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ON THE STABILITY OF CURRENT-CARRYING SHELL

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Abstract: The stability of oscillations of an ideal conducting shell with a longitudinal current containing the flow of an ideal incompressible fluid with respect to radial disturbance of the shape of the shell is studied.

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1. Introduction

Recently, there has been an intensive implementation of the ideas and methods of magnetism into applied industries - in the energetics, metallurgy, technology, etc. In the first place we are talking about the use of strong magnetic fields and materials with different conductivities. In theoretical and practical studies related to these areas, an important role is played by taking into account the properties of actual media, such as finite and infinite conductivity, superconductivity, internal friction, relaxation, etc. They often lead to new results that are of both theoretical and practical interest.

Various problems of the theory of magnetism lead to the study of systems of quasilinear non-stationary equations. This problem, in particular, is related to the phenomenon of superconductivity of type II. In 1962, C. P. Bean [1] proposed a mathematical model to describe this phenomenon. We should note that by its very statement, the Bean model implies a solution with a compact

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support, as well as the study of local and global properties of the solution of the corresponding mathematical problem. Some of these questions were studied and solved in [2] , [5] , [6] , [12] – [14] .

In this paper, we touch upon the second equally important aspect of this problem - stability, namely, the question of stability of oscillations of a magnetoelastic system. Earlier, the problem of the stability of a tangential discontinuity formed by two elastic superconducting half-spaces in a magnetic field was considered [10], [11]. Now a conducting shell in a magnetic field is used as a magnetoelastic system.

2. Preliminary Notes

The equations of magnetoelastic oscillations of the shell represent a set of equations of electrodynamics and exact or approximate equations of shell mechanics. The relationship between the equations of mechanics and electrodynamics is due to the Ampere force with which the magnetic field acts on the surface current.

At a disturbance of the shape of the shell, there appears a disturbance of the magnetic field and hence the "magnetic" pressure on the surface of the shell. Thus, the oscillations will be described by agreed equations in which this change in the shape of the shell leads to a perturbation of the magnetic field of the currents. Here we will consider thin cylindrical shells of an infinite length.

As it is known [4], [9] cylindrical shells can have a significant number of the form of oscillations, which are determined by the number of nodal lines along the generatrix of the cylinder. We confine ourselves to considering only radial oscillations, that is, we will study such a change in the shape of the shell at which its radius experiences disturbance of the form:

$$r = R + \xi\left(\varphi, z, t\right) = R + \xi_0 \exp\left(in\varphi + ikz - i\omega t\right). \qquad (1)$$

Here, the axis of the shell coincides with the axis z of the cylindrical coordinate system, the thickness of the shell is equal to h and its radius - R. The shell material has density ρ, elastic modulus E and Poisson's coefficient v. We will consider the conductivity of the shell as equal to infinity, and the magnetic permeability 1.

3. Main results

Consider the problem of the stability of oscillations of a current-carrying shell containing an incompressible fluid flow [10], [11]. Let the radius of the shell experience small disturbance of the form (1). The fluid flow rate is equal V and its density ρ_0. Current by force I flowing along the shell in the direction of the axis$z$.

The unperturbed field of shell has in this case only the azimuthal component equal to

$$H_{0\varphi} = \frac{2I}{cr}, \quad r > R.$$

The disturbance of the shell leads to a disturbance of the magnetic field, which can be written in the form:

$$\overline{H} = \overline{H_0} + \overline{h} \exp\left(-i\omega t\right), \quad h \ll H_0.$$

To determine the disturbance $\overline{h}$, we use the magnetostatic approximation [9], [10]. Then we have

$$\begin{gathered} \overline{h} = grad\psi, \quad div\overline{h} = 0, \\ \Delta\psi = 0 \text{ at } r > R. \end{gathered}$$

The solution of the Laplace equation in the outer area has the form [7]:

$$\psi = \alpha K_n\left(kr\right) \exp\left(ikz + in\varphi\right),$$

where α is the constant determined from the boundary condition and K_n is the modified Bessel function of the second kind. Thus, the components of the perturbation $\bar{h}$ are equal to:

$$h_r = \alpha k K_n'\left(kr\right) \exp\left(ikz + in\varphi\right),$$

$$h_\varphi = \alpha \frac{in}{r} K_n\left(kr\right) \exp\left(ikz + in\varphi\right),$$

$$h_z = ik\alpha K_n\left(kr\right) \exp\left(ikz + in\varphi\right).$$

The condition at the boundary with an ideal conductor requires the absence of a normal component of the magnetic field [8], i.e.

$$\overline{n}\,\bar{H} = 0 \quad \text{at } r = R,$$

where $\overline{n}$ is the unit vector of the external normal to the perturbed surface of shell. For a disturbance of the form (1), its components are equal to:

$$n_r = 1, \;\; n_\varphi = -\frac{in\xi}{R}, \;\; n_z = -ik\xi.$$

Using the boundary conditions, we find

$$\alpha = inH_{0\varphi}\frac{1}{kK'_n(kR)}\cdot\frac{\xi_0}{R}.$$

The magnetic pressure on the perturbed surface of the shell in this case is determined by the formula

$$P_m = P_{om}\left(1 - 2\frac{\xi}{R}\left(1 + n^2\frac{K_n(kR)}{kRK'_n(kR)}\right)\right), \tag{2}$$

where $P_{om} = \frac{1}{8\pi}\cdot H_{0\varphi}^2$ is the magnetic pressure on the non-perturbed shell. If the disturbance depends only on z, then

$$P_m = P_{om}\left(1 - 2\frac{\xi}{R}\right). \tag{3}$$

With a disturbance of the shape of the shell, which depends only on φ, the solution of the Laplace equation in the area $r > R$ is the function of

$$\psi = \alpha r^{-n}\exp(in\varphi).$$

In this case, the pressure is equal to:

$$P_m = P_{om}\left(1 + 2\frac{\xi}{R}(n-1)\right). \tag{4}$$

For simplicity, we consider the problem separately for perturbations of the form $\exp(ikz)$ and $\exp(in\varphi)$. In the first case, the equation for small radial oscillations of the shell has the form ([4], [9])

$$\rho h\frac{\partial^2\xi}{\partial t^2} + D\frac{\partial^4\xi}{\partial z^4} + \frac{2vD}{R^2}\cdot\frac{\partial^2\xi}{\partial z^2} + \frac{12\left(1-v^2\right)}{h^2R^2}D\xi = P_r - P_m,$$

where $D = \frac{Eh^2}{R(1-v^2)}$ is the cylindrical rigidity, P_r and P_m is the hydrodynamic and magnetic pressure on the surface of the shell.

To find the hydrodynamic pressure, we solve the corresponding boundary value problem

$$\Delta P = 0 \text{ at } r > R,$$

$$\frac{\partial P}{\partial r} = -\rho_0\left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial z}\right)^2\xi \text{ at } r = R.$$

Hence we find that

$$P_r = \rho_0\frac{(\omega - kV)^2}{k}\cdot\frac{I_0(kR)}{I'_0(kR)}\xi, \tag{5}$$

where I_0 is the modified Bessel function of the first kind, [3]. Substituting the expressions for the pressures (3) and (5) and taking into account that in the case under consideration $\xi \sim \exp\left(i\left(kz - \omega t\right)\right)$, we obtain the dispersion equation of the problem in the form:

$$\omega^2 - 2\omega\frac{kV}{1+\varepsilon} + \frac{k^2V^2}{1+\varepsilon} - \Omega^2 = 0, \tag{6}$$

where the parameter ε is determined by the expression

$$\varepsilon = \rho k h I_0'\left(kR\right) \cdot \frac{1}{\rho_0 I_0\left(kR\right)}$$

and

$$\Omega^2 = \left(kD\left(k^4 - \frac{2vk^2}{R^2} + \frac{12\left(1-v^2\right)}{h^2R^2}\right) - 2k\frac{P_{om}}{R}\right) \cdot \frac{1}{\rho k h + \rho_0 \frac{I_0(kR)}{I'_0(kR)}}.$$

If the fluid is stationary, then equation (6) is simplified and its roots are equal

$$\omega = \sqrt{\Omega^2}\,.$$

This shows that a sufficiently strong current can lead to instability of the oscillations of the type under consideration. This type of instability is of the same nature as the instability of the “constriction” type in the ideally diamagnetic direct pinch. The presence of a fluid flow changes the condition of instability. Indeed, the roots of the dispersion equation in this case have the form

$$\omega = \frac{kV}{1+\varepsilon} \pm \sqrt{\Omega^2 - \frac{\varepsilon}{\left(1+\varepsilon\right)^2}k^2V^2}\,.$$

Therefore, instability can occur even in the case, where $\Omega^2 > 0$. Thus, the longitudinal current and the flow fluid reduce the stability of the shell to the considered form of disturbance. In the second case, for the disturbance

$$\xi = \xi_0 \exp\left(in\varphi - i\omega t\right)$$

the equation of radial oscillations of the shell has the form [8], [9]

$$\rho h\frac{\partial^2\xi}{\partial t^2} + \frac{D}{R^4}\frac{\partial^4\xi}{\partial\varphi^4} = P_r - P_m. \tag{7}$$

To determine the hydrodynamic pressure, we solve the boundary value problem

$$\Delta P = 0 \ \text{ at } \ r < R,$$

$$\frac{\partial P}{\partial r} = -\rho_0 \frac{\partial^2 \xi}{\partial t^2} \text{ at } r = R.$$

Hence we get that

$$P_r = \rho_0 \frac{\omega^2}{n} R\xi. \tag{8}$$

The flow rate does not affect the disturbance of the type in question. Inserting into the oscillation equation (7) the expressions for P_r from (4) and (8) and taking into account that $\xi \sim \exp\left(i\left(n\varphi - \omega t\right)\right)$, we obtain the dispersion equation of the problem in the form

$$\omega^2 \left(n\rho h + \rho_0 R\right) = \frac{Dn^5}{R^4} + 2P_{om}\frac{n\left(n-1\right)}{R}.$$

The roots of this equation are equal to:

$$\omega = \sqrt{\frac{Dn^5 + 2P_{om}R^3 n\left(n-1\right)}{R^4\left(n\rho h + \rho_0 R\right)}}.$$

This means that disturbance proportional to $\exp\left(in\varphi\right)$ turns out to be stable.

4. Conclusion

It is shown that the disturbances of the form $\exp\left(i\omega t - in\varphi\right)$ are always stable, and also define the conditions of instability for perturbations of the form $\exp\left(i\omega t - ikz\right)$.

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