in (W^{1,\gamma(x)}(\Omega_1))' \oplus (W^{1,\gamma(x)}(\Omega_2))'$ we may think of it as an element of X' (the dual space of X):

$$\langle (u_1^*, u_2^*), (u_1, u_2)\rangle = \langle u_1^*, u_1\rangle + \langle u_2^*, u_2\rangle.$$

Then we have $X' \cong (W^{1,\gamma(x)}(\Omega_1))' \oplus (W^{1,\gamma(x)}(\Omega_2))'$ (isometric isomorphism), where the norm in X' is given by

$$\|(u_1^*, u_2^*)\|_{X'} = \|u_1^*\| + \|u_2^*\|.$$

The function space for the weak formulation of (1) is

$$E = \{(u_1, u_2) \in X : u_1 = u_2 \text{ on } \Gamma_2\}.$$

It is quite easy to prove that E is a closed subspace of X hence E is reflexive, and separable as product of separable spaces. From now on, we denote the norm and the seminorm in E inherited from X by $|.|_E$ and $\|.\|_E$.

In the following, $\alpha \in C_+(\overline{\partial\Omega_2})$, $\alpha(x) < p^\partial(x)$ for $x \in \overline{\partial\Omega_2}$, and $\beta \in C_+(\overline{\Omega_2})$ with $\beta(x) < p^*(x)$, where

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N \end{cases}, \quad p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Definition 1. Let X be a reflexive real Banach space. The operator $T : X \to X^*$ is said to satisfy condition (S_2) iff, as $\nu \to +\infty$ the following holds

$$u_\nu \rightharpoonup u, \qquad Tu_\nu \to Tu \qquad \text{implies} \qquad u_\nu \to u.$$

We have denoted by " $\rightharpoonup$ " (respectively " $\to$ ") the convergence in the weak (respectively strong) topology.

The following theorem due to Dinca and Jebelean [10] allows us to solve problem (1).

Theorem 2. *Let $T : X \to X^*$ be a monotone, hemicontinuous, coercive operator, satisfying condition (S_2) and let $K : X \to X^*$ be compact. If there is a constant $k > 0$ such that $Tv = Ku$ and $\|u\| \le k$ implies $\|v\| \le k$, then the equation $Tu = Ku$ has a solution $u \in X$ with $\|u\| \le k$.*

We need the following auxiliary results.

Lemma 3. *Assume that (A_0) holds. Then for any $k, l > 0$ we have*

$$\left(ka(|\xi|^p)|\xi|^{p-2}\xi - la(|\eta|^p|)||\eta|^{p-2}\eta\right).(\xi - \eta) \ge C|\xi - \eta|^p, \ \forall\ \xi, \eta \in \mathbb{R}^n. \qquad (2)$$

Proof. We follow the idea presented by Peral in [19]. By the homogeneity of norm we may assume that $|\xi| = 1$ and $|\eta| \le 1$. Furthermore, with a convenient basis in $\mathbb{R}^n$ we can take

$$\xi = (1, 0, 0, 0, \dots, 0), \eta = (\eta_1,\ \eta_2, 0, \dots, 0) \text{ and } \sqrt{\eta_1^2 + \eta_2^2}.$$

We will prove that

$$\frac{(A_\xi - A_\eta|\eta|^{p-2}\eta_1)(1 - \eta_1) + A_\eta|\eta|^{p-2}\eta_2}{\left[(1 - \eta_1)^2 + \eta_2^2\right]^{p/2}} \ge C,$$

where $A_\xi = ka(|\xi|^p)$ and $A_\eta = la(|\eta|^p)$.

Put $t = \frac{|\eta|}{|\xi|}$ and $s = \frac{\langle \eta, \xi \rangle}{|\eta||\xi|}$, and we just need to show that the function

$$h(t, s) = \frac{A_\xi - (A_\xi t + A_\eta t^{p-1})s + A_\eta t^p}{(1 - 2st + t^2)^{p/2}}$$

is bounded from below by a positive constant. Direct calculations show that fixed t, $\frac{\partial h}{\partial s} = 0$, if

$$A_\xi - (A_\xi t + A_\eta t^{p-1})s + A_\eta t^p = \frac{2}{p}(A_\xi + A_\eta t^{p-2})(1 - 2st + t^2).$$

Then for the critical s for f we get

$$\begin{aligned} h(t,s) =& \frac{2}{p}\frac{(A_\xi + A_\eta t^{p-2})(1-2st+t^2)}{(1-2st+t^2)^{\frac{p-2}{2}}} \\ \geq& \frac{2a_0}{p}\min\{k,l\}\min_{0\leq t\leq 1}\frac{1+t^{p-2}}{(1+t)^{p-2}} \geq \frac{a_0}{p}\min\{k,l\}. \end{aligned}$$

This gives our conclusion. □

Lemma 4. *Let $s \geq 1, r > 1$. Then there exists a positive constant $c_2 > 0$ such that*

$$|v|^r_E + \frac{\|v\|^r_E}{\|v\|^s_E}|v_2|^s_{\alpha(x),\Gamma_3} \geq c_2\|v\|^r_E, \tag{3}$$

for all $v = (v_1, v_2) \in E$.

Proof. First, we prove that there exists $c_2 > 0$ such that

$$|v|^r_E + |v_2|^s_{\alpha(x),\Gamma_3} \geq c_2, \tag{4}$$

for all $v = (v_1, v_2) \in E$ with $\|v\|_E = 1$. Let us assume that (4) is not valid. Then there exists a sequence $\{v_\nu\} \subset E$ such that:

$$\begin{aligned} &a) \quad \|v_\nu\|_E = 1, \\ &b) \quad v_\nu \rightharpoonup v = (v_1, v_2) \text{ weakly in } E, \\ &c) \quad |v_\nu|^r_E + |v_{2,\nu}|^s_{\alpha(x),\Gamma_3} \leq \frac{1}{\nu}. \end{aligned}$$

From the compactness of embedding $W^{1,\gamma(x)}(\Omega_i) \hookrightarrow L^{\alpha(x)}(\partial\Omega_i), i = 1, 2$ and $b)$ it follows that

$$v_\nu \to v = (v_1, v_2) \text{ strongly in } L^{\alpha(x)}(\partial\Omega_1) \times L^{\alpha(x)}(\partial\Omega_2). \tag{5}$$

Using (5), the weak lower semicontinuity of the seminorm $|v|_E$ and $c)$ we get

$$|v|^r_E + |v_2|^s_{\alpha(x),\Gamma_3} = 0.$$

Then, $v_1 = k_1, v_2 = k_2$, for some constants k_1, k_2. So $v_2 \mid_{\Gamma_3}= k_2$. As $|v_2|_{\alpha(x),\Gamma_3} = 0$ we have $k_2 = 0$ and , from the transmission condition, we get $v_1 \mid_{\Gamma_2}= 0$. Therefore $v = 0$. This is a contradiction to $a)$.

Finally, to prove (3) let $v \in E, v \neq 0$ and $\hat{v} = \frac{v}{\|v\|_E}$. From (4) we have

$$\frac{|v|_E^r}{\|v\|_E^\beta} + \frac{1}{\|v\|_E^s}|v_2|^s_{\alpha(x),\Gamma_3} \geq c_2.$$

Multiplying this inequality by $\|v\|_E^r$ the assertion (3) follows. □

3. Existence of solutions

In this section, we shall state the existence of solution to the elliptic problem (1). A key role in the proof of our result is played by Theorem 2. For simplicity, we use C, C_i, C_i', $i = 1, 2, \dots$ to denote the general positive constant (the exact value may change from line to line). Let us define the operators $T, S : E \to E^*$ by

$$\begin{aligned}\langle Tu, v\rangle =& \sum_{i=1}^{2} M(u_i) \int_{\Omega_i} a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i \nabla v_i \, dx \\ &+ \int_{\Gamma_3} |u_2|^{\alpha(x)-2} u_2 v_2 \, dS, \\ \langle Su, v\rangle =& \int_{\Omega_1} f(x, u_1, \nabla u_1)|u_1|^{t(x)}_{s(x)} v_1 \, dx + \int_{\Omega_2} |u_2|^{\beta(x)-2} u_2 v_2 \, dx, \\ u =&(u_1, u_2), v = (v_1, v_2) \in E.\end{aligned}$$

Definition 5. A function $u \in E$ is said to be a weak solution of (1) if

$$Tu = Su \quad \text{in} \quad E^*.$$

Proposition 6. *Assume that* (M_0) *and* (A_0) *hold. Then:*

(i) $T : E \to E^*$ *is a continuous, bounded and strictly monotone operator.*

(ii) T *is coercive.*

(iii) T *is of type* (S_2).

Proof. Define the mappings B_0, B and $C : E \to E^*$ respectively by

$$\langle B_0(u_i), v_i\rangle = \int_{\Omega_i} a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i \nabla v_i \, dx,$$

$$\langle B(u), v\rangle = \sum_{i=1}^{2} M(u_i)\langle B_0(u_i), v_i\rangle,$$

$$\langle C(u), v\rangle = \int_{\Gamma_3} |u_2|^{\alpha(x)-2} u_2 v_2\, dS, \quad u = (u_1, u_2),\ v = (v_1, v_2) \in E.$$

So, $T = B + C$.

i) We first show the boundedness of T. From the Hölder's inequality, properties in $W^{1,p(x)}(\Omega)$ and assumptions $(M_0), (A_0)$ we obtain

$$\begin{aligned} |\langle B_0(u_i), v_i\rangle| &\le |a_1|\nabla u_i|^{p(x)-1} + a_3|\nabla u_i|^{qx-1}|_{\gamma'(x)}|\nabla v_i|_{\gamma(x)} \\ &\le C\left(|\nabla u_i|^{p^+-1}_{(p(x)-1)\gamma'(x)} + |\nabla u_i|^{q^+-1}_{(q(x)-1)\gamma'(x)} + 2\right)|\nabla v_i|_{\gamma(x)} \\ &\le C\left(|u_i|^{p^+-1}_{1,\gamma(x),\Omega_i} + |u_i|^{q^+-1}_{1,\gamma(x),\Omega_i} + 4\right)|v_i|_{1,\gamma(x),\Omega_i}, \end{aligned} \tag{6}$$

$$\begin{aligned} |\langle C(u), v\rangle| &\le ||u_2|^{\alpha(x)-1}|_{\alpha'(x),\Gamma_2}|v_2|_{\alpha(x),\Gamma_3} \\ &\le C\left(|u_2|^{\alpha^+-1}_{1,\gamma(x),\Omega_2} + 1\right)|v_2|_{1,\gamma(x),\Omega_2}. \end{aligned} \tag{7}$$

Then,

$$|\langle Tu, v\rangle| \le m_1 C(\|u\|_E^{p^+-1} + \|u\|_E^{q^+-1} + \|u\|_E^{\alpha^+-1} + 1)\|v\|_E.$$

To show that T is continuous it is sufficient to prove that B and C are continuous. Indeed, let $u_\nu \to u$ in E, $u_\nu = (u_{1,\nu}, u_{2,\nu}), u = (u_1, u_2)$.

Then, up a subsequence, we have

$$\begin{aligned} &u_{i,\nu} \to u_i,\ \nabla u_{i,\nu} \to \nabla u_{i,\nu} \ \text{ and } \ |\nabla u_{i,\nu}| \le g_i(x) \text{ a.e. in } \ \Omega, \\ &u_{2,\nu} \to u_2 \ \text{ and } \ |u_{2,\nu}| \le h_2(x) \text{ a.e. in } \ \Gamma_2, \end{aligned} \tag{8}$$

for some $g_i \in L^{p(x)}(\Omega_i),\ i = 1, 2,\ h_2 \in L^1(\Gamma_2)$.

Since a is continuous,

$$a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-2}\nabla u_{i,\nu} \to a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i \text{ a.e. in } \Omega_i. \tag{9}$$

Further,

$$\begin{aligned} &\int_{\Omega_i} |a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-2}\nabla u_{i,\nu}|^{\gamma'(x)}\, dx \\ &\le C\int_{\Omega_i} ||\nabla u_{i,\nu}|^{p(x)-1} + |\nabla u_{i,\nu}|^{q(x)-1}|^{\gamma'(x)}\, dx \\ &\le C\left(|\nabla u_{i,\nu}|^{((p-1)\gamma')^+}_{(p(x)-1)\gamma'(x)} + |\nabla u_{i,\nu}|^{((q-1)\gamma')^+}_{(q(x)-1)\gamma'(x)} + 2\right) \\ &\le C\left(|\nabla u_{i,\nu}|^{((p-1)\gamma')^+}_{\gamma(x),\Omega_i} + |\nabla u_{i,\nu}|^{((q-1)\gamma')^+}_{\gamma(x),\Omega_i} + 2\right) \\ &\le C\left(\|u_\nu\|_E^{((p-1)\gamma')^+} + \|u_\nu\|_E^{((q-1)\gamma')^+} + 2\right). \end{aligned} \tag{10}$$

Thus, the boundedness of (u_ν) in E implies that $\{a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-2}\nabla u_{i,,\nu}\}_\nu$ is bounded in $(L^{\gamma'(x)}(\Omega_i))^N$.

In virtue of Lemma 3.3 in [4], we obtain

$$\int_{\Omega_i} a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-2}\nabla u_{i,\nu}\nabla v_i\,dx \to \int_{\Omega_i} a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i\nabla v_i\,dx.$$

The arguments above for (u_ν) hold in fact for any of its subsequences. Hence $B_0(.)$ is continuous in E.

Let us show that T is monotone. For any $u, v, w \in E$, using conditions $(M_0), (A_0)$, the elementary inequality

$$|x|^p - |x|^{p-1}|y| \geq \frac{1}{2}|x|^{p-2}(x^2 - y^2) \quad \text{for all} \quad x, y \in \mathbb{R}$$

and Lemma 3, we get

$$\begin{aligned} &\langle Tu, u-v\rangle - \langle Tv, u-v\rangle \\ &= \sum_{i=1}^{2}\int_{\Omega_i}\Big(M(u_i)a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i \\ &- M(v_i)a(|\nabla v_i|^{p(x)})|\nabla v_i|^{p(x)-2}\nabla v_i\Big).\Big(\nabla u_i - \nabla v_i\Big)\,dx \\ &+ \int_{\Gamma_3}(|u_2|^{\alpha(x)-2} - |v_2|^{\alpha(x)-2})(|u_2|^2 - |v_2|^2)\,dS \geq 0, \end{aligned} \tag{11}$$

i.e. T is monotone.

If $\langle Tu, u-v\rangle - \langle Tv, u-v\rangle = 0$ then the terms in the right-hand side of (16) are equal to zero. Hence, $u_i - v_i = k_i = const.$ a.e. in Ω_i, $i = 1, 2$ and $u_2 - v_2 = 0$ a.e. on Γ_3. Therefore $k_i = 0$ and $u = v$ a.e.

ii) For any $u = (u_1, u_2) \in E$, we have

$$\begin{aligned} \langle Tu, u\rangle &= \sum_{i=1}^{2} M(u_i)\int_{\Omega_i} a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)}\,dx + \int_{\Gamma_3}|u_2|^{\alpha(x)}\,dS \\ &\geq m_0\sum_{i=1}^{2}\Big[a_0\min\{|\nabla u_i|^{p^-}_{p,\Omega_i}, |\nabla u_i|^{p^+}_{p,\Omega_i}\} \\ &+ a_2H(a_3)\min\{|\nabla u_i|^{q^-}_{q,\Omega_i}, |\nabla u_i|^{q^+}_{q,\Omega_i}\}\Big] + \min\{|u_2|^{\alpha^-}_{p,\Gamma_3}, |u_2|^{\alpha^+}_{p,\Gamma_3}\}. \end{aligned} \tag{12}$$

Now, if

$$\begin{aligned}\min\{|\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i}\} &= |\nabla u_i|^{p^-}_{p(x),\Omega_i},\\ \min\{|\nabla u_i|^{q^-}_{q(x),\Omega_i}, |\nabla u_i|^{q^+}_{q(x),\Omega_i}\} &= |\nabla u_i|^{q^-}_{q(x),\Omega_i},\\ \min\{|u_2|^{\alpha^-}_{p(x),\Gamma_3}, |u_2|^{\alpha^+}_{p(x),\Gamma_3}\} &= |u_2|^{\alpha^-}_{p(x),\Gamma_3},\end{aligned} \tag{13}$$

using inequalities (12) and (13), it follows that

$$\langle Tu, u\rangle \geq m_0 c_4 |u|^{p^-}_E + |u_2|^{\alpha^-}_{p(x),\Gamma_3}.$$

Provided that $\|u\|_E > 1$, putting $r = p^-, s = \alpha^-$ in (3), and noting that $\|u\|^{p^- - \alpha^-}_E \leq 1$ we obtain

$$\langle Tu, u\rangle \geq m_0 c_4 |u|^{p^-}_E + |u_2|^{\alpha^-}_{p(x),\Gamma_3} \geq c_2' \|u\|^{p^-}_E,$$

for some $c_{2'} > 0$. For other cases, the proofs are similar and we omit them here. So we have

$$\langle Tu, u\rangle \geq c_3 \min\{\|u\|^{p^-}_E, \|u\|^{p^+}_E\} = c_3 \|u\|^{p^-}_E. \tag{14}$$

iii) To prove that T is an operator of type (S_2) in E, first we will prove that $B_0(.)$ is of type (S_+).

In fact, let $\{u_\nu = (u_{1,\nu}, u_{2,\nu})\} \subset E$ be such that $u_\nu \rightharpoonup u = (u_1, u_2)$ in X. Then $u_{i,\nu} \rightharpoonup u_i$ in $W^{1,\gamma(x)}(\Omega_i)$, $i = 1, 2$ and

$$\limsup_{\nu\to\infty} \langle B_0 u_\nu - B_0 u, u_\nu - u\rangle \leq 0.$$

Thanks to Lemma 3 we infer that $B_0(.)$ is monotone, then

$$\lim_{\nu\to\infty} \langle B_0 u_\nu - B_0 u, u_\nu - u\rangle = 0. \tag{15}$$

Hence, by a standard argument (see e.g. V.K. Le, [16], Theorem 4.1) we deduce that

$$\nabla u_{i,\nu}(x) \to \nabla u_i(x) \quad \text{a.e. in } \Omega_i. \tag{16}$$

For a moment, we suppose that for $\epsilon > 0$ there exist $\delta_i > 0$ and $\nu_0 \in \mathbb{N}$ such that if H_i is a measurable subset of Ω_i, with $|H_i| < \delta_i$ then

$$a_0 \int_{H_i} |\nabla u_{i,\nu}|^{p(x)}\, dx + a_1 H(a_3) \int_{H_i} |\nabla u_{i,\nu}|^{q(x)}\, dx < \epsilon \text{ for all } \nu \geq \nu_0. \tag{17}$$

Consequently, $\{|\nabla u_{i,\nu}|^{p(x)}\}$ and $\{|\nabla u_{i,\nu}|^{q(x)}\}$ are uniformly integrable and hence, so is $\{|\nabla u_{i,\nu} - \nabla u_i|^{p(x)}\}$ and $\{|\nabla u_{i,\nu} - \nabla u_i|^{q(x)}\}$. From this and (17), we get by Vitali's theorem that

$$\lim_{\nu\to\infty}\int_{\Omega_i}|\nabla u_{i,\nu}-\nabla u_i|^{p(x)}\,dx=0. \tag{18}$$

Thanks to the compact embedding $W^{1,p(x)}(\Omega_i)\hookrightarrow L^{p(x)}(\Omega_i)$, we have from the assumptions that

$$\lim_{\nu\to\infty}\int_{\Omega_i}|u_{i,\nu}-u_i|^{p(x)}\,dx=0,$$

which together with (18) implies that $u_{i,\nu}\to u_i$ in $W^{1,p(x)}(\Omega_i)$. By arguments similar to the previous ones, we can prove that $u_{i,\nu}\to u_i$ in $W^{1,q(x)}(\Omega_i)$. Therefore $u_\nu\to u$ in X. Since E is a closed subspace of X, we get $u_\nu\to u$ in E.

Now, let us show (17). Let $\epsilon>0$ be arbitrary. By (15), there exists $\nu_0\in\mathbb{N}$ such that, for any measurable subset H_i of Ω_i and all $\nu\geq\nu_0$

$$\int_{H_i}\Big(a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-2}\nabla u_{i,\nu}-a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-2}\nabla u_i)\Big) \times\,(\nabla u_{i,\nu}-\nabla u_i)\,dx\leq\frac{k_0\epsilon}{4},$$

where $k_0=\min\{a_0,a_1\}$.

By this and (A_0), we infer

$$\begin{aligned}I\equiv&k_0\left(\int_{H_i}|\nabla u_{i,\nu}(x)|^{p(x)}\,dx+H(a_3)\int_{H_i}|\nabla u_{i,\nu}(x)|^{q(x)}\,dx\right)\\ &\leq\frac{k_0\epsilon}{4}+\int_{H_i}|a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-1}||\nabla u_i|\,dx\\ &\qquad+\int_{H_i}|a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-1}||\nabla u_{i,\nu}|\,dx.\end{aligned} \tag{19}$$

Here, we note that the boundedness of $\{u_\nu\}$ in E and (10) imply that there exists $k_1>0$ such that

$$|a(|\nabla u_{i,\nu}|^{p(x)})|\nabla u_{i,\nu}|^{p(x)-1}|_{\gamma'(x)}\leq k_1\ \text{ for all } \nu.$$

Using this fact with (19) and the Young and Holder inequalities, we obtain

$$\begin{aligned}I\leq\frac{k_0\epsilon}{4}+4k_1|\nabla u_i|_{\gamma(x),H_i}+C\int_{H_i}|a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-1}|^{\gamma'(x)}\,dx\\ +\frac{k_0}{2}\int_{H_i}|\nabla u_{i,\nu}|^{\gamma(x)}\,dx,\ \text{for all } \nu\geq\nu_0.\end{aligned} \tag{20}$$

Noting that $|\nabla u_i|^{\gamma(x)}$, $|a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-1}|^{\gamma'(x)} \in L^1(\Omega)$, we can find $\delta > 0$ such that for any $H_i \subset \Omega$ with $|H_i| < \delta$, we have

$$4k_1|\nabla u_i|_{\gamma(x),H_i} + \int_{H_i} |a(|\nabla u_i|^{p(x)})|\nabla u_i|^{p(x)-1}|^{\gamma'(x)}\, dx < \frac{k_0\epsilon}{4}.$$

Thus, we get from this inequality and (20) that

$$I < \frac{k_0\epsilon}{2} + \frac{k_0}{2}\int_{H_i} |\nabla u_{i,\nu}|^{\gamma(x)}\, dx, \text{ for all } \nu \geq \nu_0,$$

which easily yields (17).

It follows that $B(u) = \sum_{i=1}^{2} M(u_i)B_0(u_i)$ is of type (S_+). To prove this, assuming that $\{u_\nu\} \subset E$, $u_\nu \rightharpoonup u$ in E and $\limsup_{\nu\to+\infty}\langle B(u_\nu), u_\nu - u\ \rangle \leq 0$ it is sufficient to show that any subsequence of $\{u_\nu\}$ has a strongly convergent subsequence . Let $\{u_{\nu_k}\}$ be a subsequence of $\{u_\nu\}$. From the boundedness there exists a subsequence de of $\{u_{\nu_k}\}$, denoted still by $\{u_{\nu_k} = (u_{1\nu_k}, u_{2\nu_k})\}$ such that $M(u_{i\nu_k}) \to t_{i0}$ with $t_{i0} > 0$. Hence

$$\limsup_{k\to+\infty}\langle\sum_{i=1}^{2} t_{i0}\langle B_0(u_{i\nu_k}), v_i\rangle = \limsup_{k\to+\infty}\langle\sum_{i=1}^{2} M(u_{i\nu_k})B_0(u_{i\nu_k}), v_i\rangle \leq 0,$$

which implies $\limsup\limits_{k\to+\infty}\langle B_0(u_{i\nu_k}), v_i\rangle \leq 0$, i=1, 2. Since B_0 is of type (S_+) (in X), we get $u_{i\nu_k} \to u_i$ $i = 1, 2$, so $u_\nu \to u$ in X. Since $(u_\nu) \subseteq E$ and E is a closed subspace of X, we have $u \in E$, so $u_\nu \to u$ in E.

Moreover, using the compact embedding $W^{1,p(x)}(\Omega_2) \hookrightarrow L^{\alpha(x)}(\Gamma_3)$ we deduce that the operator C is compact. Noting that the sum of an (S_+) type mapping and a compact mapping is of type (S_+), it follows that the mapping $T = B + C$ is of type (S_+). So it is of type (S_2). $\square$

Now, we state the assumptions imposed on the nonlinearity f, which appears in problem (1). We assume that:

(F_1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory condition in the sense that $f(., u, \xi)$ is measurable for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, ., .)$ is continuous for almost all $x \in \Omega$.

(F_2) $|f(x, u, \xi)| \leq k(x) + |u|^{\eta(x)} + |\xi|^{\delta(x)}$ a.e. $x \in \Omega$, all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k : \mathbb{R} \to \mathbb{R}^+$, $k \in L^{p'(x)}(\Omega)$ and $0 \leq \eta(x) < p(x) - 1, 0 \leq \delta(x) < (p(x) - 1)/p'(x)$.

Lemma 7. *Assume* $(F1), (F_2)$. *Then, the operator* $S : E \to E^*$ *given by*

$$\begin{aligned}\Big\langle S(u_1,u_2),(v_1,v_2)\Big\rangle =& \int_{\Omega_1} f(x,u_1,\nabla u_1)|u_1|_{s(x)}^{t(x)} v_1\,dx \\ &+ \int_{\Omega_2} |u_2|^{\beta(x)-2} u_2 v_2\,dx \\ :=& \langle S_1u_1,v_1\rangle + \langle S_2u_2,v_2\rangle,\ \ (u_1,u_2),(v_1,v_2)\in E\end{aligned}$$

is continuous and compact.

Proof. The arguments are very similar to those of Lemma 2.2 of [5] for S_1 and Theorem 2.1 of [20] for S_2, bearing in mind that E is a closed subspace of X and taking into account the compact embedding $W^{1,p(x)}(\Omega_2)\hookrightarrow L^{\beta(x)}(\Omega_2)$, so we omit them. □

Theorem 8 (The Main result)**.** *Let* $(M_0),(A_0)$, (F_1) *and* (F_2) *hold. If* $1<\beta(x)<p(x)-1$, *then problem* (1) *has a weak solution.*

Proof. In virtue of Proposition 6, T has all the properties in Theorem 2. On the other hand, S is compact by Lemma 7. Let us prove that there exists some $k>0$ such that $T(z)=S(w)$ and $\|w\|_E\le k$ implies $\|z\|_E\le k$, for $z,w\in E$.

Let $z=(v_1,v_2)$ with $\|z\|_E\ge 1$ and $w=(u_1,u_2)\in E$ with $T(z)=S(w)$.

Thus, from (14) (we have removed the dependency on x for simplicity)

$$
\begin{aligned}
c_3\|z\|_E^{p^-} &\le \langle T(z), z\rangle = \langle S(w), z\rangle \\
&= \int_{\Omega_1} f(x, u_1, \nabla u_1)|u_1|_s^t v_1\, dx + \int_{\Omega_2} |u_2|^{\beta-2} u_2 v_2\, dx \\
&\le (\frac{1}{p^-} + \frac{1}{p^+})|f(x, u_1, \nabla u_1)|_{p'}|v_1|_p + (\frac{1}{\beta^-} + \frac{1}{\beta^+})||u_2|^{\beta-1}|_{\beta'}|v_2|_\beta \\
&\le C\Big(\int_{\Omega_1} |k(x)|^{p'}\, dx + \int_{\Omega_1} |u_1|^{\eta p'}\, dx + \int_{\Omega_1} |\nabla u_1|^{\delta p'}\, dx\Big)^{1/\alpha} \\
&\qquad\qquad \times \|v_1\|_{1,p,\Omega_1} + C(|u_2|_\beta^{\hat q}\|v_2\|_{1,p,\Omega_2}) \\
&\le C\Big[\Big(|k|_{p'}^\tau + |u_1|_{\eta p'}^\beta + |\nabla u_1|_{\delta p'}^\theta\Big)^{1/\alpha} \|v_1\|_{1,p,\Omega_1} \\
&\qquad\qquad + |u_2|_\beta^{\hat q}\|v_2\|_{1,p,\Omega_2}\Big] \\
&\le C\Big[(1 + \|u_1\|_{1,p,\Omega_1}^\beta + \|u_1\|_{1,p,\Omega_1}^\theta)^{1/\alpha}\|v_1\|_{1,p,\Omega_1} \\
&\qquad\qquad + \|u_2\|_{1,p,\Omega_2}^{\hat q}\|v_2\|_{1,p,\Omega_2}\Big] \\
&\le C\left(1 + \|w\|_E^{\beta/\alpha} + \|w\|_E^{\theta/\alpha} + \|w\|_E^{\hat q}\right)\|z\|_E,
\end{aligned}
$$

where

$$
\alpha = \begin{cases} p'^-, & \text{if } |f(x,u,\nabla u)|_{p'} > 1, \\ p'^+, & \text{if } |f(x,u,\nabla u)|_{p'} \le 1, \end{cases}, \quad \tau = \begin{cases} p'^-, & \text{if } |k|_{p'} > 1, \\ p'^+, & \text{if } |k|_{p'} \le 1, \end{cases}
$$

$$
\beta = \begin{cases} (\eta p')^+, & \text{if } |u|_{\eta p'} > 1, \\ (\eta p')^-, & \text{if } |u|_{\eta p'} \le 1, \end{cases}, \quad \theta = \begin{cases} (\delta p')^+, & \text{if } |\nabla u|_{\delta p'} > 1, \\ (\delta p')^-, & \text{if } |\nabla u|_{\delta p'} \le 1 \end{cases}
$$

and $\hat q$ is some constant with $\hat q \in [\beta^-, \beta^+]$ such that $||u_2|^{\beta(x)-1}|_{\beta'(x)} \le |u_2|_{\beta(x)}^{\hat q}$. So, we have obtained

$$
\begin{aligned}
\|z\|_E^{p^- -1} &\le C_1' + C_2'\|w\|_E^{\beta/\alpha} + C_3'\|w\|_E^{\theta/\alpha} + C_4'\|w\|_E^{\hat q} \\
&\le C_1' + C_2' k^{\beta/\alpha} + C_3' k^{\theta/\alpha} + C_4' k^{\hat q}.
\end{aligned} \tag{21}
$$

We note that, by our assumptions on η, δ and β we have $\beta/\alpha,\ \theta/\alpha,\ \hat q < p^- - 1$.

Moreover, using the fact that $t^{p^- -1} - C_2' t^{\beta/\alpha} - C_3' t^{\theta/\alpha} - C_4' t^{\hat q} - C_1' \to +\infty$ as $t \to +\infty$, there is some $R_0 > 0$ such that for all $t > R_0$

$$
t^{p^- -1} - C_2' t^{\beta/\alpha} - C_3' t^{\theta/\alpha} - C_4' t^{\hat q} - C_1' \ge 0. \tag{22}
$$

Let $k > \max\{1, R_0\}$. Hence, from (21) and (22) we deduce that if $T(z) = S(w)$ and $\|w\|_E \leq k$, then

$$\|z\|_E^{p^- -1} \leq C_1' + C_2' k^{\beta/\alpha} + C_3' k^{\theta/\alpha} + C_4' k^{\hat{q}} \leq k^{p^- -1},$$

which implies $\|z\|_E \leq k$. Being all the assumptions fulfilled, the conclusion follows from Theorem 2. □

Example. The following functions satisfy the conditions on our work:
$a(t) = 1 + t^{\frac{q(x)-p(x)}{p(x)}} + \frac{1}{(1+t^{\frac{p(x)-2}{p(x)}})}$ with $a_0 = 1, a_1 = 2, a_2 = a_3 = 1$,
$M(u) = \int_\Omega (\sin^2 u + 1)\, dx$,
$f(x, u, \xi) = k(x) + |u|^{\eta(x)} + |\xi|^{\delta(x)}$, where $p \in C_+(\overline{\Omega})$, $p(x) < N$,
$k \in L^{p'(x)}(\Omega), 0 \leq \delta(x) < \frac{p(x)-1}{p'(x)}$, $0 \leq \eta(x) < p(x) - 1$.

In this case we obtain a nonlocal capillary transmission problem of the type $p(x)\&q(x)$ Laplacian, with a convection term.

References

[1] E. Acerbi, G. Mingione, Regularity results for stationary electro-rheological fluids, *Archive for Rational Mechanics and Analysis*, **164** (2002), 213-259.

[2] R. Ayazoglu (Mashiyev), M. Avci, N. Thanh Chung, Existence of solutions for nonlocal problems in Orlicz-Sobolev spaces via Monotone method, *Electronic J. of Mathematical Analysis and Applications*, **4** (1), No 6 (2016), 63-73.

[3] A. Ayoujil, M. Moussaoui, Multiplicity results for nonlocal elliptic transmission problem with variable exponent, *Boletim da Sociedade Paranaense de Matemática*, **3** (33), No 2 (2015), 185–199; doi: 10.5269/bspm.v33i2.23875.

[4] M. B. Benboubker, E. Azroul, A. Barbara, Quasilinear elliptic problems with nonstandard growth, *Electron. J. Differential Equations*, **62** (2011), 1–16.

[5] E. Cabanillas L., J. Bernui B., R.J. de la Cruz M., Z. Huaringa S, Existence of slutions for a class of $p(x)$-Kirchhoff type equation with dependence on the gradient, *Kyungpook Mathematical J. Kyungpook Nat. Univ.*, **58**, No 3 (2018), 533-546.

[6] F. Cammaroto, L. Vilasi, Existence of three solutions for a degenerate Kirchhoff–type transmission problem, *Numerical Functional Analysis and Optimization*, **35**, No 7-9 (2014), 911-931.

[7] B. Cekic, R.A. Mashiyev, Nontrivial Solution for a nonlocal elliptic transmission problem in variable exponent Sobolev spaces, *Abstract and Applied Analysis* **2010** (2010), Article ID 385048, 12 pp.; doi: 10.1155/2010/385048.

[8] M.I.M. Copetti, J. Durany, J.R. Fernández, L. Poceiro, Numerical analysis and simulation of a bio-thermal model for the human foot, *Applied Mathematics and Computation*, **305** (2017), 103-116.

[9] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. on Applied Mathematics*, **66** (2006), 1383-1406.

[10] G. Dinca, P. Jebelean, Some existence results for a class of nonlinear equations involving a duality mapping, *Nonlinear Analysis*, **46** (2001), 347-363.

[11] X.L. Fan, J.S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. of Mathematical Analysis and Applications*, **262** (2001), 749-760.

[12] X.L. Fan, Q.H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, *Nonlinear Analysis*, **52** (2003), 1843-1852.

[13] X.L. Fan, D. Zhao, On the Spaces $L^{p(x)}$ and $W^{m,p(x)}$, *J. of Mathematical Analysis and Applications*, **263** (2001), 424-446; doi: 10.1006/jmaa.2000.7617.

[14] X.L. Fan, Y.Z. Zhao, Q.H. Zhang, A strong maximum principle for $p(x)$-Laplace equations, *Chinese J. of Contemporary Mathematics*, **24**, No 3 (2003), 277-282.

[15] G.M. Figueiredo, Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^n$, *J. Math. Anal. Appl.*, **378**, No 2 (2011), 507-518.

[16] V.K. Le, On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces, *Nonlinear Analysis*, **71** (2009), 3305-3321.

[17] F. Li, Y. Zhang, X. Zhu, Z. Liang, Ground-state solutions to Kirchhoff-type transmission problems with critical perturbation, *J. of Mathematical Analysis and Applications*, **482**, No 2 (2020), Art. 123568.

[18] T.F. Ma, J. E. Muñoz Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, *Applied Mathematics Letters*, **16** (2003), 243-248.

[19] I. Peral, Multiplicity of solutions for the p-Laplacian, *International Center for Theoretical Physics*, Lecture Notes, Trieste (1997).

[20] L. Wang, Y. Fan, W. Ge, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, *Nonlinear Analysis*, **71** (2009), 4259–4270.

[21] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Mathematics of the USSR-Izvestiya*, **9** (1987), 33-66; doi: 10.1070/IM1987v029n01ABEH000958.