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APPROXIMATE SOLUTIONS AND EXISTENCE OF SOLUTION FOR A CAPUTO NONLOCAL FRACTIONAL VOLTERRA FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

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Abstract: We use a recent approach to establish the existence and uniqueness results of Caputo fractional Volterra Fredholm integro-differential equation. We derive some sufficient conditions for the existence of solutions of fractional integrodifferential equations with nonlocal conditions. the modified Adomian decomposition method is applied to obtain the approximate solution of proposed problem. Moreover, the Krasnoselskii's and Banach's fixed point theorems are employed to analyze our results. An example is given to justify the adduced results.

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1. Introduction

In this work, we study the existence and uniqueness of solution by using some fixed point theorems of Krasnoselskii and Banach, then we apply the modified Adomian decomposition method (MADM) for the following Caputo fractional Volterra Fredholm Integro-Differential Equation (Caputo fractional VFIDE)

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$$\begin{cases} {}^{C}D_{0^+}^{\rho}x(t) = g(t) + \chi_1 x(t) + \chi_2 x(t), \ t \in I = [0,1], \\ x(0) = x_0 + h(x), \end{cases} \tag{1}$$

where $0 < \rho < 1$, ${}^{C}D_{0^+}^{\rho}$ is fractional derivative of order ρ in the Caputo sense, $g : I \to \mathbb{R}$, $h : C(I, \mathbb{R}) \to \mathbb{R}$, K_1, $K_2 : I \times I \to \mathbb{R}$ are continuous functions and $F_1, F_2 : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are Lipschitz continuous functions. For short, we set

$$\chi_1 x(t) := \int_0^t K_1(t,\xi)F_1(x(\xi))d\xi$$

and

$$\chi_2 x(t) := \int_0^1 K_2(t,\xi)F_2(x(\xi))d\xi.$$

In recent years, the fractional integrodifferential equation emerges in a lot of phenomena of different fields of science and engineering, [9, 22, 21, 24].

Some results on the existence of solutions of fractional integrodifferential equations have been studied by many authors by employing the fixed point techniques. For recent papers, see [1, 2, 3, 4, 5, 6, 31, 11, 14, 16, 20, 10, 29, 30, 25, 32]. Furthermore, much researches on the approximate solution of this kind of equations have occurred through the method of Adomian decomposition introduced by George Adomian [7] and other numerical methods for more details see [8, 19, 15, 18, 34] . The method of Adomian decomposition has the feature of style and easiness of use. The solution is provided as a series in which every expression can be easily calculated by means of Adomian polynomials appropriated to nonlinear terms see [7, 12, 13, 17, 26, 27].

Wazwaz in [33] presented the method of modified Adomian decomposition (MADM) that contains decomposing the 1^{st} term of the series into 2^{nd} terms, one remains in the 1^{st} term while the other is assigned to define the 2^{nd} term of series. The main aim of this method is to reduce the number of operations used and quicken the convergence towards the exact solution of the proposed problem. For instance on the application of the MADM, we refer to [23].

In this article, we use a recent approach to establish the existence and uniqueness results of Caputo fractional VFIDE (1). We derive some sufficient conditions for the existence of solutions of fractional integrodifferential equations with nonlocal conditions. the MADM is applied to obtain the approximate solution of Moreover, the Krasnoselskii's and Banach's fixed point theorems are employed to analyze our results.

The paper is organized as follows. In Section 2, we give some basic results related the hypothesis and several lemmas needed throughout this work. In Section 3, we prove the existence and uniqueness of solutions to the porposed problem by means of fixed point theorems of Krasnoselskii and Banach. In section 4, we discuss the modified Adomian decomposition method and establish the convergence of the series built by the MADM to the exact solution of the Caputo fractional VFIDE. Finally, we give an example to illustrate our results.

2. Preliminaries

In this section, we need the following basic definitions and Lemmas used throughout this paper. For more details, see [24].

Definition 1. Let $\rho > 0$, and $\omega \in L^1([0,T],\mathbb{R})$. The Riemann-Liouville fractional integral of order ρ is defined by

$$I_{0^+}^{\rho}\omega(t) = \begin{cases} \frac{1}{\Gamma(\rho)}\int_0^t (t-\xi)^{\rho-1}\omega(\xi)d\xi, & \rho > 0 \\ \omega(t), & \rho = 0 \end{cases},$$

where Γ is the Euler's Gamma function satisfies

$$\Gamma(\rho) = \int_0^{\infty} t^{\rho-1}e^{-t}dt, \text{ and } \frac{\Gamma(\rho)\Gamma(\beta)}{\Gamma(\rho+\beta)} = \int_0^1 (1-t)^{\rho-1}t^{\beta-1}dt.$$

Moreover, The operator $I_{0^+}^{\rho}$ is bounded on $C([0,T],\mathbb{R})$, *i.e.*, for a positive constant κ

$$\left\| I_{0^+}^{\rho}\omega \right\|_{\infty} \leq \kappa \left\| \omega \right\|_{\infty}, \quad \text{for all } \omega \in C([0,T],\mathbb{R}).$$

Definition 2. Let $\rho > 0$, $\omega \in AC^n([0,T],\mathbb{R})$. The Caputo fractional derivative of order ρ is defined by

$$^{C}D_{0^+}^{\rho}\omega(t) = D_{0^+}^{\rho}\left[\omega(t) - \sum_{k=0}^{n-1}\frac{\omega^{(k)}(0)}{k!}t^k\right] \quad t \in [0,T], \tag{2}$$

where $n = [\rho]+1$, $[\rho]$ is the integer part of ρ and $D_{0^+}^{\rho}$ is the fractional derivative of order ρ in the Riemann-Liouville sense defined by

$$\begin{aligned} D_{0^+}^{\rho}\omega(t) &= \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\rho}\omega(t) \\ &= \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\rho)}\int_0^t (t-\xi)^{n-\rho-1}\omega(\xi)d\xi. \end{aligned}$$

Lemma 1. *If $\rho, \beta > 0$, then*

$$I_{0^+}^{\rho} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\rho+\beta)} t^{\rho+\beta-1}. \tag{3}$$

Lemma 2. *Let $\omega \in AC^n(I, \mathbb{R})$, then the Caputo fractional derivative of order $\rho > 0$*

$$I_{0^+}^{\rho} \, {}^C D_{0^+}^{\rho} \omega(t) = \omega(t) - \sum_{k=0}^{n-1} \frac{\omega^{(k)}(0)}{k!} t^k, \tag{4}$$

where $n = [\rho] + 1$. In a special case, if $0 < \rho < 1$, then $I_{0^+}^{\rho} \, {}^C D_{0^+}^{\rho} \omega(t) = \omega(t) - \omega(0)$. Furthermore, if ω is a continuous on I, we have ${}^C D_{0^+}^{\rho} \, I_{0^+}^{\rho} f(t) = f(t)$.

Theorem 3. ([28]) (Banach fixed point theorem) *Let (U, d) be a Banach space with $T : U \to U$ is a contraction mapping. Then mapping T has a fixed point in U.*

Theorem 4. ([28]) (Krasnoselskii fixed point theorem) *Let U be a Banach space, let S be a nonempty bounded closed convex subset of U and let T_1, T_2 be mapping from S into U such that $T_1 x + T_2 v \in S$ for any $x, v \in S$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1 z + T_2 z = z$ has a solution on S.*

3. Existence result via Krasnoselkii's fixed point theorem

In this part, we study existence of solution of Caputo fractional VFIDE (1) by using Krasnoselkii's fixed point theorem.

First we make the following assumptions.

(H$_1$) Let $F_1(x(t)), F_2(x(t))$ can be considered as continuous nonlinearity terms and there exist constants $L_{F_1}(> 0)$ and $L_{F_1}(> 0)$ such that

$$|F_i(x_1(t)) - F_i(x_2(t))| \leq L_{F_i} |x_1 - x_2|, \; i = 1, 2, \; \forall x_1, x_2 \in \mathbb{R}.$$

(H$_2$) The kernels $K_1(t, \xi)$ and $K_1(t, \xi)$ are continuous on $I \times I$, and there exist two positive constants K_1^* and K_2^* in $I \times I$

such that

$$K_i^* = \sup_{t \in I} \int_0^t |K_i(t, \xi)| \, d\xi < \infty, \; i = 1, 2.$$

(H$_3$) $g : I \to \mathbb{R}$ is continuous on I.

(H$_4$) $h : C(I,\mathbb{R}) \to \mathbb{R}$ is continuous on $C(I)$ and there exist constant $0 < L_h < 1$ such that

$$|h(x_1(t)) - h(x_2(t))| \leq L_h |x_1 - x_2|, \ \forall x_1, x_2 \in C(I,\mathbb{R}), \ t \in I.$$

The following lemma yields the equivalence between the problem (1) and the integral equation. The proof for this lemma is neglected because it is similar to some classical proofs in the literature.

Lemma 3. *The function $x \in C(I,\mathbb{R})$ be a solution of the Caputo fractional VFIDE (1) if and only if x is a solution of the integral equation*

$$\begin{aligned} x(t) &= x_0 + h(x) + \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \left\{ \int_0^s K_1(s,\tau) F_1(x(\tau)) d\tau \right. \\ &\quad \left. + \int_0^1 K_2(s,\tau) F_2(x(\tau)) d\tau \right\} ds. \end{aligned}$$

Our first result is concerned with existence based on Theorem 4.

Theorem 5. *Suppose $(H_1) - (H_4)$ hold. If*

$$\Delta_1 := \left(L_h + \frac{\sum_{i=1}^{2} L_{F_i} K_i^*}{\Gamma(\rho+1)} \right) < 1, \tag{5}$$

then the Caputo fractional VFIDE (1) has at least one solution on I.

Proof. Let $C(I,\mathbb{R})$ be the space of the continuous functions x on I with the usual norm defined by

$$\|x\|_\infty = \sup_{t \in I} |x(t)|$$

Consider the ball

$$B_\gamma = \{x \in C(I,\mathbb{R}) : \|x\|_\infty \leq \gamma\} \subset C(I,\mathbb{R}). \tag{6}$$

Clearly, B_γ is nonempty convex closed subset of $C(I,\mathbb{R})$. Choose γ such that $\gamma \geq \frac{\Delta_2}{1-\Delta_1}$, where $\Delta_1 < 1$,

$$\Delta_2 := \mu_0 + \frac{\mu_g + \sum_{i=1}^{2} \mu_{F_i} K_i^*}{\Gamma(\rho+1)}, \tag{7}$$

$\mu_g := \sup_{t\in[0,1]} |g(t)|$, $\mu_0 := |x_0| + \mu_h$, $\mu_h = |h(0)|$, $\mu_{F_1} := |F_1(0)|$ and $\mu_{F_2} := |F_2(0)|$.

In view of Lemma 3, the equivalent fractional integral equation to Caputo fractional VFIDE (1) can be written as operator equation as follows

$$x = Px + Qx, \quad x \in B_\gamma \subset C(I, \mathbb{R}), \tag{8}$$

where P and Q are two operators defined on B_γ by

$$\begin{aligned}(Px)(t) &= \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \Big(\int_0^s K_1(s,\tau) F_1(x(\tau)) d\tau \\ &\quad + \int_0^1 K_2(s,\tau) F_2(x(\tau)) d\tau \Big) ds,\end{aligned}$$

and

$$(Qx)(t) = x_0 + h(x) + \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} g(s) ds.$$

Now, we obtain the fixed point of the operator equation (8) by using the conditions of Theorem 6 as in the following steps:

Step 1: In this step, we show that, $Px + Qv \in B_\gamma$ for each $x, v \in B_\gamma$.

By (H_1) and for any $x, v \in B_\gamma$, we have

$$\begin{aligned}|F_i(x(t))| &\leq |F_i(x(t)) - F_i(0)| + |F_i(0)| \\ &\leq L_{F_i} \|x\|_\infty + |F_i(0)| \\ &\leq L_{F_i}\gamma + \mu_{F_i}, \quad \text{for all } i = 1, 2,\end{aligned}$$

and

$$\begin{aligned}|h(v(t))| &\leq |h(v(t)) - h(0)| + |h(0)| \\ &\leq L_h \|v\|_\infty + |h(0)| \\ &\leq L_h\gamma + \mu_h.\end{aligned}$$

Let $x, v \in B_\gamma$. Then

$$\begin{aligned}&|(Px)(t) + (Qv)(t)| \\ &\leq \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \Big(\int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^1 |K_2(s,\tau)|\, |F_2(x(\tau))|\, d\tau\Big) ds \\
&\quad + |x_0| + |h(v)| + \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} |g(s)|\, ds \\
&\le \mu_0 + L_h\gamma + \frac{\mu_g + \sum_{i=1}^2 (L_{F_i}\gamma + \mu_{F_i}) K_i^*}{\Gamma(\rho+1)} t^\rho,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|Px + Qv\|_\infty \\
&\le \mu_0 + \frac{\mu_g + \sum_{i=1}^2 \mu_{F_i} K_i^*}{\Gamma(\rho+1)} + \left(L_h + \frac{\sum_{i=1}^2 L_{F_i} K_i^*}{\Gamma(\rho+1)} \right)\gamma \\
&\le \Delta_2 + \Delta_1\gamma \le \gamma.
\end{aligned}
$$

Consequently,

$$
Px + Qv \in B_\gamma.
$$

Step 2: In this step, we show that Q is contraction on B_γ.
Let $x, x^* \in B_\gamma$. It follows from (H_4) that

$$
\begin{aligned}
\|Qx - Qx^*\|_\infty &= \sup_{t\in I} |Qx(t) - Qx(t)| = \sup_{t\in I} |h(x(t)) - h(x^*(t))| \\
&\le L_h \|x - x^*\|_\infty .
\end{aligned}
$$

Since $L_h < 1$, Q is contraction mapping.

Step 3: In this step, we show that, P is completely continuous on B_γ.

First, we show that P is continuous. Let (x_n) be a sequence such that $x_n \to x$ in $C(I, \mathbb{R})$. Then for each $x_n, x \in C(I, \mathbb{R})$ and for any $t \in I$, we have

$$
\begin{aligned}
&|(Px_n)(t) - (Px)(t)| \\
&\le \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \Big(\int_0^s |K_1(s,\tau)|\, |F_1(x_n(\tau)) - F_1(x(\tau))|\, d\tau \\
&\quad + \int_0^1 |K_2(s,\tau)|\, |F_2(x_n(\tau)) - F_2(x(\tau))|\, d\tau \Big) ds \\
&\le \frac{\sum_{i=1}^2 L_{F_i} K_i^*}{\Gamma(\rho+1)} \|x_n - x\|_\infty .
\end{aligned}
$$

Since $x_n \to x$ as $n \to \infty$, $\|Px_n - Px\|_\infty \to 0$, as $n \to \infty$. This proves that P is continuous on $C(I, \mathbb{R})$.

Next, from Step 1, we observe that

$$\begin{aligned} & |(Px)(t)| \\ \leq \quad & \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \Big(\int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau \\ & + \int_0^1 |K_2(s,\tau)| \, |F_2(x(\tau))| \, d\tau \Big) ds \\ \leq \quad & \frac{\sum_{i=1}^2 \left(L_{F_i}\gamma + \mu_{F_i} \right) K_i^*}{\Gamma(\rho+1)} t^\rho . \end{aligned}$$

Thus

$$\|Px\|_\infty \leq \frac{\sum_{i=1}^2 \left(L_{F_i}\gamma + \mu_{F_i} \right) K_i^*}{\Gamma(\rho+1)}.$$

This shows that (PB_γ) is uniformly bounded.

Finally, we prove that (PB_γ) is equicontinuous. Let $x \in B_\gamma$. Then for $t_1, t_2 \in I$ with $t_1 \leq t_2$, we have

$$\begin{aligned} & |(Px)(t_2) - (Px)(t_1)| \\ = \quad & \Bigg| \frac{1}{\Gamma(\rho)} \int_0^{t_2} (t_2 - s)^{\rho-1} \left(\int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau \right. \\ & \left. + \int_0^1 |K_2(s,\tau)| \, |F_2(x(\tau))| \, d\tau \right) ds \\ & - \frac{1}{\Gamma(\rho)} \int_0^{t_1} (t_1 - s)^{\rho-1} \left(\int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau \right. \\ & \left. + \int_0^1 |K_2(s,\tau)| \, |F_2(x(\tau))| \, d\tau \right) ds \Bigg| \\ \leq \quad & \frac{1}{\Gamma(\rho)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\rho-1} \int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau ds \right. \\ & \left. + \int_0^{t_1} \left| (t_2 - s)^{\rho-1} - (t_1 - s)^{\rho-1} \right| \int_0^s |K_1(s,\tau)| \, |F_1(x(\tau))| \, d\tau ds \right) \\ & + \frac{1}{\Gamma(\rho)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\rho-1} \int_0^s |K_2(s,\tau)| \, |F_2(x(\tau))| \, d\tau ds \right. \\ & \left. + \int_0^{t_1} \left| (t_2 - s)^{\rho-1} - (t_1 - s)^{\rho-1} \right| \int_0^s |K_2(s,\tau)| \, |F_2(x(\tau))| \, d\tau ds \right), \end{aligned}$$

which implies

$$|(Px)(t_2) - (Px)(t_1)| \quad \leq \quad \frac{(L_{F_1}\gamma + \mu_{F_1}) K_1^*}{\Gamma(\rho)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\rho-1} ds \right.$$

$$
\begin{aligned}
& +\int_0^{t_1} \left|(t_2-s)^{\rho-1}-(t_1-s)^{\rho-1}\right| ds\Big) \\
& +\frac{(L_{F_2}\gamma+\mu_{F_2})K_2^*}{\Gamma(\rho)}\Big(\int_{t_1}^{t_2}(t_2-s)^{\rho-1}ds \\
& +\int_0^{t_1} \left|(t_2-s)^{\rho-1}-(t_1-s)^{\rho-1}\right| ds\Big) \\
\leq\ & \left(\frac{(L_{F_1}\gamma+\mu_{F_1})K_1^*}{\Gamma(\rho)}+\frac{(L_{F_2}\gamma+\mu_{F_2})K_2^*}{\Gamma(\rho)}\right) \\
& \times\left(\frac{(t_2-t_1)^\rho}{\rho}+\frac{t_1^\rho}{\rho}-\frac{t_2^\rho}{\rho}+\frac{(t_2-t_1)^\rho}{\rho}\right) \\
\leq\ & \frac{2\sum_{i=1}^{2}\left(L_{F_i}\gamma+\mu_{F_i}\right)K_i^*}{\Gamma(\rho+1)}(t_2-t_1)^\rho,
\end{aligned}
$$

which tends to zero as $t_2-t_1\to 0$. So, (PB_γ) is equicontinuous. Hence along with the Arzela-Ascoli theorem, it is concluded that $P: C(I,\mathbb{R})\to C(I,\mathbb{R})$ is continuous and completely continuous.

An application of Theorem 4 shows that P has a fixed point x in B_γ which is a solution of the Caputo fractional VFIDE (1). □

The uniqueness result for the Caputo fractional VFIDE (1) will be proved by using Theorem 3.

Theorem 6. *Suppose* $(H_1)-(H_4)$ *hold. If*

$$
\left(L_h+\frac{\sum_{i=1}^{2}L_{F_i}K_i^*}{\Gamma(\rho+1)}\right)<1, \tag{9}
$$

then the Caputo fractional VFIDE (1) has a unique solution on I.

Proof. Thanks to Lemma 3, the equivalent fractional integral equation to Caputo fractional VFIDE (1) can be written as operator equation as follows

$$
x=Tx,\quad x\in C(I,\mathbb{R}),
$$

where the operator $T: C(I,\mathbb{R})\to C(I,\mathbb{R})$ defined by

$$
\begin{aligned}
(Tx)(t)\ =\ & x_0+h(x)+\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}g(s)ds \\
& +\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}\Big(\int_0^s K_1(s,\tau)F_1(x(\tau))d\tau
\end{aligned}
$$

$$+\int_0^1 K_2(s,\tau)F_2(x(\tau))d\tau\Big)\,ds,$$

for all $t \in I$. Let $x, x^* \in C(I,\mathbb{R})$. Then for for each $t \in I$ we have

$$\begin{aligned}
& |Tx(t) - Tx^*(t)| \\
\leq\ & |h(x(t)) - h(x^*(t))| \\
& +\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\Big(\int_0^s K_1(s,\tau)\,|F_1(x(\tau)) - F_1(x^*(\tau))|\,d\tau\Big)ds \\
& +\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\Big(\int_0^1 K_2(s,\tau)\,|F_2(x(\tau)) - F_2(x^*(\tau))|\,d\tau\Big)ds \\
\leq\ & L_h\,\|x - x^*\|_\infty + \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}K_1^*L_{F_1}\,\|x - x^*\|_\infty\,ds \\
& +\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}K_2^*L_{F_2}\,\|x - x^*\|_\infty\,ds \\
\leq\ & \left(L_h + \frac{K_1^*L_{F_1} + K_2^*L_{F_2}}{\Gamma(\rho+1)}t^\rho\right)\|x - x^*\|_\infty\,,
\end{aligned}$$

which implies

$$\|Tx - Tx^*\|_\infty \leq \left(L_h + \frac{\sum_{i=1}^2 L_{F_i}K_i^*}{\Gamma(\rho+1)}\right)\|x - x^*\|_\infty\,.$$

The relation (9) shows that T is contraction on $C(I,\mathbb{R})$. Hence, by the conclusion of Theorem 3, T has a unique fixed point, which is solution of the Caputo fractional VFIDE (1). □

4. Approximate solution

Here, we provide the approximate solution of the Caputo fractional VFIDE (1) which relies on the fractional Adomian decomposition technique.

First, we recall the classical Adomian decomposition technique where the solution of the proposed problem is obtained in the form of a series as

$$x = \sum_{n=0}^{\infty} x_n, \tag{10}$$

and the nonlinear terms F_1, F_2 and h are decomposed as

$$F_1 = \sum_{n=0}^{\infty} A_n, \quad F_2 = \sum_{n=0}^{\infty} B_n, \; h = \sum_{n=0}^{\infty} D_n \tag{11}$$

where A_n, B_n, D_n are Adomian polynomials for all $n \in \mathbb{N}$, and write

$$x = x(\lambda) = \sum_{n=0}^{\infty} \lambda^n x_n = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots + \lambda^k x_k + \cdots \tag{12}$$

$$F_1 = F_1(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k + \cdots \tag{13}$$

$$F_2 = F_2(\lambda) = \sum_{n=0}^{\infty} \lambda^n B_n = B_0 + \lambda B_1 + \lambda^2 B_2 + \cdots + \lambda^k B_k + \cdots \tag{14}$$

$$h = h(\lambda) = \sum_{n=0}^{\infty} \lambda^n D_n = D_0 + \lambda D_1 + \lambda^2 D_2 + \cdots + \lambda^k D_k + \cdots \tag{15}$$

By utilizing the previous formulas (12) (13), (14) and (15), we deduce that

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(F_1 \sum_{i=0}^{\infty} \lambda^i x_i \right) \right]_{\lambda=0},$$

$$B_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(F_2 \sum_{i=0}^{\infty} \lambda^i x_i \right) \right]_{\lambda=0},$$

$$D_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(h \sum_{i=0}^{\infty} \lambda^i x_i \right) \right]_{\lambda=0},$$

where x_0, x_1, $x_2, \ldots$ are repeatedly specified by

$$\begin{cases} x_0(t) = x_0 + I^{\rho}_{0+}\,(g(t)) \\ x_{k+1}(t) = D_k + I^{\rho}_{0+}\left(\int_0^t K_1(t,\xi) A_k d\xi\right) \\ \quad +I^{\rho}_{0+}\left(\int_0^1 K_2(t,\xi) B_k d\xi\right), \quad k \geq 1. \end{cases} \tag{16}$$

Now, we apply the modified Adomian decomposition method, Therefore, the scheme (16) gives

$$\begin{cases} x_0(t) = x_0 + R_1(t), \\ x_1(t) = R_2(t) + D_0 + I^{\rho}_{0+}\left(\int_0^t K_1(t,\xi)A_0 d\xi\right) \\ \qquad + I^{\rho}_{0+}\left(\int_0^1 K_2(t,\xi)B_0 d\xi\right), \\ x_{k+1}(t) = D_k + I^{\rho}_{0+}\left(\int_0^t K_1(t,\xi)A_k d\xi\right) \\ \qquad + I^{\rho}_{0+}\left(\int_0^1 K_2(t,\xi)B_k d\xi\right), \quad k \geq 1. \end{cases} \tag{17}$$

Now, we will study the convergence theorem of the solution based on the MADM.

Theorem 7. *Assume that $(H_1)-(H_4)$ and (5) are satisfied, if the solution $x(t) = \sum_{i=0}^{\infty} x_i(t)$ and $\|x\|_\infty < \infty$ is convergent, then it converges to the exact solution of the Caputo fractional VFIDE (1).*

Proof. The proof is similar to some works found in the literature see [8], so we omit it. □

Example 1. Consider an integro-differential equation with Caputo fractional derivative

$$\begin{cases} {}^C D^{\frac{1}{2}}_{0+} x(t) = \frac{2}{\sqrt{\pi}}\left(\frac{4t^{\frac{3}{2}}}{\Gamma(6)} + t^{\frac{1}{2}}\right) + \frac{t^3}{\Gamma(7)} + \frac{t}{\Gamma(8)} \\ + \frac{1}{4}\int_0^t (1+t-s)x(s)ds + \frac{5}{18}\int_0^1 e^{s-t}x^2(s)ds, \end{cases} \tag{18}$$

with the nonlocal condition

$$x(0) = \frac{1}{4}x(\frac{1}{3}) \tag{19}$$

where

$$\begin{aligned} \rho &= \frac{1}{2},\ x_0 = 0,\ h(x(t)) = \frac{1}{4}x(\frac{1}{3}), \\ g(t) &= \frac{2}{\sqrt{\pi}}\left(\frac{4t^{\frac{3}{2}}}{\Gamma(6)} + t^{\frac{1}{2}}\right) + \frac{t^3}{\Gamma(7)} + \frac{t}{\Gamma(8)}, \\ K_1(t,\xi) &= \frac{1}{4}(1+t-\xi),\ K_2(t,\xi) = \frac{5}{18}e^{\xi-t}. \end{aligned}$$

Clearly, $L_{F_1} = L_{F_2} = 1$, $L_h = \frac{1}{4}$.

$$\mu_g \quad : \quad = \sup_{t\in[0,1]} |g(t)| = \|g\|_\infty$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{4}{\Gamma(6)}+1\right)+\frac{1}{\Gamma(7)}+\frac{1}{\Gamma(8)}$$
$$= \frac{1302+\sqrt{\pi}}{630\sqrt{\pi}},$$

$$K_1^* = \frac{1}{4}\sup_{t\in I}\int_0^t |1+t-\xi|\,d\xi = \frac{1}{8}.$$
$$K_2^* = \frac{5}{18}\sup_{t\in I}\int_0^t \left|e^{\xi-t}\right|d\xi = \frac{5}{18}\sup_{t\in I} e^{-t}\int_0^t \left|e^{\xi}\right|d\xi$$
$$= \frac{5}{18}(1-\frac{1}{e}).$$

Hence,

$$\Delta_1 := \left(L_h + \frac{\sum_{i=1}^2 L_{F_i}K_i^*}{\Gamma(\rho+1)}\right) \approx 0.6 < 1.$$

As consequence of Theorem 6, then the problem (18)-(19) has a unique solution on $[0,1]$.

Applying the operator $I_{0^+}^{\frac{1}{2}}$ to both sides of equation (18-a), we get

$$x(t) = \frac{1}{4}x(\frac{1}{3}) + I_{0^+}^{\frac{1}{2}}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4t^{\frac{3}{2}}}{\Gamma(6)}+t^{\frac{1}{2}}\right)+\frac{t^3}{\Gamma(7)}+\frac{t}{\Gamma(8)}\right)$$
$$+I_{0^+}^{\frac{1}{2}}\left(\frac{1}{4}\int_0^t (1+t-s)x(s)ds\right) + I_{0^+}^{\frac{1}{2}}\left(\frac{5}{18}\int_0^1 e^{s-t}x^2(s)ds\right).$$

Suppose

$$R(t) = I_{0^+}^{\frac{1}{2}}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4t^{\frac{3}{2}}}{\Gamma(6)}+t^{\frac{1}{2}}\right)+\frac{t^3}{\Gamma(7)}+\frac{t}{\Gamma(8)}\right)$$
$$= \frac{2}{\sqrt{\pi}}\frac{4}{\Gamma(6)}\left(I_{0^+}^{\frac{1}{2}}s^{\frac{3}{2}}\right)(t) + \frac{2}{\sqrt{\pi}}\left(I_{0^+}^{\frac{1}{2}}s^{\frac{1}{2}}\right)(t)$$
$$+\frac{1}{\Gamma(7)}\left(I_{0^+}^{\frac{1}{2}}s^3\right)(t) + \frac{1}{\Gamma(8)}\left(I_{0^+}^{\frac{1}{2}}s\right)(t)$$
$$= \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)}t^2 + \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}}t + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{9}{2})}t^{\frac{7}{2}} + \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}.$$

Now, we apply the modified Adomian decomposition method,

$$R(t) = R_1(t) + R_2(t)$$

where

$$R_1(t) = \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)} t^2,$$

and

$$R_2(t) = \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}} t + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}.$$

The modified recursive relation

$$x_0(t) = R_1(t) = \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)} t^2,$$

$$\begin{aligned}
x_1(t) &= R_2(t) + I_{0^+}^{\frac{1}{2}} \left(\frac{1}{4} \int_0^t (1+t-s) A_0(s) ds \right) \\
&\quad + I_{0^+}^{\frac{1}{2}} \left(\frac{5}{18} \int_0^1 e^{s-t} B_0(s) ds \right) + D_0(t) \\
&= \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}} t + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} \\
&\quad + I_{0^+}^{\frac{1}{2}} \left(\frac{1}{4} \int_0^t (1+t-s) x_0(s) ds \right) \\
&\quad + I_{0^+}^{\frac{1}{2}} \left(\frac{5}{18} \int_0^1 e^{s-t} x_0(s) ds \right) + \frac{1}{4} x_0(\frac{1}{3}) \\
&= \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}} t + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{9}{2})} t^{\frac{7}{2}} + \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} \\
&\quad + I_{0^+}^{\frac{1}{2}} \left(\frac{1}{4} \int_0^t (1+t-s) \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)} s^2 ds \right) \\
&\quad + I_{0^+}^{\frac{1}{2}} \left(\frac{5}{18} \int_0^1 e^{s-t} \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)} s^2 ds \right) \\
&\quad + \frac{1}{4} \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)} \left(\frac{1}{3} \right)^2 \\
&= 0,
\end{aligned}$$

$$x_2(t) = 0,$$

$$\vdots$$

$$x_n(t) = 0.$$

Therefore, the obtained solution is

$$x(t)=\sum_{i=0}^{\infty}x_i(t)=\frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(3)}t^2.$$

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