ON FAITHFUL MATRIX REPRESENTATIONS OF
q-DEFORMED LIE ALGEBRA FOR COUPLED
QUANTIZED OSCILLATORS

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Abstract: The polynomially deformed Lie algebra, $\mathfrak{l}_q : [K_0, K_+]_q = G(K_+)$, $[K_-, K_0]_q = G(K_-), [K_+, K_-]_q = P(K_0)$, is introduced as a generalized model of the coupled quantized oscillators model, where G and P are real polynomial functions, subject to the physical properties: K_0 is a real diagonal operator, and $K_- = K_+^\dagger$. Matrix representations are discussed and conditions are given for G and P to guarantee the existence of the faithful representations.

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1. Introduction

Modeling of nonlinear quantum optical physics phenomena (like, multimode radiation fields on multiphoton processes mediums, many -body physics models [1], [2], investigating the statistics of particles that interpolate between Bosons and Fermions, [13]-[17], or study of q-deformed oscillators [12], [16]) is expressed as Hamiltonians whose non-linearities are of higher order than the quadratic forms. In turn, such highly nonlinear Hamiltonians are associated with nonlinear q-deformed Lie algebra or polynomially deformed Lie algebras $SU_{pd}(2)$, [3].

On the other hand, direct methods to solve for the evolution parameters or Schrödinger's wave equations of such nonlinear Hamiltonian models are tedious

in many cases . However, the use of Lie algebraic approach (e.g., [3] and references therein) in such nonlinear models has some advantages as compared with direct methods, (similar to the familiar Lie algebraic decomposition method for bilinear Hamiltonians, [21], [22]).

For such useful Lie algebraic decomposition methods, one requires faithful matrix representation for the generators of such deformed Lie algebra.

The intention of this paper is to consider the q-deformed algebra in a more general form that considers both types of deformation, namely, q-deformed Lie brackets and polynomially deformed Lie algebras, and investigate the faithful matrix representations of the generators of such deformed Lie algebras. Representations of degree 2 are the least possible degrees.

2. Preliminaries

Definition 1. Let X and Y be $n \times n$ matrices. The q-deformed Lie bracket of X and Y is defined as

$$[X, Y]_q = XY - qYX \quad \text{for} \quad q \in \mathbb{R}.$$

For $q = 0$, it is the ordinary matrix multiplication of X and Y. Thus, throughout this paper, q is supposed to be a nonzero real number. Whereas, for $q = 1$, it is the ordinary Lie bracket. Thus, we always, write $[X, Y]_1$ as $[X, Y]$. In [11], e.g., q was considered as $0 \leq q \leq 1$ for the model of Fermion Oscillators.

Faithful matrix representations of least degree of the Lie algebra $\mathfrak{l}$, were considered in [4]-[10], namely,

$$\mathfrak{l} : [K_0, K_\pm] = \pm r K_\pm \text{ and } [K_+, K_-] = P(K_0), \tag{1}$$

where P is a real polynomial function, subject to the physical properties, namely, K_0 is a real diagonal operator and $K_- = K_+^\dagger$ ($\dagger$ is for Hermitian conjugation), with $K_+ + K_-$ is a real operator, in order to satisfy the Hermiticity of the Hamiltonian of the coupled quantized optical atoms, namely, [10]

$$H = \omega K_0 + \lambda(t)(K_+ + K_-). \tag{2}$$

In this paper, we discuss, the faithful matrix representations for $\mathfrak{l}_q$, where

$$\mathfrak{l}_q : [K_0, K_+]_q = G(K_+), \tag{3}$$

$$[K_-, K_0]_q = G(K_-), \tag{4}$$

$$[K_+, K_-]_q = P(K_0), \tag{5}$$

subject to the physical properties, namely, $K_- = K_+^\dagger$ and K_0 is a real diagonal operator with $K_+ + K_-$ is a real operator, in order to satisfy the Hermiticity of the Hamiltonian (2), where P and are real polynomials.

In (3)-(5), $\mathfrak{l}_q$ is introduced as a more generalization of the models considered in [4]-[10]. It is the q-deformed and polynomially deformed model of $\mathfrak{l}$ in (1). As on setting $q = 1$, then $\mathfrak{l}_1$ becomes the Lie algebra $\mathfrak{l}$, where $G(x) = rx$. So, for $q = 1$ and with particular choices of G and P, one gets the models in [4]-[9] and their special models, e.g., in [27], [28] and also, obtains their matrix representations.

3. Basic properties of the q-deformed Lie Bracket for quantized Hamiltonians

Some basic properties of the q-deformed Lie bracket are listed in the following theorem,whose proof can be driven immediately from the definition.

Theorem 2. Let X,Y and Z be $n \times n$ matrices, $q \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$. Then:

1. $tr\left([X, Y]_q\right) = tr\left([Y, X]_q\right).$

2. $[X, Y]_q^\dagger = \left[Y^\dagger, X^\dagger\right]_q.$

3. $[X, Y]_q = q[X, Y] + (1 - q)XY.$

4. $[\alpha A, \beta B]_q = \alpha\beta[A, B]_q.$

5. $[X, Y]_q = [X, Y] + (1 - q)YX.$

6. $[X, X]_q = (1 - q)X^2.$

7. $[X, Y + Z]_q = [X, Y]_q + [X, Z]_q.$

8. $[Y, X]_q = [X, Y]_q - (1 + q)[X, Y].$

9. $[X + Y, Z]_q = [X, Z]_q + [Y, Z]_q.$

10. $[X, Y]_q = (1 + q)[X, Y] + [Y, X]_q.$

11. $[X, Y]_q = (1 + q) [X, Y] + [Y, X]_q$.

12. $[X, Y]_q = -q [Y, X]_q + (1 - q^2) XY$.

13. $[Y, X]_q = -q [X, Y]_q + (1 - q^2) YX$.

14. $[X, Y]_q - [Y, X]_q = (1 + q) [X, Y]$.

15. $\left[[X, Y]_q, Z \right]_q = XYZ - qYXZ - qZXY - q^2 ZYX$.

16. $\left[X, [Y, Z]_q \right]_q = XYZ - qYZX - qXZY + q^2 ZYX$.

17. $\left[[X, Y]_q, Z \right]_q - \left[X, [Y, Z]_q \right]_q = q [X, Z] Y + qY [Z, X]$.

18. $\left[X, [Y, Z]_q \right] - \left[[X, Y]_q, Z \right]_q = qY [X, Z] + q [Z, X] Y$.

19. $\left[X, [Y, Z]_q \right]_q + \left[Y, [Z, X]_q \right]_q + \left[Z, [X, Y]_q \right]_q$

$= (1 - q) [(XYZ + YZX + ZXY)$

$- q (XZY + YXZ + ZYX)]$.

Proposition 3. *The polynomial function $G(x)$ of x satisfies*

$$G(K_-) = [K_0, K_-]_q - (1 + q) [K_0, K_-]$$
$$= -q [K_0, K_-]_q + (1 - q^2) K_- K_0.$$

Proof. From (3), by using parts 2 and 10, of Theorem 2, we have

$$G(K_-) = (G(K_+))^\dagger = [K_0, K_+]_q^\dagger = \left[K_{++}^\dagger, K_0^\dagger \right]_q = [K_-, K_0]_q$$
$$= [K_0, K_-]_q + (1 + q) [K_-, K_0]$$

and from part (12) of Theorem 2, we have

$$[K_-, K_0]_q = -q [K_0, K_-]_q + (1 - q^2) K_- K_0.$$

$\square$

Theorem 4. *The defining relations of $\mathfrak{l}_q$ can be either:*

$$[K_0, K_+]_q = G(K_+), \text{ and } [K_+, K_-]_q = P(K_0), \tag{6}$$

or

$$[K_-, K_0]_q = G(K_-), \text{ and } [K_+, K_-]_q = P(K_0),$$

where P and G are real polynomials, subject to the physical properties, namely, $K_- = K_+^\dagger$ and K_0 is a real diagonal operator with $K_+ + K_-$ is a real operator.

4. Faithful representations of $\mathfrak{l}_q$

Unless otherwise stated, A, B and C are representation matrices for K_+, K_- and K_0, respectively. All representations into consideration are supposed to satisfy the physical properties, namely, $B = A^\dagger$ and C is a real diagonal matrix with $A + B$ is real a matrix. Also, $P(x)$ and $G(x)$ are polynomial functions in $\mathbb{R}[x]$.

The following lemma shows that, interchanging the position of two diagonal elements of C, leads to a conjugate representation of $\mathfrak{l}_q$ satisfying the physical requirements.

Lemma 5. *Let $\delta = (uv)$ be a permutation in the symmetric group S_n. When applying δ to the rows as well as to the columns of the $n \times n$ matrices A, B and C, we obtain a conjugate matrix representation of $\mathfrak{l}_q$ satisfying the physical requirements of degree n.*

Proof. Let E be the elementary matrix obtained by applying δ to the rows of I_n. Since $E = E^{-1} = E^T = E^\dagger$, then $A' = E^{-1}AE, B' = E^{-1}BE$ and $C' = E^{-1}CE$ are representation matrices for K_+, K_- and K_0, respectively. For, $[C', A']_q = (E^{-1}CE)(E^{-1}AE) - q(E^{-1}AE)(E^{-1}CE) = E^{-1}(CA - qAC)E = E^{-1}[C, A]_q E = E^{-1}G(A)E = G(E^{-1}AE) = G(A')$. Similarly, $[A', B']_q = P(C')$. The physical properties of $\mathfrak{l}_q$ are also satisfied in respective with the matrices A', B' and C'. $\square$

So, we consider C consists of k diagonal blocks of scalar matrices corresponding to different scalars. That is $C = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, ...,$

$\delta_k I_{n_k}$), for different real numbers $\delta_1, \delta_2, ..., \delta_k$ with $\sum_{i=1}^{k} n_i = n$, where n is the degree of the representation.

The following equations (7)-(9) are necessary relations for A, B and C, which are obtained from (3) and (5), respectively. For $i, j = 1, 2, ..., n$, we have

$$(G(A))_{ij} = a_{ij}(c_{ii} - qc_{jj}) \tag{7}$$

and

$$\sum_{t=1}^{n} (a_{it}\bar{a}_{jt} - qa_{tj}\bar{a}_{ti}) = \quad \text{for } i \neq j \tag{8}$$

$$\sum_{t=1}^{n} \left(|a_{it}|^2 - q|a_{ti}|^2 \right) = P(c_{ii}). \tag{9}$$

5. Faithful matrix representations of $\mathfrak{l}_q$ of degree 2, where F and G are real linear polynomials

Since $\mathfrak{l}_q$ is generated by 3 generators, namely, $K_\pm$ and K_0, then the least possible degree of a faithful matrix representation is 2. In such a case, A, $B = A^\dagger$ and $C = diag(c_1, c_2)$ with $c_1, c_2 \in \mathbb{R}$, are linearly independent 2×2 matrices satisfying the physical conditions. It should be noticed that when a contradiction occurs in a representation of degree 2, when $c_1 \neq c_2$, a contradiction is going to occur in representations of higher degrees between two different diagonal blocks of the representation matrix of K_0. Also, if a contradiction may occur when $c_1 = c_2$, then it will occur in representations of higher degrees in the same diagonal block of the representation matrix of K_0. This is why representations of degree 2 are of particular importance. The q-deformed Lie algebra (3)-(5) is a generalization to the Lie algebras in [4] and [9] and many other models. This is why the case where F and G are linear polynomials is practically important.

Unless otherwise stated we consider, from now on, $P(x) = \mu x + \nu$ and $G(x) = \alpha x + \beta$ for real α, β, μ, ν and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = A^\dagger \text{ and } C = diag(c_1, c_2), \tag{10}$$

where $c_1, c_2 \in \mathbb{R}$. Thus, we have from (5) and (3), respectively, that $[A, B]_q = \mu C + \nu I$

$$= \begin{bmatrix} (1-q)|a|^2 + |b|^2 - |c|^2 q & (a\bar{c} + b\bar{d}) - (b\bar{a} + d\bar{c})q \\ \overline{(a\bar{c} + b\bar{d}) - (b\bar{a} + d\bar{c})q} & |c|^2 - |b|^2 q + (1-q)|d|^2 \end{bmatrix} \tag{11} \text{ and}$$

$$[C, A]_q = \begin{bmatrix} (1-q)\, ac_1 & b\,(c_1 - qc_2) \\ c\,(c_2 - qc_1) & (1-q)\, dc_2 \end{bmatrix} = \alpha A + \beta I. \qquad (12)$$

In the following theorem we state some basic and necessary properties of the representation matrices, enough to find and decide whether the Lie algebra $\mathfrak{l}_q$ has a faithful representation of degree 2.

Theorem 6. *Consider the representation matrices $A, B = A^\dagger$ and $C = \mathrm{diag}\,(c_1, c_2)$ of K_+, K_- and K_0, respectively as in (10), and $P\,(C) = \mu C + \nu I$ and $G\,(A) = \alpha A + \beta I$. Then*

1. *a and d are real and $b \neq \bar{c}$.*

2. *$\alpha = (c_1 - qc_2)$, if $b \neq 0$ or $\alpha = (c_2 - qc_1)$ if $c \neq 0$.*

   If $bc \neq 0$, then either $q = -1$ or C is a scalar matrix.

3. *$\beta = [(1-q)\, c_1 - \alpha]\, a = [(1-q)\, c_2 - \alpha]\, d$*
   $= \frac{1}{2}\, [(1-q)\, (ac_1 + dc_2) - \alpha\, (a+d)]$.

4. *$\mu c_1 + \nu = (1-q)\, a^2 + \left(|b|^2 - q\, |c|^2 \right)$ and*

   $\mu c_2 + \nu = (1-q)\, d^2 + \left(|c|^2 - q\, |b|^2 \right)$.

5. *$\mu\,(c_1 - c_2) = (1-q)\, \left(a^2 - d^2 \right) + (1+q)\, \left(|b|^2 - |c|^2 \right)$.*

6. *$b\,(d - qa) + \bar{c}\,(a - qd) = 0$.*

Moreover, if A, B and C are linearly independent, then the representation is faithful.

Proof. On comparing the diagonal elements on both sides of (12), we get, $a\,[(1-q)\, c_1 - \alpha] = \beta = d\,[(1-q)\, c_2 - \alpha]$. This shows that, a and d must be real numbers. If $b = \bar{c}$, then A is Hermitian matrix and $B = A$ and the representation is not faithful. That proves parts 1 and 3, which is completed by adding the first two equations of it. Similarly, on comparing the non-diagonal elements of on both sides of (12) and solving for α we prove part 2. If $bc \neq 0$, then $(c_1 - c_2)\,(q + 1) = 0$. Thus, $c_1 = c_2$ or $q = -1$. Parts 4 - 6 are by comparing elements of matrices in both sides of the matrices $P\,(C)$ in (11). $\qquad \square$

Theorem 7. *If* $\alpha\beta \neq 0$ *and* $bc \neq 0$, *then* $\mathfrak{l}_q$ *has no faithful real represen-tation of degree* 2.

Proof. The proof depends on Theorem 6. So, from part 3 of Theorem 6 as $bc \neq 0$, then either $q = -1$ or $c_1 = c_2 = r \in \mathbb{R}$.

If $q = -1$, then from part 2, $\alpha = c_1 + c_2$ and from part 3, we have, $(a + d)(c_1 - c_2) = 0$.

Case 1: $a = -d$. Thus, from part 5, $\mu(c_1 - c_2) = 0$. If $\mu = 0$, then $P(C) = \nu I$, which leads to $c_1 = c_2$.

Case 2: $c_1 = c_2 = r$. Thus, from part 2, as $q = -1$, then $\alpha = 2r$, and hence, from part 3, we get $\beta = 0$, which is a contradiction.

On the other hand if $q \neq -1$ and $C = rI$, a scalar matrix, then from part 2, $\alpha = (1 - q)r$ and from part 3 $\beta = [(1 - q)r - \alpha]a = 0$, contradicting $\beta \neq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It should be pointed out that if $b = c = 0$, then the representation is not faithful. Also, as results of parts 2 and 3 of Theorem 6, if $C = rI$, a scalar matrix and $b \neq 0$ and $c = 0$, then $\beta = 0$. Representation matrices obtained, when $b = 0$ and $c \neq 0$ are similar to those when $b \neq 0$ and $c = 0$, by changing B and A.

So, we always discuss representations with $c = 0$ and $b \neq 0$.

In the following theorem we introduce conditions and a procedure to calculate A and C to guarantee a faithful matrix representations for $\mathfrak{l}_q$.

Theorem 8. *Let* $q^2 \neq 1$ *and* $\alpha \neq 0$. *A necessary and sufficient condition for* $\mathfrak{l}_q$ *to have a faithful representation with* $c = 0$ *and* $ab \neq 0$, *is that* a *is a real root of equation (13), satisfying that* $|b|^2 > 0$, *where the elements of these representation matrices are given by the following sequence of equations:*

1. a *is calculated as a real root of the following equation satisfying that* $|b|^2 > 0$,

2. $a^3\left(-q^2 + q^3 + q^4 - q^5\right) + a\left(q\nu - q^3\nu + q\alpha\mu + q^2\alpha\mu\right)$
    $+\beta\mu\left(1 + q^2\right) = 0,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13)$

3. $|b|^2 = \dfrac{a^3\left(-q+2q^2-2q^3+2q^4-q^5\right)+2a\left(q\alpha\mu+q\nu-q^2\nu\right)+\beta\mu(1+q)}{aq(q-1)^2},$ $\qquad (14)$

4. $d = aq,$

5. $c_1 = \dfrac{\beta+a\alpha}{a(1-q)}$ *and* $c_2 = \dfrac{\beta+\alpha aq}{aq(1-q)}.$

Proof. This proof is directly driven from Theorem 6. So, from part 6, $d = aq$. And from part 3, $c_2 = \frac{\beta + \alpha aq}{aq(1-q)}$, and substitution in part 3, we get $c_1 = \frac{\beta + a\alpha}{a(1-q)}$. And from part 4, we get, by addition, $(1-q)\left(1+q^2\right)a^2 + (1-q)|b|^2 = \mu\left(\frac{\beta + a\alpha}{a(1-q)} + \frac{\beta + \alpha aq}{aq(1-q)}\right) + 2\nu$. Thus,

$$(1-q)|b|^2 = \mu\left(\frac{\beta + a\alpha}{a(1-q)} + \frac{\beta + \alpha aq}{aq(1-q)}\right) + 2\nu - (1-q)\left(1+q^2\right)a^2.$$

Therefore, $|b|^2 = \frac{\mu\left(\frac{\beta + a\alpha}{a(1-q)} + \frac{\beta + \alpha aq}{aq(1-q)}\right) + 2\nu - (1-q)\left(1+q^2\right)a^2}{1-q}$, which must be positive and after some reductions, we prove (14). From part 4 $\mu\left(\frac{\beta + a\alpha}{a(1-q)}\right) + \nu = (1-q)a^2 + |b|^2$. Thus, $\mu\left(\frac{\beta + a\alpha}{a(1-q)}\right) + \nu - (1-q)a^2 - |b|^2 = 0$. On using (14), we have

$$\mu\left(\frac{\beta + a\alpha}{a(1-q)}\right) + \nu - (1-q)a^2$$
$$-\frac{a^3\left(-q + 2q^2 - 2q^3 + 2q^4 - q^5\right) + 2a\left(q\alpha\mu + q\nu - q^2\nu\right) + \beta\mu(1+q)}{aq(q-1)^2} = 0.$$

Therefore, $\frac{\left(\beta\mu - a^3 q^2 + a^3 q^3 + a^3 q^4 - a^3 q^5 + aq\nu - aq^3\nu + q^2\beta\mu + aq\alpha\mu + aq^2\alpha\mu\right)}{aq(q-1)^2} = 0$. Which yields to $\beta\mu - a^3 q^2 + a^3 q^3 + a^3 q^4 - a^3 q^5 + aq\nu - aq^3\nu + q^2\beta\mu + aq\alpha\mu + aq^2\alpha\mu = 0$, and so is (13). Therefore, $\mathfrak{l}_q$ has a faithful representation if and only if, a is calculated as a real root of equation (13) satisfying equation (14), with $|b|^2 > 0$. $\square$

The following examples are to illustrate the above procedure.

Example 9. Given that $q = \frac{1}{2}, \alpha = 1, \beta = -\frac{31}{40}, \mu = 3$, and $\nu = 2$.

Substitute in (13), we have, $a^3 - 32a + 31 = 0$, then $(a-1)\left(a^2 + a - 31\right) = 0$. The roots of this equation are: $1, \frac{5}{2}\sqrt{5} - \frac{1}{2}$ and $-\frac{5}{2}\sqrt{5} - \frac{1}{2}$. But $|b|^2$ is only positive for $a = 1$, thus $a = 1$. And since from (14) and assuming that the representation is real, we have $b^2 = \frac{57}{20}$. So, take $b = \pm\sqrt{\frac{57}{20}}, c = 0, d = \frac{1}{2}$, $c_1 = \frac{9}{20}$ and $c_2 = -\frac{11}{10}$.

Example 10. Given that $q = 2, \alpha = 2, \beta = 2, \mu = 3, \nu = -1$.

We have from (13), the equation $2a^3 - 7a - 5 = 0$ has roots: $\frac{1+\sqrt{11}}{2}, \frac{1-\sqrt{11}}{2}$ and -1. But $|b|^2 < 0$ for $\frac{1+\sqrt{11}}{2}$ and $\frac{1-\sqrt{11}}{2}$, rejected. So, $a = -1$. Thus, $d = -2$. And

from (14), we have $|b|^2 = 0$ and $c_1 = 0, c_2 = -1$. Thus, $A = B = diag\,(-1, -2)$ and $C = diag\,(0, -1)$. Which gives a representation that is not faithful.

6. $G\,(x)$ is a polynomial of zero absolute term

Practically, as in [10], a particular interest is given for representations where A is a nilpotent matrix or a triangular matrix of zero diagonal elements. This is when $\beta = 0$ as in the following theorem, where G is considered as a linear polynomial.

Theorem 11.   Let $q \neq -1$ and $G\,(x) = \alpha x$, where $\alpha \neq 0$. Then the representation of $\mathfrak{l}_q$ is conjugate to a representation where A is $n \times n$ real upper (or lower) triangular matrix with zero diagonal entries. Furthermore, if $A + B$ is real matrix then the representation is real.

Proof. From (3), (4), we have for $i, j = 1, ..., n$,

$$\alpha a_{ij} = \sum\nolimits_{t=1}^{n} \left(\delta_{it} c_{it} a_{tj} - q a_{it} \delta_{tj} c_{tj} \right),$$

i.e.,

$$a_{ij}\,(c_{ii} - q c_{jj} - \alpha) = 0,$$

and

$$a_{ji}\,(c_{jj} - q c_{ii} - \alpha) = 0.$$

Suppose, $a_{ij} a_{ji} \neq 0$ for $i \neq j$, we have

$$c_{ii} - q c_{jj} = \alpha, \tag{15}$$

$$c_{jj} - q c_{ii} = \alpha. \tag{16}$$

By subtraction in (15), (16), we get $(1 + q)\,(c_{ii} - c_{jj}) = 0$. Since $q \neq -1$, then for two different diagonal blocks in C, where, $c_{ii} \neq c_{jj}$, we have that a_{ij} and a_{ji} cannot both be nonzero. Thus, one can arrange the matrix C, which is equivalent to getting a conjugate representation, in order to obtain A as an upper (or lower) triangular matrix. $\qquad\square$

Now, consider representation of degree 2, where A, B and C are as in (10). If $\beta = 0$, $c = 0$ and $b \neq 0$, then from parts 2 and 3 of Theorem 6, we get $aq\,(c_2 - c_1) = 0$, and $d\,(c_2 - c_1) = 0$. Then, $a = 0$ and $d = 0$, when C is not a scalar matrix. And we get, A as a nilpotent matrix of degree of nilpotency 2 and

$G(0) = 0$. This allows to consider G of a degree greater than 1. In the following theorem, the necessary and sufficient conditions are exhibited to have faithful representation for $\mathfrak{l}_q$, and formulae are given to calculate the representation matrices, even when $P(x)$ is not linear and $G(x) = \alpha x + \sum_{i \geq 2} \alpha_i x^i$ and A as a nilpotent matrix.

Theorem 12. If $G(A) = \alpha A + \sum_{i \geq 2} \alpha_i A^i$, where the matrices $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$ and $C = diag(c_1, c_2)$ are representation matrices of K_+, K_- and K_0, respectively, then $\mathfrak{l}_q$ has a faithful representation if and only if there exists a real numbers c_1 such that $P(c_1) > 0$. In this case, $b = \pm\sqrt{P(c_1)}$ and $c_2 = \dfrac{c_1 - \alpha}{q}$.

Proof. Since $b \neq 0$, then from part 1 of Theorem 6, $c_2 = \dfrac{c_1 - \alpha}{q}$. From (11), as $a = c = d = 0$, we have $P(c_1) = \left([A, B]_q\right)_{11} = |b|^2 > 0$, for a faithful representation. Thus, $b = \sqrt{P(c_1)}$. $\qquad\square$

Actually, From (11), $tr\left([A, B]_q\right) = P(c_1) + P(c_2) = -|b|^2(q-1) = -P(c_1)(q-1)$. Therefore,

$$P(c_2) + qP(c_1) = 0. \tag{17}$$

Example 13. Let $q = -1$, $P(x) = x^2 - 2x - 3$ and $G(x) = 2x + 11x^4$. On choosing, $c_1 = 4$, since $P(4) = 5 > 0$. Then $c_2 = \frac{4-2}{-1} = -2$. Then $A = \begin{bmatrix} 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$, and $C = diag(4, -2)$ are the representation matrices of $\mathfrak{l}_{-1} : [K_0, K_+]_{-1} = 2K_+ + 11K_+^4$, and $[K_+, K_-]_{-1} = K_0^2 - 2K_0 - 3K_0^0$.

While on choosing, $c_1 = 5$, since $P(5) = 12 > 0$. Then $c_2 = \frac{5-2}{-1} = -3$ and $P(c_2) = 12$. Then $A = \begin{bmatrix} 0 & \sqrt{12} \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$, and $C = diag(4, -2)$ are another representation matrices of $\mathfrak{l}_{-1}$.

Example 14. If $q = \frac{1}{3}$, $P(x) = 2x^2 - 3x - \frac{11}{4}$ and $G(x) = 3x^4 - 5x^3 + 3x^2 - \frac{5}{3}x$. Choose, $c_1 = -1$, since $P(-1) = \frac{9}{4} > 0$. Then $c_2 = 2$. Then $A = \begin{bmatrix} 0 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$ and $C = diag(-1, 2)$ are representation matrices of a faithful representation of $\mathfrak{l}_{1/3} : [K_0, K_+]_{1/3} = G(K_+)$, and $[K_+, K_-]_{1/3} = P(K_0)$.

Example 15. If $q = 1, P(x) = -x^2 + 2x$ and $G(x) = x^2$. Choose, $c_1 = 1$ since $P(1) = 1 > 0$. Then $c_2 = \frac{1-0}{2} = \frac{1}{2}$. Then $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = A^\dagger, C = diag\left(1, \frac{1}{2}\right)$.

7. Generalized coupling quantized oscillators model

We conclude this paper by introducing a generalized model for the coupling oscillators model. and considering its faithful representations of least degree of the q-deformed Lie algebra $\mathfrak{l}_q$, which is of a practically important case. This case in a generalization of the q-deformed $sl_{pd}(2, \mathbb{R})$, [26]. Also, in [9], it was considered that $q = 1$, while $G(K_+) = \alpha K_+$. While it was considered in [5], [7], that $q = 1, G(K_0) = \alpha K_+$ and $P(K_0) = \mu K_0$.

Theorem 16. *The q-deformed lie algebra generated by $K_+, K_- = K_+^\dagger$, and real diagonal operator K_0, satisfying*

$$\mathfrak{l}_q: \begin{aligned} [K_0, K_+]_q &= \alpha K_+, \\ [K_-, K_0]_q &= \alpha K_-, \\ [K_+, K_-]_q &= \mu K_0, \quad where \quad \alpha\mu > 0, \end{aligned}$$

and $K_+ + K_-$ is real operator, has real faithful representations of degree 2 as least degree, whose representation matrices A, B and C for K_+, K_- and K_0, respectively, where,

1. $A = \begin{bmatrix} 0 & \sqrt{\frac{\alpha\mu}{q^2+1}} \\ 0 & 0 \end{bmatrix}, B = A_+^\dagger$ and $C = \begin{bmatrix} \frac{\alpha}{q^2+1} & 0 \\ 0 & -\frac{\alpha q}{q^2+1} \end{bmatrix}$,

2. $A = \begin{bmatrix} 0 & -\sqrt{\frac{\alpha\mu}{q^2+1}} \\ 0 & 0 \end{bmatrix}, B = A_+^\dagger$ and $C = \begin{bmatrix} \frac{\alpha}{q^2+1} & 0 \\ 0 & -\frac{\alpha q}{q^2+1} \end{bmatrix}$,

3. $A = \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{\alpha\mu}{q^2+1}} & 0 \end{bmatrix}, B = A_+^\dagger$ and $C = \begin{bmatrix} -\frac{\alpha q}{q^2+1} & 0 \\ 0 & \frac{\alpha}{q^2+1} \end{bmatrix}$, and

4. $A = \begin{bmatrix} 0 & 0 \\ -\sqrt{\frac{\alpha\mu}{q^2+1}} & 0 \end{bmatrix}, B = A_+^\dagger$ and $C = \begin{bmatrix} -\frac{\alpha q}{q^2+1} & 0 \\ 0 & \frac{\alpha}{q^2+1} \end{bmatrix}$.

Proof. Since $P(c_1) = \mu c_1 = |b|^2$ and $P(c_2) = \mu c_2 = \mu\left(\frac{c_1-\alpha}{q}\right)$, from Theorem 10. Using (17), we have $\mu\left(\frac{c_1-\alpha}{q}\right) + q(\mu c_1) = 0$. Then $\mu\left[c_1\left(q^2+1\right)-\alpha\right] = 0$. Thus we have, $c_1 = \frac{\alpha}{q^2+1}$ and hence, $c_2 = -\frac{\alpha q}{q^2+1}$ and also, $|b|^2 = \frac{\alpha\mu}{q^2+1}$. $\qquad\square$

8. Summary

The q-deformed Lie algebra $\mathfrak{l}_q$, is a generalization of deformed Lie algebras for quantized Hamiltonians. It was shown that $\mathfrak{l}_q$ has two types of faithful matrix representations. If the absolute term of the polynomial $G(K_+)$ is zero, then the matrices representing K_+ and K_- are nilpotent matrices of degree 2 as least degree. The formulae to compute the representation matrices are given together with the conditions to guarantee that these matrices satisfy the physical properties and the representation is faithful. If the absolute term of $G(K_+)$ is not zero, then the representation matrix of K_+ is a triangular matrix. The conditions and formulae are given to evaluate the faithful representations for linear polynomials $P(K_0)$ and $G(K_+)$. So,

(1) If $G(K_+) = \alpha K_+ + \beta$, $\beta \neq 0$, then $\mathfrak{l}_q$, has faithful representation of degree 2 as least degree, if and only if the upper triangular matrix $A = [a_{ij}]$ representing K_+, is such that: a_{11} is a real root of equation (13), satisfying that $|b|^2$ is positive form (14), where $q \neq 1$.

(2) If $G(K_+) = \alpha K_+ + \sum_{\ell>1} \alpha_\ell K_+^\ell$ then the representation matrix of K_+ is nilpotent matrices if and only if $\alpha = c_1 - qc_2$ for some $c_1, c_2 \in \mathbb{R}$ such that $P(c_1) > 0$.

(3) The q-deformed Lie algebra $\mathfrak{l}_q$, was introduced as a generalized coupling oscillators model, that is when $P(K_0) = \mu K_0$ and $G(K_+) = \alpha K_+$. It was found that $\mathfrak{l}_q$ has faithful representation if and only if $\alpha\mu > 0$.

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