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JUMP-DIFFUSION PROCESS OF INTEREST RATES AND THE MALLIAVIN CALCULUS

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Abstract: In this paper, we employ the existing Hull-White short rate model to derive an interest rate model driven by jump-diffusion process. Interest rates experience both positive and negative jumps at some intervals as a result of several factors which include natural disasters and presence of pandemics such as corona virus. Much has been done in the modelling of interest rates driven by Brownian motion process whereas little emphasis are laid on jumps inherent in the interest rates. For efficient modelling and pricing of financial derivatives, there is need to consider the aspect of jumps. Hence, this paper bridges the gap by focusing on an improved model. Sensitivities namely 'delta', 'vega', 'Theta' and 'Gamma' of the new model are also derived using Malliavin calculus.

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1. Introduction

Adequate modelling of interest rates is as important as what engine is to a vehicle. Thus, the importance of accurate modelling of interest rates cannot be

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overemphasized. A lot of factors contribute to abnormal behaviour of interest rates. These factors include government policies, attitude of investors, inflation, natural disaster, etc. For example, the presence of corona virus which started towards the end of year 2019 and entered the year 2020, affected the economy of almost every country. This has contributed to the presence of negative jumps. A good investor requires adequate understanding of movements of interest rates to avoid unnecessary risks. In a quest of handling jumps in financial assets, jump-diffusion processes were considered for better modelling. Thus, Merton [14] derived an option pricing formula for underlying stock returns generated by combination of continuous and jump processes. He suggested that equal analysis applied to the options can be extended to the valuation of corporate liabilities. Kou [11] proposed a double exponential jump-diffusion model for the pricing of options while in Kou [12], he discussed application of a jump-diffusion model in valuing assets in financial engineering. Since then, different researchers have worked on the applications of the jump-diffusion model in different fields. Cartea and Figueroa [6], Meyer-Brandis and Tankov [15] and Kegnenlezom et al. [10] applied the model in the evaluation of electricity prices. Berhane et al. [3] used the model in the modelling of Ethiopian commodity prices to reduce risks caused by spikes in the prices. Carr and Mayo [5] discussed numerical pricing of options with partial integral differential equation when density functions of jump processes involve Gaussian, exponential and polynomials. Feng and Linetsky [8] proposed a computational technique involving partial integro differential equation for valuing options involving jump-diffusion processes. Bates [1] introduced a method for valuing stochastic volatility models driven by the jump-diffusion processes. Su et al. [19] applied the jump-diffusion model in the valuation of warrant bond. Ruf and Scherer [17] discussed valuing of a corporate bond and how stochastic recovery rates are modelled using the model. Novat et al. [16] applied Merton's jump-diffusion model to stock market by considering certain stocks of three East African countries. Salmi and Toivanen [18] discussed an iterative procedure for valuing American options driven by jump-diffusion processes, while Wang et al. [20] discussed valuation of vulnerable American options driven by such processes. Moreover, Jiahui et al. [9] derived a closed form valuation formula for that of European option involving credit and jump risks under inadequate information.

In this paper, we shall apply the jump-diffusion model in the modelling of interest rates by considering Hull-White [13] short rate model. The choice of the Hull-White model is because it has the property of mean-reversion of the Ornstein-Uhlenbeck processes, and such property is common in interest rates. The Hull-White [13] model was derived under a Brownian motion process, hence it does not capture possibility of jumps inherent in the interest rates. Malliavin calculus will be applied in deriving expressions for greeks, namely delta, vega, Theta and Gamma.

The rest of the paper is arranged as follows: in Section 2, we discuss the important mathematical tools used in the work. We present and discuss our result in Section 3. Conclusion is drawn in Section 4. In the following section, we discuss the important mathematical tools needed for the success of the paper.

2. Mathematical Tools

In this section, we discuss Itô's formula for jump-diffusion process given by Cont and Tankov [7], and the Malliavin calculus (Bavouzet and Messaoud [2]). The jump-diffusion process involves a compound Poisson process given by

$$J_t = \sum_{i=1}^{N_t} \Delta_i,$$

where N_t is a Poisson process that counts the random number of jumps up to time t.

Here $\Delta_i = \Lambda(t_i) - \Lambda(t_{i_-})$ represents the jump size or amplitude, and the jump sizes are independent and identically distributed. $E[(\cdot)]$, W_t and $C^{1,2}$ denote expectation of $(\cdot)$, Wiener process and twice differentiable functions, respectively.

2.1. Itô's formula for jump-diffusion process

Let Λ be a diffusion process with jumps defined as

$$\Lambda_t = \Lambda_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta\Lambda_i,$$

where a_s and σ_s are non-anticipating processes with $E[\int_0^T \sigma_t^2 dt] < \infty$.

Then, for each $C^{1,2}$ function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process $f(t, \Lambda_t)$ can be written as

$$\begin{aligned} f(t,\Lambda_t) = f(0,\Lambda_0) &+ \int_0^t \left[\frac{\partial f}{\partial s}(s,\Lambda_s) + a_s \frac{\partial f}{\partial \wedge}(s,\Lambda_s)\right] ds \\ &+ \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial \wedge^2}(s,\Lambda_s) ds + \int_0^t \frac{\partial f}{\partial \wedge}(s,\Lambda_s) dW_s \end{aligned}$$

$$+ \sum_{\{i\geq 1, T_i \leq t\}} [f(\Lambda_{T_{i\_}} + \Delta\Lambda_i) - f(\Lambda_{T_{i\_}})].$$

2.2. The Hull-White [13] model

This is given by

$$dr(t) = (\rho(t) - \eta(t)r(t))dt + \sigma(t)dW_t,$$

where $\rho(t)$, $\eta(t)$ and $\sigma(t)$ are deterministic functions of time. It is one of the short rate models with mean-reversion which is a common property of interest rates, that is, interest rates tends to revert back to certain mean after a long run. It has been shown in Brigo and Mercurio [4](pg. 73) that

$$r(t) = r(s)e^{-\eta(t-s)} + \int_s^t e^{-\eta(t-u)}\rho(u)du + \sigma \int_s^t e^{-\eta(t-u)}dW(u).$$

In the following section, we discuss Malliavin calculus stating some of its lemmas and theorem (without proof) needed in this paper. The proof of the lemmas and theorem can be seen in Bavouzet and Messaoud [2]. The notations used are mainly from Bavouzet and Messaoud [2].

2.3. The Malliavin calculus

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω, $\mathcal{F}$ and $\mathbb{P}$ denote the sample space, set of filtrations and probability measure. Let $(Y_n, n \in \mathbb{N})$ be a sequence of independent random variables having moments of any order where Y_n has density $\widehat{f}$ which is continuously differentiable on $\mathbb{R}$ such that $\varphi(z) = \frac{\partial \widehat{f}}{\partial z}$ has at most a polynomial growth. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the space of functions that are k times continuously differentiable ($f \in C^k(\mathbb{R}^n)$) with f and its derivatives up to order k having polynomial growth. Z denotes a Gaussian random variable.

Definition 1. A *simple functional* is a random variable $F = f(Y_1, ..., Y_n)$, where $f : \mathbb{R}^n \to \mathbb{R}$ represent specific measurable functions, for certain $n \in \mathbb{N}$. The set of simple functional $f \in C^k(\mathbb{R}^n)$ is represented as $S_{(n,k)}$.

Definition 2. A *simple process* of length n is a sequence of random variables $U = (U_i, i \leq n)$ such that $U_i = u_i(Y_1(w), ..., Y_n(w))$. The space of simple processes of length n where $u_i \in C^k(\mathbb{R}^n), i = 1, 2, ..., n$ is denoted by $P_{(n,k)}$. $U \in P_{(n,k)}$ implies that $U_i \in S_{(n,k)}$ for $i \in \mathbb{N}$.

Definition 3. An operator $D : S_{(n,1)} \to P_{(n,0)}$ is called the *Malliavin derivative operator*, if for $F = f(Y_1, ..., Y_n)$, we have

$$D_i f = \frac{\partial f}{\partial y_i}(Y_1(w), ..., Y_n(w)), \text{ where } DF := (D_i F)_{i \leq n} \in P_{(n,0)}.$$

Definition 4. An operator $\delta : P_{(n,1)} \to S_{(n,0)}$ is called the *Skorohod integral operator*, if

$$\begin{aligned}\delta(U) &= -\sum_{i=1}^{n}(D_i U_i + \varphi_i(Z)U_i) \\ &= -\sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial y_i}(Y_1, ..., Y_n) + \varphi_i(Z)u_i(Y_1, ..., Y_n)\right),\end{aligned}$$

where

$$\varphi_i(z) = \frac{\partial \ln \widehat{f}_i(z)}{\partial z} \text{ if } \widehat{f}_i(z) > 0$$
$$0 \text{ if } \widehat{f}_i(z) = 0.$$

Definition 5. Let $L : S_{(n,2)} \to S_{(n,0)}$. The operator L is called an *Ornstein-Uhlenbeck operator* given by

$$LF = -\sum_{i=1}^{n}[D_i D_i F + \varphi_i D_i F], \ i = 1, 2, ..., n.$$

Lemma 6. *(i) For $F \in S_{(n,1)}$ and $U \in P_{(n,1)}$, we have*

$$E[\langle DF, U\rangle] = E[F\delta(u)],$$

where $\langle \cdot, \cdot \rangle$ represents the scalar product in $\mathbb{R}^n$.

(ii) If $\Phi \in C_b^1(\mathbb{R}^d)$ where b denotes bounded derivatives then $\Phi(F) \in S_{(n,1)}$ and the Malliavin derivative satisfies

$$D\Phi(F) = \sum_{k=1}^{d} \partial_k \Phi(F) DF^k.$$

(iii) If $F, G \in S_{(n,2)}$ then

$$L[F, G] = FLG + GLF - 2\langle DF, DG\rangle,$$

where L denotes Ornstein-Uhlenbeck operator.

Definition 7. Let $F = (F^1, ..., F^d)$ and $F^i \in S^d_{(n,1)}$, then the *Malliavin covariance matrix* is defined as

$$\varpi_F^{i,j} := \langle DF^i, DF^j \rangle, \text{ where } F^i = f^i(Y_1, ..., Y_n).$$

Theorem 8. (cf. Bavouzet-Morel and Messaoud [2] 'The Malliavin integration by parts formula') *For* $F = (F^1, ..., F^d) \in S^d_{(n,2)}$ *and* $G \in S_{(n,1)}$, *let the matrix* ϖ_F *be invertible written as* ϖ_F^{-1}. *Let* $E[(\det \varpi_F^{-1})^4] < \infty$, *then for each smooth function* $\Phi : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2, ..., d$,

$$E[\partial_i \Phi(F)G] = E[\Phi(F)H^i(F, G)],$$

where

$$H^i(F, G) = \sum_{j=1}^{d} G\varpi_{ji}^{-1}(F)LF^j - \varpi_{ji}^{-1}(F)\langle DF^j, DG\rangle - G\langle DF^j, D\varpi_{ji}^{-1}(F)\rangle$$

is known as the Malliavin weight.

3. Results and Discussion

In our results, we consider a modified Hull-White model for the interest rate $r(t) = r_t$ given by

$$dr(t) = (\rho - \eta r(t))dt + \sigma dW_t + dJ_t,$$

where ρ, η and σ are constants denoting long-term mean rate, speed of reversion of mean and volatility of the interest rate, respectively. W_t represents Wiener process while J_t is a compound Poisson process given by $J_t = \sum_{i=1}^{N_t} \Delta_i$ where Δ_i and N_t denote jump size at time t_i and number of jumps, respectively. T_i represent the times that the jumps occurred.

3.1. Interest rate dynamics driven by pure jump and jump-diffusion processes

Theorem 9. *Let* $r(t) = r_t$ *be an interest rate driven by pure jump process where* ρ, σ *and* η *represent its long-run mean rate, volatility and mean-reversion rate, respectively. Then, its dynamics defined as*

$$dr(t) = (\rho - \eta r(t))dt + \sigma dJ_t$$

satisfies

$$r(t) = r_0 e^{-\eta t} + \frac{\rho}{\eta}(1 - e^{-\eta t}) + \sigma \sum_{i=1}^{n} \Delta r_i e^{-\eta(t-T_i)}. \tag{1}$$

Proof. Using Itô's formula,

$$f(t,r) = r_t e^{\eta t} = r_0 + \int_0^t \frac{\partial f}{\partial s} ds + \int_0^t (\rho - \eta r_s) \frac{\partial f}{\partial x} ds + \sigma \sum_{i \geq 1, T_i \leq t} [f(s, r_{s_-} + \Delta r_s) - f(s, r_{s_-})].$$

Thus,

$$\begin{aligned} r_t e^{\eta t} &= r_0 + \int_0^t \eta r_s e^{\eta s} ds + \int_0^t (\rho - \eta r_s) e^{\eta s} ds \\ &\quad + \sigma \sum_{i \geq 1, T_i \leq t} [r_{s_-} e^{\eta s -} + \Delta r_s e^{\eta s} - r_{s_-} e^{\eta s -}] \\ &= r_0 + \frac{\rho}{\eta}(e^{\eta t} - 1) + \sigma \sum_{i \geq 1, T_i \leq t} \Delta r_i e^{\eta T_i} \\ &= r_0 + \frac{\rho}{\eta}(e^{\eta t} - 1) + \sigma \sum_{i=1}^{N_t} \Delta r_i e^{\eta T_i}. \end{aligned}$$

Hence,

$$r_t = r_0 e^{-\eta t} + \frac{\rho}{\eta}(1 - e^{-\eta t}) + \sigma \sum_{i=1}^{N_t} \Delta r_i e^{-\eta(t-T_i)}$$

which can be written as Eq. (1) □

Theorem 10. *Let an interest rate dynamics driven by pure jump-diffusion process be given by*

$$dr(t) = (\rho(t) - \eta r(t))dt + \sigma(t) dW_t + dJ_t,$$

where $\rho(t)$, $\eta(t)$ and $\sigma(t)$ are deterministic functions of time. Then

$$r(t) = r_0 e^{-\eta t} + \int_0^t e^{-\eta(t-s)} \rho(s) ds + \int_0^t \sigma_s e^{-\eta(t-s)} dW_s + \sum_{i=1}^{n} \Delta r_i e^{-\eta(t-T_i)}. \tag{2}$$

Proof. Let $f(t, r_t) = re^{\eta t}$. Then, $\dfrac{\partial f}{\partial t} = \eta r e^{\eta t}$, $\dfrac{\partial f}{\partial r} = e^{\eta t}$, $\dfrac{\partial^2 f}{\partial t^2} = 0$.
By Itô's formula,

$$\begin{aligned} r_t e^{\eta t} &= r_0 + \int_0^t \big[r_s e^{\eta s}\eta + e^{\eta s}(\rho(s) - \eta r_s)\big]ds + \int_0^t \sigma_s e^{\eta s} dW_s \\ &\quad + \sum_{i\geq 1, T_i \leq t} \big[r_s e^{\eta s} + \Delta r_i e^{\eta T_i} - r_s e^{\eta s}\big] \\ &= r_0 + \int_0^t e^{\eta s}\rho(s)ds + \int_0^t \sigma_s e^{\eta s} dW_s + \sum_{i=1}^{N_t} \Delta r_i e^{\eta T_i}. \end{aligned}$$

With $N_t = n \neq 0$, we have

$$r_t = r_0 e^{-\eta t} + \int_0^t e^{\eta(s-t)}\rho(s)ds + \int_0^t \sigma_s e^{\eta(s-t)} dW_s + \sum_{i=1}^{n} \Delta r_i e^{\eta(T_i - t)}.$$

□

Remark 11. To compute European option price with maturity T on the interest rate, from Eq. (2) we obtain

$$\begin{aligned} r_T = r_0 e^{-\eta T} &+ \int_0^T e^{-\eta(T-s)}\rho(s)ds + \int_0^T \sigma_s e^{-\eta(T-s)} dW_s \\ &+ \sum_{i=1}^{n} \Delta r_i e^{-\eta(T-T_i)}. \end{aligned}$$

Theorem 12. *The interest rate dynamics driven by jump-diffusion process given by*

$$dr(t) = (\rho - \eta r(t))dt + \sigma dW_t + dJ_t,$$

where ρ, η and σ are constants, satisfy

$$\begin{aligned} r(t) = r_0 e^{-\eta t} &+ \frac{\rho}{\eta}(1 - e^{-\eta t}) + \sigma \sum_{j=1}^{m} e^{-\eta(t-t_j)}(W(t_j) - W(t_{j-1})) \\ &+ \sum_{i=1}^{n} \Delta r_i e^{-\eta(t-T_i)}. \end{aligned} \tag{3}$$

Proof. From the condition given in the theorem,

$$dr_t = (\rho - \eta r_t)dt + \sigma dW_t + dJ_t.$$

Applying Itô's formula,

$$\begin{aligned} r_t &= r_0 e^{-\eta t} + e^{-\eta t}\rho\int_0^t e^{\eta s}ds + \int_0^t \sigma e^{-\eta(t-s)}dW_s + \sum_{i\geq 1, T_i\leq t}\Delta r_i e^{-\eta(t-T_i)} \\ &= r_0 e^{-\eta t} + \frac{\rho}{\eta}e^{-\eta t}(e^{\eta t}-1) + \sigma\int_0^t e^{-\eta(t-s)}dW_s + \sum_{i=1}^{N_t}\Delta r_i e^{-\eta(t-T_i)} \\ &= r_0 e^{-\eta t} + \frac{\rho}{\eta}(1-e^{-\eta t}) + \sigma\sum_{j=1}^{m} e^{-\eta(t-t_j)}(W(t_j)-W(t_{j-1})) \\ &\qquad + \sum_{i=1}^{n}\Delta r_i e^{-\eta(t-T_i)}. \end{aligned}$$

□

Remark 13. To compute European option price with maturity T on the interest rate, we obtain from Eq. (3) that

$$\begin{aligned} r_T = r_0 e^{-\eta T} &+ \frac{\rho}{\eta}(1-e^{-\eta T}) + \sigma\sum_{j=1}^{m} e^{-\eta(T-t_j)}(W(t_j)-W(t_{j-1})) \\ &+ \sum_{i=1}^{n}\Delta r_i e^{-\eta(T-T_i)}. \end{aligned} \tag{4}$$

We proceed to compute the sensitivities delta '$\triangle$', vega '$\mathcal{V}$' and 'Theta' of an European call option on the interest rate. We denote $E[\cdot]$ as the expected pay-off function and $\Phi(r_t) = \max(0, r_t - K)$ where K is the strike price. Also, $W(t) = \sqrt{t}Z$.

3.2. Sensitivity analysis of the interest rates

To compute the greeks, namely 'delta', 'vega', 'Theta' and 'Gamma' of the interest rate, $F = r_T \in S_{(n,2)}$ and $G = \dfrac{\partial r_T}{\partial \alpha} \in S_{(n,1)}$ are functions of $(t_1, ..., t_n, Z_1, ..., Z_n, \Delta_1, ..., \Delta$
where α denotes parameter of the interest rate whose effect is to be determined.

Lemma 14. *Let an interest rate be given by Eq. (4). Then the following holds:*

(i) Its Malliavin derivative gives

$$DF^W = \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)}, \tag{5}$$

$$DF^{\Delta_i} = \sum_{i=1}^{n} e^{-\eta(T-T_i)}. \tag{6}$$

(ii) The Ornstein-Uhlenbeck operator on the interest rate gives

$$Lr_T = Lr_T^W + Lr_T^{\Delta_i}$$

$$= \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j + \sum_{i=1}^{n} \Delta_i e^{-\eta(T-T_i)},$$

where

$$Lr_T^W = \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j \tag{7}$$

and

$$Lr_T^{\Delta_i} = \sum_{i=1}^{n} \Delta_i e^{-\eta(T-T_i)}. \tag{8}$$

Proof. (i) Eq.(5) and Eq.(6) are obtained when the Malliavin derivative operator acts on the interest rate given by Eq. (4).

(ii) The Ornstein-Uhlenbeck operator is given by

$$Lr_T = -[DDr_T + \varphi Dr_T] = -\sum_{i=1}^{n} [D_i D_i r_T + \varphi_i D_i r_T],$$

where $\varphi_i = \frac{\partial \ln f(z)}{\partial z} = \frac{\partial}{\partial z} \ln \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right) = -z.$

Thus, Eq. (7) and Eq. (8) are obtained as

$$Lr_T^W = \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j$$

and

$$Lr_T^{\Delta_i} = \sum_{i=1}^{n} \Delta_i e^{-\eta(T-T_i)}, \text{ respectively.}$$

Hence, the Ornstein-Uhlenbeck operator on r_T gives

$$\begin{aligned} Lr_T &= Lr_T^W + Lr_T^{\Delta_i} \\ &= \sum_{j=1}^{m} \sigma \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j + \sum_{i=1}^{n} \Delta_i e^{-\eta(T-T_i)}. \end{aligned}$$

□

Lemma 15. *Let r_t be given by Eq. (4), then its inverse Malliavin covariance matrix satisfies the following:*

$$(\varpi_j^{-1})^W = \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)}}{\sigma^2(t_j - t_{j-1})}, \tag{9}$$

$$(\varpi_i^{-1})^{\Delta_i} = \sum_{i=1}^{n} e^{2\eta(T-T_i)}. \tag{10}$$

Furthermore, the Malliavin derivative of the inverse Malliavin covariance Matrix is given by

$$D(\varpi^{-1})^W = 0 = D(\varpi^{-1})^{\Delta_i}. \tag{11}$$

Proof. The Malliavin covariance matrix of r_T is given by

$$\varpi = \langle DF^W, DF^W \rangle + \langle DF^{\Delta_i}, DF^{\Delta_i} \rangle = \varpi^W + \varpi^{\Delta_i}.$$

The Malliavin covariance matrix gives

$$\varpi^W = \langle DF^W, DF^W \rangle = \sigma^2 \sum_{j=1}^{m} (t_j - t_{j-1}) e^{-2\eta(T-t_j)}$$

and

$$\varpi^{\Delta_i r} = \langle DF^{\Delta_i}, DF^{\Delta_i} \rangle = \sum_{i=1}^{n} e^{-2\eta(T-T_i)}.$$

Whence,

$$(\varpi^{-1})^W = \frac{1}{\sigma^2} \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)}}{(t_j - t_{j-1})}, \quad (\varpi^{-1})^{\Delta_i} = \sum_{i=1}^{n} e^{2\eta(T-T_i)}.$$

Hence, the Malliavin derivative on the inverse covariance matrix gives

$$D(\varpi^{-1})^W = D(\varpi^{-1})^{\Delta_i} = 0.$$

□

3.2.1. Derivation of delta

The greek 'delta' measures the sensitivity of the interest rate to change in its initial value. The value of an European call option price on r_T is given by

$$V = E[\Phi(r_T)] = E[\max(r_T - K, 0)],$$

where r_0 is the initial initial interest rate, $\Phi(r_T) = \max(r_T - K, 0)$ is payoff of the European call option price, T is the maturity date and K is the call option price.

The greek 'delta $\triangle$' is given by

$$\triangle = \frac{\partial}{\partial r_0}\big(E[\Phi(r_T)]\big) = E\left[\Phi'(r_T)\frac{\partial r_T}{\partial r_0}\right] = E\left[\Phi(r_T)H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right],$$

where $H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)$ is for the Malliavin weight.

From r_T given by Eq. (4), we obtain

$$G = \frac{\partial r_T}{\partial r_0} = e^{-\eta T}, \tag{12}$$

$$DG = 0. \tag{13}$$

We proceed to derive the Malliavin weight for the greek 'delta'.

Theorem 16. *Let r_T be an interest rate given by Eq. (4). Then, the Malliavin weight for the greek 'delta' of the interest rate is given by*

$$H\left(r_T, \frac{\partial r_T}{\partial r_0}\right) = \sum_{j=1}^{m} \frac{e^{-\eta T}}{\sigma\sqrt{t_j - t_{j-1}}} e^{\eta(T-t_j)} Z_j + e^{-\eta T}\sum_{i=1}^{n} e^{\eta(T-T_i)}\Delta_i.$$

Proof. The Malliavin weight $H(F, G)$ satisfies

$$H(F, G) = G\varpi^{-1}(F)LF - \varpi^{-1}(F)\langle DF, DG\rangle - G\langle DF, D\varpi^{-1}(F)\rangle,$$

where $G = \frac{\partial r_T}{\partial r_0}$ and $F = r_T$.

Since $D\varpi^{-1}(F) = D\varpi^{-1}(r_T) = 0$ in Eq. (11) and $DG = 0$ in Eq. (13), it follows that

$$H(F, G) = G\varpi^{-1}(F)LF.$$

From Equations (12), (9) and (7), we obtain

$$H(F,G)^W = e^{-\eta T} \cdot \frac{1}{\sigma^2} \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)}}{(t_j - t_{j-1})} \cdot \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j$$

$$= \sum_{j=1}^{m} \frac{e^{-\eta T}}{\sigma\sqrt{t_j - t_{j-1}}} e^{\eta(T-t_j)} Z_j.$$

From Equations (12), (10) and (8), we obtain

$$H(F,G)^{\Delta_i} = e^{-\eta T} \cdot \sum_{i=1}^{n} e^{2\eta(T-T_i)} \cdot \Delta_i e^{-\eta(T-T_i)} = e^{-\eta T} \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i.$$

Therefore,

$$\begin{aligned} H(F,G) &= H(F,G)^W + H(F,G)^{\Delta_i} \\ &= \sum_{j=1}^{m} \frac{e^{-\eta T}}{\sigma\sqrt{t_j - t_{j-1}}} e^{\eta(T-t_j)} Z_j + e^{-\eta T} \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i. \end{aligned}$$

□

3.2.2. Derivation of vega

The greek 'vega $\mathcal{V}$' measures the sensitivity of the interest rate to changes in its volatility. It is given by

$$\mathcal{V} = \frac{\partial}{\partial \sigma}\big(E[\Phi(r_T)]\big) = E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial \sigma}\right)\right],$$

where $H\left(r_T, \frac{\partial r_T}{\partial \sigma}\right)$ is the Malliavin weight for vega.

From Eq. (4),

$$G_\sigma = \frac{\partial G}{\partial \sigma} = \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j, \tag{14}$$

$$DG_\sigma = \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}}. \tag{15}$$

We derive the Malliavin weight for the greek 'vega'.

Theorem 17. *Let r_T be an interest rate given by Eq.(4). Then, the Malliavin weight for the greek 'vega' is given by*

$$H(F, G_\sigma) = \sum_{j=1}^{m} \frac{1}{\sigma}(Z_j^2 - 1) + \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i$$
$$- \sigma \sum_{i=1}^{n} e^{2\eta(T-T_i)} \sum_{j=1}^{m} e^{-2\eta(T-t_j)}(t_j - t_{j-1}).$$

Proof. The Malliavin weight is given by

$$H(F, G_\sigma) = G_\sigma \varpi^{-1}(F) LF - \varpi^{-1}(F)\langle DF, DG_\sigma\rangle - G_\sigma \langle DF, D\varpi^{-1}(F)\rangle,$$

where $G_\sigma = \dfrac{\partial r_T}{\partial \sigma}$ and $F = r_T$.

Since $D\varpi^{-1}(F) = D\varpi^{-1}(r_T) = 0$ in Eq. (11), it follows that

$$H(F, G_\sigma) = G_\sigma \varpi^{-1}(F) LF - \varpi^{-1}(F)\langle DF, DG_\sigma\rangle.$$

From Equations (14), (10), (7), (5) and (15), we have

$$H(F, G_\sigma)^W = \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j \cdot \frac{1}{\sigma^2} \frac{e^{2\eta(T-t_j)}}{(t_j - t_{j-1})}$$
$$\cdot \sigma \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j \; - \; \frac{1}{\sigma^2} \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)}}{t_j - t_{j-1}}$$
$$\cdot \sigma e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} \cdot e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}}$$

$$= \sum_{j=1}^{m} \frac{Z_j^2}{\sigma} - \frac{1}{\sigma}.$$

Furthermore, from Equations (14), (10), (8), (6) and (15), we have

$$H(F, G_\sigma)^{\Delta_i} = \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j \cdot \sum_{i=1}^{n} e^{2\eta(T-T_i)} \cdot \Delta_i e^{-\eta(T-T_i)}$$
$$- \sum_{i=1}^{n} e^{2\eta(T-T_i)} \cdot \sigma \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}}$$

$$\cdot \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}}$$

$$= \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i$$

$$- \sigma \sum_{i=1}^{n} e^{2\eta(T-T_i)} \sum_{j=1}^{m} e^{-2\eta(T-t_j)} (t_j - t_{j-1}).$$

Therefore,

$$H(F, G_\sigma) = H(F, G_\sigma)^W + H(F, G_\sigma)^{\Delta_i}$$

$$= \sum_{j=1}^{m} \frac{Z_j^2}{\sigma} - \frac{1}{\sigma} + \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i$$

$$- \sigma \sum_{i=1}^{n} e^{2\eta(T-T_i)} \sum_{j=1}^{m} e^{-2\eta(T-t_j)} (t_j - t_{j-1}).$$

□

3.2.3. Derivation of Theta

The greek 'Theta' measures how the option price on the interest rate depreciates as time to maturity draws near. It is given by

$$\text{Theta} = \Theta = \frac{\partial}{\partial T} E[\Phi(r_T)] = E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial T}\right)\right],$$

where $H(r_T, G_T) = H\left(r_T, \frac{\partial r_T}{\partial T}\right)$ is the Malliavin weight for Theta.

We proceed to derive the Malliavin weight for Theta. From Eq. (4),

$$G_T = (\rho - \eta r_0) - \sigma\eta \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}} Z_j - \eta \sum_{i=1}^{n} \Delta_i e^{-\eta(T-T_i)}. \tag{16}$$

The Malliavin derivative on B_T gives

$$DG_T^W = -\eta\sigma \sum_{j=1}^{m} e^{-\eta(T-t_j)} \sqrt{t_j - t_{j-1}}, \tag{17}$$

$$DG_T^{\Delta_i} = -\eta \sum_{i=1}^{n} e^{-\eta(T-T_i)}. \tag{18}$$

Theorem 18. *Let the interest rate r_T be given by Eq.(4), then its Malliavin weight with respect to the greek 'Theta' gives*

$$H(F,G_T) = H\left(r_T, \frac{\partial r_T}{\partial T}\right) = \left(\left(\frac{e^{-\eta T}}{\sigma}(\rho - \eta r_0) - \eta\left[\sum_{j=1}^{m} e^{-\eta(T-t_j)}\right.\right.\right.$$

$$\left.\left.\left. \cdot\sqrt{t_j - t_{j-1}}Z_j - \frac{1}{\sigma}\sum_{i=1}^{n}\Delta_i e^{-\eta(T-T_i)}\right]\right)\sum_{j=1}^{m}\frac{1}{\sqrt{t_j - t_{j-1}}}e^{\eta(T-t_j)}Z_j\right)$$

$$+ \left((\rho - \eta r_0)e^{-\eta T} - \sigma\eta\sum_{j=1}^{m} e^{-\eta(T-t_j)}\sqrt{t_j - t_{j-1}}Z_j\right.$$

$$\left. - \eta\sum_{i=1}^{n}\Delta_i e^{-\eta(T-T_i)}\right)\cdot\sum_{i=1}^{n}\Delta_i e^{\eta(T-T_i)} + 2\eta.$$

Proof. $H(F,G_T) = H(r_T, G_T)$

$$= G_T\varpi^{-1}(F)LF - \varpi^{-1}(F)\langle DF, DG_T\rangle - G_T\langle DF, D\varpi^{-1}(F)\rangle$$

$$= H\left(r_T, \frac{\partial r_T}{\partial T}\right) = H^W(F,G_T) + H^{\Delta_i}(F,G_T).$$

Recall from Eq. (11) that $D\varpi^{-1}(F) = 0$, thus

$$H(F,G_T) = G_T\varpi^{-1}(F)LF - \varpi^{-1}(F)\langle DF, DG_T\rangle$$
$$= H^W(F,G_T) + H^{\Delta_i}(F,G_T).$$

From Equations (16), (10), (7), (5) and (17), we get

$$H^W(F,G_T) = \left(\left(\frac{e^{-\eta T}}{\sigma}(\rho - \eta r_0) - \eta\left[\sum_{j=1}^{m} e^{-\eta(T-t_j)}\sqrt{t_j - t_{j-1}}Z_j\right.\right.\right.$$

$$\left.\left.\left. -\frac{1}{\sigma}\sum_{i=1}^{n}\Delta_i e^{-\eta(T-T_i)}\right]\right)\sum_{j=1}^{m}\frac{1}{\sqrt{t_j - t_{j-1}}}e^{\eta(T-t_j)}Z_j\right) + \eta.$$

Moreover, from Equations (16), (10), (9), (6) and (18), we get

$$H^{\Delta_i}(F,G_T) = \left((\rho - \eta r_0)e^{-\eta T} - \sigma\eta\sum_{j=1}^{m} e^{-\eta(T-t_j)}\sqrt{t_j - t_{j-1}}Z_j\right.$$

$$\left. - \eta\sum_{i=1}^{n}\Delta_i e^{-\eta(T-T_i)}\right)\cdot\sum_{i=1}^{n}\Delta_i e^{\eta(T-T_i)} + \eta.$$

Adding $H^W(F,G_T)$ and $H^{\Delta_i}(F,G_T)$ gives the Malliavin weight. □

3.2.4. Derivation of Gamma

The greek 'Gamma Γ' measures how sensitive the option price on the interest rate is to change in 'delta'. It is given by

$$\Gamma = \frac{\partial^2 V}{\partial r_0^2} = \frac{\partial}{\partial r_0} E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right]$$

$$= E\left[\Phi'(r_T)\frac{\partial r_T}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right] + E\left[\Phi(r_T)\frac{\partial}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right]$$

$$= E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right)\right] + E\left[\Phi(r_T)\frac{\partial}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right].$$

But

$$E\left[\Phi(r_T)\frac{\partial}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right] = 0.$$

Hence,

$$\Gamma = E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right)\right] = E[\Phi(r_T) H(r_T, G_\Gamma)],$$

where

$$G_\Gamma = \frac{\partial r_T}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right).$$

We proceed to obtain the Malliavin weight for the greek 'Gamma'.

From equation (4) and Malliavin weight for the greek 'delta', we obtain

$$G_\Gamma^W = \frac{e^{-2\eta T}}{\sigma} \sum_{j=1}^{m} \frac{e^{\eta(T-t_j)} Z_j}{\sqrt{t_j - t_{j-1}}}. \tag{19}$$

$$G_\Gamma^{\Delta_i} = e^{-2\eta T} \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i. \tag{20}$$

$$DG_\Gamma^W = \frac{e^{-2\eta T}}{\sigma} \sum_{j=1}^{m} \frac{e^{\eta(T-t_j)}}{\sqrt{t_j - t_{j-1}}}. \tag{21}$$

$$DG_\Gamma^{\Delta_i} = e^{-2\eta T} \sum_{i=1}^{n} e^{\eta(T-T_i)}. \tag{22}$$

Theorem 19. *Let the interest rate r_T be given by Eq.(4), then its Malliavin weight with respect to the greek 'Gamma' is given by*

$$H\left(r_T, \frac{\partial r_T}{\partial r_0} H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right) = H(F, G_\Gamma) = \frac{e^{-2\eta T}}{\sigma}\left(\frac{1}{\sigma}\sum_{j=1}^{m} e^{2\eta(T-t_j)} \cdot\left(\frac{Z_j^2}{t_j - t_{j-1}} - 1\right)\right) + e^{-2\eta T}\left(\sum_{i=1}^{n} e^{2\eta(T-T_i)}(\Delta_i^2 - 1)\right).$$

Proof. $H(F, G_\Gamma)$

$$= G_\Gamma \varpi^{-1}(F)LF - \varpi^{-1}(F)\langle DF, DG_\Gamma\rangle - G_\Gamma\langle DF, D\varpi^{-1}(F)\rangle$$
$$= H^W(F, G_\Gamma) + H^{\Delta_i}(F, G_\Gamma).$$

Recall from Eq. (10) that $D\varpi^{-1}(F) = 0$, thus

$$H(F, G_\Gamma) = G_\Gamma \varpi^{-1}(F)LF - \varpi^{-1}(F)\langle DF, DG_\Gamma\rangle$$
$$= H^W(F, G_\Gamma) + H^{\Delta_i}(F, G_\Gamma).$$

From Equations (19), (9), (7), (5) and (21), we get

$$H^W(F, G_\Gamma) = \sum_{j=1}^{m} \frac{e^{-2\eta T}}{\sigma\sqrt{t_j - t_{j-1}}} e^{\eta(T-t_j)} Z_j \cdot \frac{e^{2\eta(T-t_j)}}{\sigma^2(t_j - t_{j-1})} \cdot \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} Z_j - \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)}}{\sigma^2(t_j - t_{j-1})} \cdot \sigma \sum_{j=1}^{m} \sqrt{t_j - t_{j-1}} e^{-\eta(T-t_j)} \cdot \frac{e^{-2\eta T}}{\sigma\sqrt{t_j - t_{j-1}}} \sum_{j=1}^{m} e^{\eta(T-t_j)}$$

$$= \frac{e^{-2\eta T}}{\sigma^2} \sum_{j=1}^{m} \frac{e^{2\eta(T-t_j)} Z_j^2}{t_j - t_{j-1}} - \sum_{j=1}^{m} \frac{e^{-2\eta T}}{\sigma} \cdot e^{2\eta(T-t_j)}.$$

Hence,

$$H^W(F, G_\Gamma) = \frac{e^{-2\eta T}}{\sigma}\left(\frac{1}{\sigma}\sum_{j=1}^{m} e^{2\eta(T-t_j)}\left(\frac{Z_j^2}{t_j - t_{j-1}} - 1\right)\right). \tag{23}$$

Moreover, from Equations (20), (10), (8), (6) and (22), we get

$$H^{\Delta_i}(F, G_\Gamma) = e^{-2\eta T} \sum_{i=1}^{n} e^{\eta(T-T_i)} \Delta_i \cdot e^{2\eta(T-T_i)} \cdot \Delta_i e^{-\eta(T-T_i)}$$

$$- \sum_{i=1}^{n} e^{2\eta(T-T_i)} \cdot e^{-\eta(T-T_i)} \cdot e^{-2\eta T} e^{\eta(T-T_i)}$$
$$= e^{-2\eta T} \sum_{i=1}^{n} e^{2\eta(T-T_i)} \Delta_i^2 - e^{-2\eta T} \sum_{i=1}^{n} e^{2\eta(T-T_i)}.$$

Thus,

$$H^{\Delta_i}(F, G_\Gamma) = e^{-2\eta T} \left(\sum_{i=1}^{n} e^{2\eta(T-T_i)} (\Delta_i^2 - 1) \right). \tag{24}$$

Adding Equations (23) and (24) gives the result. □

4. Conclusion

In the paper, we have been able to derive a modified Hull-White model that takes care of pure jump and jump-diffusion processes of interest rates. Using Malliavin calculus for jump-diffusion processes as given by Bavouzet-Morel and Messaoud [2], we have derived the greeks 'delta', 'vega', 'Theta' and 'Gamma' which measures sensitivity of the interest rate to change in its initial value, the sensitivity of the interest rate with respect to change in its volatility, the sensitivity of the interest rate with respect to maturity time and the sensitivity of the interest rate with respect to change in 'delta', respectively. This is very useful in the process of reducing risk in a given portfolio. We recommend that this work can be extended to multivariate random variables.

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