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POSITIVE SOLUTIONS FOR A SECOND ORDER EXTENDED FISHER-KOLMOGOROV'S EQUATION

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Abstract: We consider the existence of positive solutions of extended Fisher-Kolmogorov second order differential equation. Using a variational method and an approach of Verzini, we obtain the positive bounded solutions of this ODE.

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1. Introduction

In the present paper we study the existence of positive solutions of a second-order ordinary differential equation (ODE)

$$u'' + cu' + f(t,u) = 0, \qquad t \in (a, +\infty), \tag{1}$$

coupled with the boundary conditions

$$u(a) = u(+\infty) = 0, \tag{2}$$

where $c > 0$ is a constant, $a \in \mathbb{R}$. We suppose that $f(t,s) : \mathbb{R}^2 \to \mathbb{R}$ and

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$f_s(t,s) = \frac{\partial f}{\partial s}(t,s), f_t(t,s) = \frac{\partial f}{\partial t}(t,s)$ are continuous functions which satisfy the following conditions:

$$c_1|s|^{1+q} \leq |f(t,s)| \leq c_2|s|^{1+q}, sf(t,s) \geq 0, \quad \forall (t,s) \in \mathbb{R}^2, \tag{H1}$$

$$f_s(t,s)s^2 - Af(t,s) \geq 0, \quad \forall (t,s) \in \mathbb{R}^2, \tag{H2}$$

$$f(t,s)s - (2+\alpha)F(t,s) \geq 0, \quad \forall (t,s) \in \mathbb{R}^2, \tag{H3}$$

$$c_3|s|^{q+2} \geq f_t(t,s)\, s \geq 0, \quad \forall (t,s) \in \mathbb{R}^2, \tag{H4}$$

where $A > 1$, $\alpha > 0$, $c_j > 0$, $j = 1,2,3$, and $F(t,s) := \int_0^s f(t,\tau)\, d\tau$.

An example of a function f, which satisfies these conditions is $f(t,s) = C|s|^q .s$ with $A \in (1, q+1]$, $\alpha = q$, $c_1 = c_2 = c_3 = C > 0$.

We will look for positive solutions u of (1.1) such that $u \in H_0^1(a,+\infty) \cap H_{c,a}$, where $H_0^1(a,+\infty)$ is the usual Sobolev space and

$$H_{c,a} := \left\{ u \in H_{loc}^1(a,+\infty) : \int_a^{+\infty} e^{ct} u'(t)^2\, dt < +\infty, \quad u(+\infty) = 0 \right\} \tag{3}$$

with norm

$$||u||_{c,a} = \int_a^{+\infty} e^{ct} \left(u'(t)^2 + u(t)^2 \right) dt$$

and $H_{c,a}^0 := H_{c,a} \cap H_0^1(a,+\infty) = \{u \in H_{c,a} : u(a) = 0\}$.

Equation (1) is obtained by the Fisher-Kolmogorov's equation $u_t = u_{xx} + f(t,u)$, looking for the traveling waves $u(x,t) = U(x-ct)$ with speed c. There is a vast studies on heteroclinic solutions of eq. (1). We refer the reader to Kolmogorov, Petrovsky and Piskunov [3], Aronson and Weinberger [2], Arias and al. [1], Nehari [7], Versini [9], Szulkin [8], Li and Wang [4] and references therain. Solutions of (1) with initial data with compact support are studied in [2]. Fast solutions of Eq. (1) are studied in the paper of Arias and al. [1] via variational methods. Heteroclinic solutions for non-autonomous second order differential equations are studied in [5, 10]. Verzini also studied equation of type (1), when $c = 0$, and she proves the existence of many oscillating solutions belonging to $L^\infty(\mathbb{R})$. She used the variational method and the approach of Nehari [7]. In the present paper we will prove the existence of positive solution of Eq. (1), belonging to $H_0^1(a,+\infty)$, using the methods of [9].

Note that by (H1) it follows

$$\frac{c_1}{q+2}|s|^{q+2} \leq F(t,s) \leq \frac{c_2}{q+2}|s|^{q+2}, \quad \forall (t,s) \in \mathbb{R}^2, \tag{4}$$

for some positive constants c_1 and c_2.

We introduce the energy functional associated with the problem (1), (2), further referred as (P):

$$J_{[a,b]}(u) := \int_a^b e^{ct}\left(\frac{u'(t)^2}{2} - F(t,u(t))\right)dt,$$

$$J(u) = J_{[a,+\infty)}(u) := \int_a^{+\infty} e^{ct}\left(\frac{u'(t)^2}{2} - F(t,u(t))\right)dt,$$

where $a \in \mathbb{R}$ is a fixed real number and $b \in (a,+\infty)$. Let

$$\mu(u) = \mu_{[a,+\infty)}(u) := \sup_{\lambda>0} J_{[a,+\infty)}(\lambda u) = \sup_{\lambda>0} J(\lambda u). \tag{5}$$

We will prove that for each nonzero function u, $\mu(u) = \sup_{\lambda>0} J(\lambda u) \geq C > 0$. There exists unique number $\lambda = \lambda(u) > 0$, for which $\sup_{\lambda>0} J(\lambda u)$ is attained, i.e. $\mu(u) = \sup_{\lambda>0} J(\lambda u) = J(\lambda(u)u)$.

We introduce the following set, analogous to the Nehari manifold:

$$N(a,+\infty) := \left\{u \in H^0_{c,a} \setminus (0) : \lambda(u) = 1\right\}$$

$$= \left\{u \in H^0_{c,a} \setminus (0) : \nabla J(u).u = 0\right\},$$

as in Verzini [9]. Since we are looking for the solutions of (1), which are non-negative on $[a,+\infty)$, we introduce the set

$$N^+(a,+\infty) := \{u \in N(a,+\infty) : u \geq 0\}.$$

Define the function

$$\varphi^+(a,+\infty) := \inf\left\{\sup_{\lambda>0} J(\lambda u) : u \in H^0_{c,a} \setminus (0), u \geq 0\right\}.$$

Our main result is:

Theorem 1. *Let the conditions (H1)-(H4) hold.*

Then $\varphi^+(a,+\infty)$ is attained by at least one function $u_+ \in N^+(a,+\infty)$, $u_+ > 0$ on $(a,+\infty)$ and $u_+(t)$ is a solution of the problem (P) for $t \in (a,+\infty)$.

The paper is organized as follows. In Section 2 we give preliminaries on the function spaces, embedding inequalities and three lemmas for the corresponding functional J. In Section 3 we give the proof of Theorem 1 and some comments.

2. Preliminaries

By [1] for each $u \in H_{c,a}$ the following inequality holds:

$$\int_a^{+\infty} e^{ct} u'(t)^2 \, dt \geq \frac{c}{2} e^{ct_0} u(t_0)^2 + \frac{c^2}{4} \int_a^{+\infty} e^{ct} u(t)^2 \, dt \tag{6}$$

for any $t_0 \in [a, +\infty)$. The inequality (6) shows that in the linear space $H_{c,a}$ we can introduce the norm

$$\|u\|_{H_{c,a}} = \left(\int_a^{+\infty} e^{ct} u'(t)^2 \, dt \right)^{\frac{1}{2}},$$

corresponding to the scalar product $\langle u, v \rangle_{H_{c,a}} = \int_a^{+\infty} e^{ct} u'(t) \, v'(t) \, dt$.

For each function $u \in H_{c,a} \subset L^\infty [a, +\infty)$, $\sup\limits_{t \in [a,+\infty)} |u(t)| < +\infty$ and it is attained, i.e.

$$\sup_{t \in [a,+\infty)} |u(t)| = \max_{t \in [a,+\infty)} |u(t)| = |u(t_1)|$$

for some point $t_1 \in [a, +\infty)$. Further by C we will denote various positive constants not depending on u.

We have the following lemma.

Lemma 1. *Let $u \in H_{c,a}$. Then the inequality*

$$\int_a^{+\infty} e^{ct} u'(t)^2 \, dt \geq C(u(t_1)^2 + \int_a^{+\infty} e^{ct} u(t)^2 \, dt + \frac{\int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt}{|u(t_1)|^q} \tag{7}$$

$$+ (\int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt)^{\frac{2}{q+2}}),$$

holds for a constant $C > 0$.

Proof. We have by (6) that

$$\begin{aligned} \int_a^{+\infty} e^{ct} u'(t)^2 \, dt &\geq \frac{c}{2} e^{ct_1} u(t_1)^2, \\ \int_a^{+\infty} e^{ct} u'(t)^2 \, dt &\geq \frac{c^2}{4} \int_a^{+\infty} e^{ct} u(t)^2 \, dt. \end{aligned}$$

By Young's inequality we get

$$\int_a^{+\infty} e^{ct} u'(t)^2 dt \geq C(u(t_1)^2 + \int_a^{+\infty} e^{ct} u(t)^2 dt)$$

$$\begin{aligned}
&\geq C\left(u(t_1)^2+\frac{\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt}{|u(t_1)|^q}\right)\\
&\geq C\left(u(t_1)^2\right)^{\frac{q}{q+2}}\frac{\left(\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt\right)^{\frac{2}{q+2}}}{\left(|u(t_1)|^q\right)^{\frac{2}{q+2}}}\\
&= C\left(\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt\right)^{\frac{2}{q+2}}, \quad C>0
\end{aligned}$$

for each nonzero function $u \in H_{c,a}$. These inequalities imply (7).

Next, we have the following

Lemma 2. *Let the function $f(t,s) \in C^1(\mathbb{R}^2)$ satisfy the conditions (H1), (H4) and $\mu(u)$ is defined by (5). Then, for every nonzero function $u(t) \in H_{c,a}$, $\mu_{[a,+\infty)}(u) \geq C > 0$, where the constant C does not depend on u.*

Proof. By (4),

$$\begin{aligned}
J(\lambda u) &= \frac{1}{2}\lambda^2\int_a^{+\infty} e^{ct}u'(t)^2\,dt-\int_a^{+\infty} e^{ct}F(t,\lambda u(t))\,dt\\
&\geq \frac{1}{2}\lambda^2\int_a^{+\infty} e^{ct}u'(t)^2\,dt-\left(\frac{c_2}{q+2}\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt\right)\lambda^{q+2}\\
&= A_1\lambda^2-B\lambda^{q+2},
\end{aligned}$$

where $A_1=\frac{1}{2}\int_a^{+\infty} e^{ct}u'(t)^2\,dt>0$, $B=\frac{c_2}{q+2}\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt>0$, since $u(t)\in H_{c,a}$ is nonzero function. We have that for every $\lambda\in[0,+\infty)$,

$$\begin{aligned}
\mu_{[a,+\infty)}(u) &= \sup_{\lambda>0} J(\lambda u)\\
&\geq c_4(q)\frac{2^{\frac{2}{q}}.q}{(q+2)^{1+\frac{2}{q}}}\left(\frac{\frac{1}{2}\int_a^{+\infty} e^{ct}u'(t)^2\,dt}{\left(\frac{c_2}{q+2}\int_a^{+\infty} e^{ct}|u(t)|^{q+2}dt\right)^{\frac{2}{q+2}}}\right)^{\frac{q+2}{q}}\\
&\geq c_5(q)>0,
\end{aligned}$$

where the constants $c_4(q)$ and $c_5(q)$ depend only on q. In the conclusion of the last inequality, we took into account (7). The lemma is proved.

Let $u(t)\in H_{c,a}$ be an arbitrary fixed nonzero function and the conditions of Lemma 2 be fulfilled. By (4) we have

$$\begin{aligned} J(\lambda u) &= \frac{1}{2}\lambda^2 \int_a^{+\infty} e^{ct} u'(t)^2\,dt - \int_a^{+\infty} e^{ct} F(t, \lambda u(t))\,dt \\ &\le \lambda^2 A_1 - \left(\frac{c_1}{q+2} \int_a^{+\infty} e^{ct} |u(t)|^{q+2}\,dt \right) \lambda^{q+2} \\ &= A_1 \lambda^2 - B_1 \lambda^{q+2}. \end{aligned}$$

Then $J(\lambda u) < 0$ for sufficiently large $\lambda > 0$. Since $J(0) = 0$ and $J(\lambda u)$ is continuous function in λ, then $\mu(u) = \sup_{\lambda > 0} J(\lambda u)$ is attained.

Moreover $\mu(u) = \mu(ku)$ for every constant $k > 0$.

Thus for the given nonzero function $u(t) \in H_{c,a}$, there exists a positive number $\lambda_0 > 0$, such that

$$\mu(\lambda_0 u) = J(\lambda_0 u) = \sup_{\lambda > 0} J(\lambda u).$$

If we denote the function $\lambda_0 u \in H_{c,a} \backslash \{0\}$ again by u, then the last equality can be written as

$$\mu(u) = J(u). \tag{8}$$

We show that for any nonzero function $v \in H_{c,a}$, i.e. function belonging to $H_{c,a} \backslash \{0\}$, there exists a function $u \in H_{c,a} \backslash \{0\}$ such that $u = kv$, with suitable constant $k > 0$, such that (8) holds. As in [9, p.2017] it follows that

$$\frac{\partial}{\partial \lambda} J(\lambda u) \,|_{\lambda=1} = \nabla J(u)\, u = \int_a^{+\infty} e^{ct} \Big(u'(t)^2 - f(t, u(t))\, u(t) \Big)\,dt = 0, \tag{9}$$

which holds for critical points of $J(\lambda u)$ as a function of λ.

Lemma 3. *Let the function $f(t,s) \in C^1(\mathbb{R}^2)$ satisfy the conditions (H1), (H2) and (H4) and $u \in H_{c,a}$ be nonzero function, for which (9) holds. Then*

$$J''(u)[u,u] < 0.$$

Moreover there exists unique number $\lambda = \lambda(u) > 0$ such that $\mu(u) = J(\lambda(u)\, u)$ and the function $u \to \lambda(u)$ is of class C^1.

Proof. We suppose that for the nonzero function $u \in H_{c,a}$, (9) holds, but $J''(u)[u,u] \ge 0$. Then

$$\int_a^{+\infty} e^{ct} \Big(u'(t)^2 - f_s(t, u(t))\, u(t)^2 \Big)\,dt \ge 0. \tag{10}$$

Subtracting (10) from (9), we obtain that

$$\int_a^{+\infty} e^{ct} \left(f_s\left(t, u\left(t\right)\right) u\left(t\right)^2 - f\left(t, u\left(t\right)\right) u\left(t\right) \right) dt \leq 0.$$

Taking into account (H1) and (H2), we get

$$\begin{aligned} 0 &\geq \int_a^{+\infty} e^{ct} \left(f_s\left(t, u\left(t\right)\right) u\left(t\right)^2 - f\left(t, u\left(t\right)\right) u\left(t\right) \right) dt \\ &\geq \int_a^{+\infty} e^{ct} \left(A-1\right) f\left(t, u\left(t\right)\right) u\left(t\right) dt \\ &\geq c_5 \left(A-1\right) \int_a^{+\infty} e^{ct} \left|u\left(t\right)\right|^{2+q} dt, \end{aligned}$$

where the constant $c_5 > 0$ and $A > 1$. Thus we proved that $\int_a^{+\infty} e^{ct} \left|u\left(t\right)\right|^{2+q} dt \leq 0$. But it is impossible for nonzero function $u\left(t\right)$. The obtained contradiction shows that the considered function $u\left(t\right)$ satisfies the inequality $J''\left(u\right)\left[u, u\right] < 0$. The rest of the proof of the lemma is as in [9, Proposition 3.1]. Exactly, the unique number $\lambda = \lambda\left(u\right) > 0$ such that $\mu\left(u\right) = J\left(\lambda\left(u\right) u\right)$, satisfies the equation

$$\Phi\left(\lambda\left(u\right), u\right) := \nabla J\left(\lambda\left(u\right) u\right).u = 0.$$

Also $\frac{\partial}{\partial\lambda}\Phi\left(\lambda, u\right) = \frac{\partial}{\partial\lambda}\left(\nabla J\left(\lambda u\right).u\right) = J''\left(\lambda u\right)\left[u, u\right] < 0$ for $\lambda = \lambda\left(u\right)$, where Φ is of class C^1 and the function $\lambda = \lambda\left(u\right)$ can be locally implicitly defined. From the implicit function theorem $\lambda\left(u\right)$ is of class C^1. Lemma 3 is proved.

The considerations in the proof of Lemma 3 show that

$$\varphi^+\left(a, +\infty\right) = \inf_{N^+\left[a, +\infty\right)} J\left(u\right).$$

3. Proof of the main result

Let $\{u_n\} \subset N^+\left(a, +\infty\right)$ be a minimizing sequence for $\varphi^+\left(a, +\infty\right)$. Without loss of generality, we can suppose that

$$\varphi^+\left(a, +\infty\right) + \varepsilon \geq J\left(u_n\right) \longrightarrow \varphi^+\left(a, +\infty\right), \qquad n \to +\infty \tag{11}$$

for sufficiently small number $\varepsilon > 0$.

Proof of Theorem 1:

Step 1. We have

$$c_6 \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt \le J(u_n) \le c_7 \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt, \tag{12}$$

$$c_8 \int_a^{+\infty} e^{ct} F(t, u_n(t))\, dt \le J(u_n) \le c_9 \int_a^{+\infty} e^{ct} F(t, u_n(t))\, dt \tag{13}$$

for some positive constants c_i, $i = 6, 7, 8, 9$. Since $\nabla J(u_n).u_n = 0$, as in (9),

$$\int_a^{+\infty} e^{ct} \left(u_n'(t)^2 - f(t, u_n(t))\, u_n(t) \right) dt = 0.$$

We have $J(u_n) = \int_a^{+\infty} e^{ct} \left(\frac{1}{2} u_n'(t)^2 - F(t, u_n(t)) \right) dt$. By (H3) and (4),

$$\begin{aligned} 2J(u_n) &= \int_a^{+\infty} e^{ct} \left(f(t, u_n(t))\, u_n(t) - 2F(t, u_n(t)) \right) dt \\ &\ge \alpha \int_a^{+\infty} e^{ct} F(t, u_n(t))\, dt \ge \frac{\alpha c_1}{q+2} \int_a^{+\infty} e^{ct} |u_n(t)|^{2+q}\, dt. \end{aligned}$$

This inequality shows that $J(u_n) \ge 0$. Hence

$$\begin{aligned} \frac{1}{2} \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt &= \int_a^{+\infty} e^{ct} F(t, u_n(t))\, dt + J(u_n) \\ &\le \frac{c_2}{q+2} \int_a^{+\infty} e^{ct} |u_n(t)|^{2+q}\, dt + J(u_n) \\ &\le \left(1 + \frac{2c_2}{\alpha c_1} \right) J(u_n) \end{aligned}$$

and

$$J(u_n) \ge \frac{\alpha c_1}{2\alpha c_1 + 4c_2} \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt.$$

By the definition of $J(u_n)$ and (4), $J(u_n) \le \frac{1}{2} \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt$. Thus, we proved the first inequality (12) with $c_6 = \frac{\alpha c_1}{2\alpha c_1 + 4c_2}$ and $c_7 = \frac{1}{2}$. The inequality (13) holds with $c_8 = \frac{\alpha}{2}$ and $c_9 = \frac{c_2(q+2)}{2c_1}$ by (H1) and (4) since

$$\begin{aligned} 2J(u_n) &\le \int_a^{+\infty} e^{ct} u_n'(t)^2\, dt = \int_a^{+\infty} e^{ct} f(t, u_n(t)) u_n(t)\, dt \\ &\le \frac{c_2(q+2)}{c_1} \int_a^{+\infty} e^{ct} F(t, u_n(t)) dt. \end{aligned}$$

These assertions show that $\int_a^{+\infty} e^{ct} u_n'(t)^2\,dt$ and $\int_a^{+\infty} e^{ct}|u_n(t)|^{2+q}\,dt$ are bounded by constant, which does not depend on n. From the inequality (6), the same is true for $\int_a^{+\infty} e^{ct} u_n(t)^2\,dt$. Hence

$$\int_a^{+\infty} e^{ct}(u_n(t)^2 + u_n'(t)^2)dt$$

and

$$\int_a^{+\infty} e^{ct}|u_n(t)|^{2+q}\,dt$$

are bounded by a constant, which does not depend on n. Then, the sequence $\{u_n\}$ is bounded in $H_{c,a}$ equipped by the norm $\|u\|_{H_{c,a}} = \left(\int_a^{+\infty} e^{ct} u'(t)^2\,dt\right)^{\frac{1}{2}}$.

Step 2. There exists a function $u_0 \in H^0_{c,a} := H_{c,a} \cap H^1_0[a,+\infty)$ and subsequence $\{u_{n_k}\}$ still denoted by $\{u_n\}$, such that $u_n \longrightarrow u_0$ for $n \to +\infty$ in the week $H^0_{c,a}-$ topology; $u_n \longrightarrow u_0$ for $n \to +\infty$ in the strong L^2- topology and on any bounded and closed subinterval of $[a,+\infty)$. Moreover $u_0 \geq 0$ is nonzero function. Let us remind that $\{u_n\} \subset N^+(a,+\infty)$ and then $u_n \geq 0, n = 1, 2, \ldots$. Hence $u_0 \geq 0$.

We will prove that u_0 is nonzero function. Suppose the contrary, i.e., that $u_0 \equiv 0$. This means that $u_n \longrightarrow 0$ for $n \to +\infty$ in the week $H^0_{c,a}-$ topology and $u_n \longrightarrow 0$ for $n \to +\infty$ in the strong L^2- topology on any bounded and closed subinterval of $[a,+\infty)$. Let $b > a$ be an arbitrary number. From [1, p.321],

$$|u_n(b)| \leq \left(\frac{e^{-cb}}{c}\int_b^{+\infty} e^{ct} u_n'(t)^2\,dt\right)^{\frac{1}{2}} \leq c_{10} e^{-\frac{c}{2}b}, \quad \forall b > a \tag{14}$$

and the constant $c_{10} > 0$ does not depend on $n \in \mathbb{N}$ and b. Let $b \in (a,+\infty)$ be a fixed (sufficiently large) number. Then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, depending on b and ε, such that

$$\int_a^b e^{ct} u_n(t)^2\,dt \leq e^{cb}\int_a^b u_n(t)^2\,dt < \varepsilon, \quad \forall n \geq n_0.$$

From (14), replacing b by t, $|u_n(t)| \leq c_{10} e^{-\frac{c}{2}t}$, $\forall t \in [a,+\infty)$ and for some $n_1 > n_0$, $n_1 \in \mathbb{N}$, depending on b, we get by $q > 0$

$$\int_a^b e^{ct}|u_n(t)|^{2+q}dt \leq c_{10}\int_a^b e^{ct}|u_n(t)|^2 dt < \frac{\varepsilon}{2}, \ \forall n \geq n_1. \tag{15}$$

From (14), we have

$$\int_b^{+\infty} e^{ct}|u_n(t)|^{2+q}dt \leq \max_{t\in[b,+\infty)} |u_n(t)|^q \cdot \int_a^{+\infty} e^{ct}|u_n(t)|^2 dt \leq c_{11} e^{-\frac{c}{2}bq} \tag{16}$$

and the constant $c_{11} > 0$ does not depend on $b \in (a, +\infty)$ and $n \in \mathbb{N}$. Now let $\varepsilon > 0$ be a fixed, sufficiently small number. We choose the number $b \in (a, +\infty)$ so large, such that $c_{11} e^{-\frac{c}{2} bq} < \frac{\varepsilon}{2}$. Then by (16),

$$\int_b^{+\infty} e^{ct} |u_n(t)|^{2+q} dt < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}.$$

We choose the number $n_1 \in \mathbb{N}$, $n_1 > n_0$, such that (15) holds and hence $\int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} dt < \varepsilon$, $\forall n \geq n_1$. Thus we prove that

$$\lim_{n \to +\infty} \int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} dt = 0.$$

By Step 1, (4) and (13) it follows $\lim\limits_{n \to +\infty} J(u_n) = 0$.This contradicts to Lemma 2, according which $J(u_n) = \mu_{[a,+\infty)}(u_n) \geq C > 0$. The contradiction shows that u_0 is a nonzero function.

Step 3. $\int_a^{+\infty} e^{ct} u_n'(t)^2\, dt \geq \int_a^{+\infty} e^{ct} u_0'(t)^2\, dt + o(1)$.
We have

$$\int_a^{+\infty} e^{ct} u_n'(t)^2\, dt = \int_a^{+\infty} e^{ct} \left[u_0'(t) + \left(u_n'(t) - u_0'(t)\right)\right]^2 dt \tag{17}$$

$$= \int_a^{+\infty} e^{ct} u_0'(t)^2\, dt + 2 \int_a^{+\infty} e^{ct} \left(u_n'(t) - u_0'(t)\right) u_0'(t)\, dt$$

$$+ \int_a^{+\infty} e^{ct} \left(u_n'(t) - u_0'(t)\right)^2 dt.$$

By $u_n \longrightarrow u_0$ for $n \to +\infty$ weakly in $H_{c,a}^0$ we have

$$\lim_{n \to +\infty} \int_a^{+\infty} e^{ct} \left(u_n'(t) - u_0'(t)\right) u_0'(t)\, dt$$
$$= \lim_{n \to +\infty} \left\langle u_n'(t) - u_0'(t), u_0'(t)\right\rangle_{H_{c,a}} = 0.$$

Then

$$2 \int_a^{+\infty} e^{ct} \left(u_n'(t) - u_0'(t)\right) u_0'(t)\, dt = o(1),$$

and by (17),

$$\int_a^{+\infty} e^{ct} u_n'(t)^2\, dt \geq \int_a^{+\infty} e^{ct} u_0'(t)^2\, dt + o(1).$$

Step 4. $\lim\limits_{n \to +\infty} \int_a^{+\infty} e^{ct} F(t, u_n(t))\, dt = \int_a^{+\infty} e^{ct} F(t, u_0(t))\, dt$.
As in Step 2, replacing u_n by $u_n - u_0$, we can prove that

$$\lim_{n \to +\infty} \int_a^{+\infty} e^{ct} \left|u_n(t) - u_0(t)\right|^{2+q} dt = 0. \tag{18}$$

By (H1) and $u_n(t) \geq 0$, $u_0(t) \geq 0$,

$$\begin{aligned}|F(t,u_n(t)) - F(t,u_0(t))| &= \left|\int_{u_0(t)}^{u_n(t)} f(t,\tau)\,d\tau\right| \leq c_2 \left|\int_{u_0(t)}^{u_n(t)} \tau^{1+q} d\tau\right| \\ &= \frac{c_2}{q+2}\left|u_n(t)^{2+q} - u_0(t)^{2+q}\right|.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\int_a^{+\infty} e^{ct} F(t,u_n(t))\,dt - \int_a^{+\infty} e^{ct} F(t,u_0(t))\,dt\right| \\ \leq\ & \frac{c_2}{q+2}\int_a^{+\infty} e^{ct}\left|u_n(t)^{2+q} - u_0(t)^{2+q}\right| dt \\ \leq\ & K_1 \int_a^{+\infty} e^{ct}\left|u_n(t) - u_0(t)\right|\left(u_n(t)^{1+q} + u_0(t)^{1+q}\right) dt.\end{aligned}$$

Using the Hölder inequality, it is easy to obtain that

$$\begin{aligned}&\left|\int_a^{+\infty} e^{ct} F(t,u_n(t))\,dt - \int_a^{+\infty} e^{ct} F(t,u_0(t))\,dt\right| \\ \leq\ & K_1 \left(\int_a^{+\infty} e^{ct}(u_n(t)^{1+q} + u_0(t)^{1+q})^{\frac{2+q}{1+q}} dt\right)^{\frac{1+q}{2+q}} \\ & \left(\int_a^{+\infty} e^{ct}\left|u_n(t) - u_0(t)\right|^{2+q} dt\right)^{\frac{1}{2+q}} \\ \leq\ & K_2 \left(\int_a^{+\infty} e^{ct} u_n(t)^{2+q} dt + \int_a^{+\infty} e^{ct} u_0(t)^{2+q} dt\right)^{\frac{1+q}{2+q}} \\ & \left(\int_a^{+\infty} e^{ct}\left|u_n(t) - u_0(t)\right|^{2+q} dt\right)^{\frac{1}{2+q}} \\ \leq\ & K_3 \left(\int_a^{+\infty} e^{ct}\left|u_n(t) - u_0(t)\right|^{2+q} dt\right)^{\frac{1}{2+q}},\end{aligned}$$

where $K_j, j = 1,2,3$ are constants not depending on u and n. Taking into account (18), we obtain the assertion of Step 4.

Step 5. There exists a nonzero function $u_+ \in N^+(a,+\infty)$, which $u_+ \geq 0$ and $J_{[a,+\infty)}(u_+) = \varphi^+(a,+\infty) = \inf_{N^+(a,+\infty)} J_{[a,+\infty)}(u) > 0$.

From (4), Step 3 and Step 4 it follows that $J(u_n) \geq J(u_0) + o(1)$ (i.e., the functional J is weakly lower semi continuous). Hence the inequality $J_{[a,+\infty)}(\lambda u_0) \leq J_{[a,+\infty)}(\lambda u_n) + o(1)$, $\forall \lambda > 0$ holds.

Define

$$u_+ := \lambda(u_0)\, u_0.$$

We obtain,that u_+ satisfies the conditions of Step 5.

We will prove that $u_+ > 0$ on $(a,+\infty)$. For this purpose, we adapt the proof of Theorem 3.1 of [9, pp. 2019-2020] to our case.

Claim 1. $u_+ \in C^1(a,+\infty)$. We assume that there exists $\tau \in (a,+\infty)$, for which $u_+(\tau) = 0$ and $u'_+(t)$ is not continuous for $t = \tau$. Let $u'_+(\tau - 0) < 0$ and the constants $\rho > 0$ and $\varepsilon > 0$ are sufficiently small numbers. For $\lambda \in [1-\varepsilon, 1+\varepsilon]$ we consider the class of problems

$$\inf \left\{ J_{[\tau-\rho,\tau+\rho]}(\upsilon) : \upsilon \in H^1(\tau-\rho,\tau+\rho), \quad \begin{array}{c} \upsilon(\tau-\rho) = \lambda u_+(\tau-\rho) \\ \upsilon(\tau+\rho) = \lambda u_+(\tau+\rho) \\ \|\upsilon\|_\infty \leq 1 \end{array} \right\} \tag{19}$$

where $\tau-\rho, \tau+\rho \in (a,+\infty)$. We have

$$\frac{d^2}{d\lambda^2} J_{[\tau-\rho,\tau+\rho]}(u+\lambda\varphi)_{\lambda=0} = \int_{\tau-\rho}^{\tau+\rho} e^{ct}\left(\varphi'(t)^2 - f'_s(t,u(t))\,\varphi(t)^2\right) dt \tag{20}$$

$$\geq \int_{\tau-\rho}^{\tau+\rho} e^{ct}\varphi'(t)^2\, dt - c_{13}\int_{\tau-\rho}^{\tau+\rho} e^{ct}\varphi(t)^2\, dt$$

$$\geq \left(\frac{c_{14}}{\rho^2} - c_3\right)\int_{\tau-\rho}^{\tau+\rho} e^{ct}\varphi(t)^2\, dt > 0,$$

where the constants $c_{13} := \sup\{f'_s(t,u) : a \leq t < +\infty, -1 \leq u \leq 1\}$, $c_{14} > 0$, and the function $\varphi(t)$ vanishes on at least one point $t = t_0 \in [\tau-\rho,\tau+\rho]$. As in [9, p. 2026, Lemma 5.1], we can conclude, that J is strictly convex, and thus the minimum of (19) is uniquely achieved by a function υ_λ. We will show that υ_λ satisfies Eq. (1.) and $\lim\limits_{\rho\to 0}\|\upsilon_\lambda\|_{H^1(\tau-\rho,\tau+\rho)} = 0$. For this purpose, we need to prove that $|\upsilon_\lambda(t)| < 1$ and $\upsilon_\lambda(t) \geq 0$ for every $t \in [\tau-\rho,\tau+\rho]$.

First, we prove that $\|\upsilon_\lambda\|_\infty = \max\limits_{t\in[\tau-\rho,\tau+\rho]} |\upsilon_\lambda(t)| < 1$. Suppose the contrary, $\|\upsilon_\lambda\|_\infty = 1$. By inclusion $H^1(\tau-\rho,\tau+\rho) \subset C[\tau-\rho,\tau+\rho]$, $u_+ \in C[\tau-\rho,\tau+\rho]$ and $\upsilon_\lambda \in C[\tau-\rho,\tau+\rho]$. By $u_+(\tau) = 0$, $\upsilon_\lambda(\tau\pm\rho) = \lambda u_+(\tau\pm\rho) = o(1)$ for $\rho \to 0+$. If $\upsilon_\lambda(\tau_1) = \pm 1$ for some $\tau_1 \in (\tau-\rho,\tau+\rho)$, then

$$
\begin{aligned}
&|\upsilon_\lambda(\tau_1)-\upsilon_\lambda(\tau-\rho)| = 1-o(1) = \left|\int_{\tau-\rho}^{\tau_1} \upsilon'_\lambda(t)\,dt\right| \\
&\le \int_{\tau-\rho}^{\tau+\rho} e^{-\frac{ct}{2}} e^{\frac{ct}{2}} \left|\upsilon'_\lambda(t)\right| dt \\
&\le \left(\int_{\tau-\rho}^{\tau+\rho} e^{-ct} dt\right)^{\frac{1}{2}} \left(\int_{\tau-\rho}^{\tau+\rho} e^{ct} \upsilon'_\lambda(t)^2\,dt\right)^{\frac{1}{2}} \\
= &\left(\frac{e^{-c(\tau-\rho)}-e^{-c(\tau+\rho)}}{c}\right)^{\frac{1}{2}} \left(\int_{\tau-\rho}^{\tau+\rho} e^{ct} \upsilon'_\lambda(t)^2\,dt\right)^{\frac{1}{2}} \\
&\le c_{15}\sqrt{\rho}\left(\int_{\tau-\rho}^{\tau+\rho} e^{ct} \upsilon'_\lambda(t)^2\,dt\right)^{\frac{1}{2}},
\end{aligned}
$$

where the constant $c_{15} > 0$ is close to $(2e^{-c\tau})^{\frac{1}{2}}$ for small ρ, i.e. c_{15} depends only on τ. Since $1-o(1) \ge \frac{1}{\sqrt{2}}$ for small ρ, then

$$
\int_{\tau-\rho}^{\tau+\rho} e^{ct} \upsilon'_\lambda(t)^2\,dt \ge \frac{1}{2c_{15}^2\rho}.
$$

Since $F(t,\upsilon_\lambda)$ is bounded for $t \in [\tau-\rho,\tau+\rho]$, and $\frac{1}{2c_{15}^2\rho} - C \ge \frac{c_{16}}{\rho}$ for small $\rho > 0$, it implies

$$
J_{[\tau-\rho,\tau+\rho]}(\upsilon_\lambda) \ge \frac{c_{16}}{\rho}. \tag{21}
$$

Remind that $u_+ \in H^1(\tau-\rho,\tau+\rho)$ and $u_+(\tau) = 0$. As above

$$
\begin{aligned}
\max_{[\tau-\rho,\tau+\rho]} |u_+(t)| &= \max_{[\tau-\rho,\tau+\rho]} |u_+(t)-u_+(\tau)| \\
&\le c_{15}\sqrt{\rho}\left(\int_{\tau-\rho}^{\tau+\rho} e^{ct} u'_+(t)^2\,dt\right)^{\frac{1}{2}} = o(1)\sqrt{\rho},
\end{aligned}
$$

because $u_+ \in H^0_{c,a}$ and then $\left(\int_{\tau-\rho}^{\tau+\rho} e^{ct} u'_+(t)^2\,dt\right)^{\frac{1}{2}} = o(1)$ when $\rho > 0$ is sufficiently small. Thus

$$
|\lambda u_+(\tau \pm \rho)| = o(\sqrt{\rho}),\ \forall \lambda \in [1-\varepsilon, 1+\varepsilon].
$$

Now we consider the linear function

$$
w(t) = u_+(\tau-\rho) - \frac{u_+(\tau-\rho)-u_+(\tau+\rho)}{2\rho}(t-\tau+\rho).
$$

Evidently

$$w(t-\rho) = u_+(\tau-\rho),$$

$w(t+\rho) = u_+(\tau+\rho)$ and $w'(t) = \frac{u_+(\tau+\rho)-u_+(\tau-\rho)}{2\rho}$. As above we have

$$\begin{aligned} |u_+(\tau-\rho) - u_+(\tau+\rho)| &= \left|\int_{\tau-\rho}^{\tau+\rho} u'(s)\,ds\right| \\ &\leq c_{15}\sqrt{\rho}\left(\int_{\tau-\rho}^{\tau+\rho} e^{ct}u_+'(t)^2\,dt\right)^{\frac{1}{2}} = \sqrt{\rho}.o(1), \end{aligned}$$

because $u_+ \in H_{c,a}^0$. Then $\left(\int_{\tau-\rho}^{\tau+\rho} e^{ct}u_+'(t)^2\,dt\right)^{\frac{1}{2}} = o(1)$ when $\rho > 0$ is sufficiently small. Thus

$$\begin{aligned} |w'(t)| &\leq \frac{C}{\sqrt{\rho}}.o(1), \\ 0 &\leq \int_{\tau-\rho}^{\tau+\rho} e^{ct}w'(t)^2\,dt \leq \frac{C}{\rho}.o(1)\int_{\tau-\rho}^{\tau+\rho} e^{ct}dt \leq o(1). \end{aligned}$$

Also,

$$\begin{aligned} |w(t)| &\leq \max(u_+(\tau-\rho), u_+(\tau+\rho)) \\ &\leq \max_{[\tau-\rho,\tau+\rho]} |u_+(t)| = o(1)\sqrt{\rho}, \quad \forall t \in [\tau-\rho, \tau+\rho]. \end{aligned}$$

Hence $\|w\|_{H^1[\tau-\rho,\tau+\rho]} = o(1)$ and

$$J_{[\tau-\rho,\tau+\rho]}(w) = o(1),\ J_{[\tau-\rho,\tau+\rho]}(\lambda w) = o(1)\ \forall\lambda \in [1-\varepsilon, 1+\varepsilon]. \tag{22}$$

From (21) and (22), $J_{[\tau-\rho,\tau+\rho]}(\lambda w) << J_{[\tau-\rho,\tau+\rho]}(\upsilon_\lambda)$, $\forall\lambda \in [1-\varepsilon,\ 1+\varepsilon]$ for sufficiently small $\rho > 0$, (where $a << b$ means that $\frac{a}{b} = o(1)$). The last inequality contradicts to the fact, that infimum in (19) is attained by the function υ_λ. The contradiction is due to the assumption, that $\|\upsilon_\lambda\|_\infty = 1$. Therefore $\|\upsilon_\lambda\|_\infty < 1$ and thus υ_λ is a solution of Eq.(1).

Now we will prove that $\upsilon_\lambda \geq 0$ for $t \in [\tau-\rho, \tau+\rho]$. Suppose the contrary. Then changing the sign, υ_λ vanish at some point of $[\tau-\rho, \tau+\rho]$ and by $\upsilon_\lambda(\tau\pm\rho) = \lambda u_+(\tau\pm\rho) \geq 0$, it follows that υ_λ' also vanish at some other point of $[\tau-\rho, \tau+\rho]$. Since υ_λ satisfies (1), then

$$\left(e^{ct}\upsilon_\lambda'\right)' + e^{ct}f(t,\upsilon_\lambda) = 0 \Longrightarrow \upsilon_\lambda'(t) = -\int_{t_1}^{t} e^{-c(t-s)}f(s,\upsilon_\lambda(s))\,ds,$$

where $t_1 \in [\tau-\rho, \tau+\rho]$ is such that $\upsilon'_\lambda(t_1) = 0$. Then

$$\begin{aligned} \left|\upsilon'_\lambda(t)\right| &\leq \int_{t_1}^{t} e^{-c(t-s)} \left|f(s, \upsilon_\lambda(s))\right| ds \\ &\leq c_{17} \int_{\tau-\rho}^{\tau+\rho} e^{-c(t-s)} ds = c_{17} \frac{e^{-c(\tau-\rho)} - e^{-c(\tau+\rho)}}{c} \leq c_{18}\rho, \end{aligned}$$

because $f(s, \upsilon_\lambda(s))$ is bounded, for $s \in [\tau-\rho, \tau+\rho]$. Thus

$$\left|\upsilon'_\lambda(t)\right| \leq c_{18}\rho \ \forall t \in [\tau-\rho, \tau+\rho], \tag{23}$$

where the constant $c_{18} > 0$ depends only on τ. Recall that $u_+(\tau) = 0$ and $u'_+(\tau-0) < 0$. Then for $\rho > 0$ sufficiently small, $u_+(\tau-\rho) = u_+(\tau) - \rho u'_+(\tau-0) = -\rho u'_+(\tau-0)$. Hence

$$\upsilon_\lambda(\tau-\rho) = \lambda u_+(\tau-\rho) = -\lambda\rho u'_+(\tau-0),$$

i.e. $c_{19}\rho \leq \upsilon_\lambda(\tau-\rho) \leq c_{20}\rho$ for $\rho > 0$ sufficiently small. Then $\upsilon_\lambda(t) = \upsilon_\lambda(\tau-\rho) + \int_{\tau-\rho}^{t} \upsilon'_\lambda(s)\, ds$ and from (23)

$$\left|\int_{\tau-\rho}^{t} \upsilon'_\lambda(s)\, ds\right| \leq c_{18}\rho^2 << \upsilon_\lambda(\tau-\rho), \quad \forall t \in [\tau-\rho, \tau+\rho].$$

Thus $\upsilon_\lambda(t)$ cannot change the sign when $t \in [\tau-\rho, \tau+\rho]$. This consideration is true also , when $u'_+(\tau-0) = -\infty$, because $\upsilon_\lambda(\tau-\rho) \geq C\rho$ for sufficiently large constant $C > 0$ and $\upsilon_\lambda(t)$ does not change the sign again. We prove, that $\upsilon_\lambda \geq 0$ for $t \in [\tau-\rho, \tau+\rho]$.

Note that $\upsilon_\lambda(t) \in C^1(\tau-\rho, \tau+\rho)$, because $\upsilon_\lambda(t)$ satisfies Eq. (1) for $t \in (\tau-\rho, \tau+\rho)$. Now we define the function

$$\widetilde{u}_\lambda(t) := \begin{cases} \upsilon_\lambda(t), & t \in [\tau-\rho, \tau+\rho], \\ \lambda u_+(t), & t \in [a, +\infty) \setminus [\tau-\rho, \tau+\rho]. \end{cases}$$

We should note that $\widetilde{u}_\lambda(t) \geq 0 \quad \forall t \in [a, +\infty)$ and $\widetilde{u}_\lambda(t) \in H^0_{c,a}$, because $\upsilon_\lambda(t)$ satisfies the boundary conditions in (19). We will prove, that $\widetilde{u}_\lambda \to \lambda u_+$ in $H^0_{c,a}$, when $\rho \to 0$. Indeed, from (22), $J_{[\tau-\rho,\tau+\rho]}(\upsilon_\lambda) \leq J_{[\tau-\rho,\tau+\rho]}(w) \leq o(1)$ and since

$$\|\upsilon_\lambda\|_\infty < 1 \Longrightarrow \int_{\tau-\rho}^{\tau+\rho} F(t, \upsilon_\lambda(t))\, dt < \frac{2c_2}{q+2}\rho,$$

then $\int_{\tau-\rho}^{\tau+\rho} e^{ct} \upsilon'_\lambda(t)^2\, dt = o(1)$ when $\rho \to 0$. This proves that $\widetilde{u}_\lambda \to \lambda u_+$ in $H^0_{c,a}$, when $\rho \to 0$. Further the proof that $u_+ \in C^1(a, +\infty)$ holds as in [9, p.2020]. Finally we should note that there exists finite right derivative $u'_+(a+0)$. Indeed, if $u_+(t) \equiv 0$ in a small right neighborhood of a, then obviously $u'_+(a+0) = 0$. If $u_+(t) > 0$ in a small right neighborhood of a, then $u_+(t)$ satisfies (1) and it is easy to show that

$$\begin{aligned} u'_+(a+0) \quad &: \quad = \lim_{t\to a+0} u'_+(t) \\ &= e^{-c(a-t_0)}\left[u'_+(t_0) + \int_a^{t_0} e^{c(s-t_0)} f(s, u_+(s))\, ds\right], \end{aligned}$$

where $t_0 > a$ is a point, close enough to a. Thus we prove, that $u_+ \in C^1[a, +\infty)$.

Claim 2. $u_+(t) > 0$, $\forall t \in [a, +\infty)$.

We assume that $u_+(t_0) = 0$ for some $t_0 > a$, and $u_+(t) > 0$ in a small right (left) neibourghood of t_0. Since $u_+ \geq 0$, then t_0 is a point of local minimum for u_+ and since $u_+ \in C^1(a, +\infty)$, then $u'_+(t_0) = 0$. Since $f(t, 0) \equiv 0$, then $u \equiv 0$ is a solution of (1) for the Cauchy conditions $u(t_0) = u'(t_0) = 0$ and from the uniqueness theorem of the local Cauchy problem, follows that $u_+(t) \equiv 0$ in a small neibourhood of t_0. Thus it implies $u_+(t) = 0$, $\forall t \in [a, +\infty)$, which contradicts to the definition of u_+ as a nonzero function. This contradiction proves the assertion of Claim 2. Theorem 1 is proved.

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