

SINGULAR VALUES AND REAL FIXED POINTS OF
ONE-PARAMETER FAMILIES ASSOCIATED WITH
FUNDAMENTAL TRIGONOMETRIC FUNCTIONS

$\sin z$, $\cos z$ and $\tan z$

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Abstract: This article is devoted to investigate the singular values as well as the real fixed points of one-parameter families of transcendental meromorphic functions which are associated with fundamental trigonometric functions $\sin z$, $\cos z$ and $\tan z$. For this purpose, we consider the functions $f_\mu(z) = \frac{\sin z}{z^2 + \mu}$, $g_\eta(z) = \frac{\cos z}{z^2 + \eta}$ and $h_\kappa(z) = \frac{\tan z}{z^2 + \kappa}$ for $\mu > 0$, $\eta > 0$ and $\kappa > 0$ respectively, and $z \in \mathbb{C}$. It is found that the functions $f_\mu(z)$ and $g_\eta(z)$ have infinite number of bounded singular values while the function $h_\kappa(z)$ has infinite number of unbounded singular values. Moreover, the real fixed points of $f_\mu(z)$, $g_\eta(z)$ and $h_\kappa(z)$ are described.

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Key Words: fixed points; transcendental meromorphic function; singular values; critical values; asymptotic value

1. Introduction

Singular values and fixed points play a vital role to study the dynamics of transcendental functions. The role of the singular values and the fixed points are explored in [1, 4, 7, 14]. The results on singular values and fixed points of transcendental functions are also found in [16]. Further, the singular values of one-parameter families are explored in [9, 10]. Moreover, for one-parameter

families of functions, the real fixed points are presented in [11, 12]. Such investigations are very important for describing Julia sets, Fatou sets as well as other properties of the dynamics of functions in the complex plane [3, 5, 8, 15]. Moreover, the bifurcation in the real dynamics of one-parameter families of transcendental functions are shown in [2, 6, 13] using the real fixed points.

A point $z^* \in \mathbb{C}$ is called a critical point of $f(z)$ if $f'(z^*) = 0$. The critical value of $f(z)$ is given by $f(z^*)$ corresponding to z^* . If there exists a continuous curve $\gamma : [0, \infty) \rightarrow \hat{\mathbb{C}}$ satisfying $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ and $\lim_{t \rightarrow \infty} f(\gamma(t)) = w$, then the point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called an asymptotic value for $f(z)$. A singular value of f is defined to be either a critical value or an asymptotic value of f . A function f is called critically bounded or it is said to be a function of bounded type if the set of all singular values of f is bounded, otherwise it is said to be unbounded type. A point x is called a fixed point of function $f(x)$ if $f(x) = x$.

The present work deals with the singular values and the real fixed points of three one-parameter families of transcendental meromorphic functions which are associated with fundamental trigonometric functions. For this purpose, we consider the following three families of transcendental functions associated with $\sin z$, $\cos z$ and $\tan z$ respectively:

$$\mathcal{S} = \left\{ f_\mu(z) = \frac{\sin z}{z^2 + \mu} : \mu > 0, z \in \mathbb{C} \right\},$$

$$\mathcal{C} = \left\{ g_\eta(z) = \frac{\cos z}{z^2 + \eta} : \eta > 0, z \in \mathbb{C} \right\},$$

$$\mathcal{T} = \left\{ h_\kappa(z) = \frac{\tan z}{z^2 + \kappa} : \kappa > 0, z \in \mathbb{C} \right\}.$$

All the functions $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$ have infinitely many zeroes but $f_\mu \in \mathcal{S}$ and $g_\eta \in \mathcal{T}$ have two poles and $h_\kappa \in \mathcal{T}$ has infinitely many poles. The functions $f_\mu \in \mathcal{S}$ and $h_\kappa \in \mathcal{T}$ are odd and the function $g_\eta \in \mathcal{C}$ is even. Such types of functions are very important in the investigations involving nonlinear equations arising from modelling of engineering and scientific problems. Often, such functions can be represented naturally through transcendental entire or meromorphic functions. Our all three families of functions are highly nonlinear in nature.

In Section 2, the singular values of functions $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$ are discussed. In Section 3, the real fixed point of functions $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$ are investigated. At the end, the conclusion of this work is provided as Section 4.

2. Singular values of $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$

In this section, the singular values of functions $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$ are obtained. In the following theorem, it is found that the function $f_\mu \in \mathcal{S}$ possesses infinitely many singular values:

Theorem 1. *Let $f_\mu \in \mathcal{S}$. Then, the function $f_\mu(z)$ has infinitely many bounded singular values.*

Proof. $f'_\mu(z) = \frac{(z^2+\mu)\cos z - 2z\sin z}{(z^2+\mu)^2} = 0$ for critical points of $f_\mu(z)$. It provides us the equation $(z^2 + \mu)\cos z - 2z\sin z = 0$. It can be written as

$$\cot z = \frac{2z}{z^2 + \mu}. \quad (1)$$

(i) If $x \neq 0$ and $y \neq 0$, then, from Equation (1),

$$\frac{\cos(x + iy)}{\sin(x + iy)} = \frac{2(x + iy)}{(\mu + x^2 - y^2) + 2ixy}.$$

Separating real and imaginary parts,

$$\frac{\sin 2x}{\cos 2x - \cosh 2y} = \frac{-2x(\mu + x^2 + y^2)}{(\mu + x^2 - y^2)^2 + 4x^2y^2}, \quad (2)$$

$$\frac{\sinh 2y}{\cos 2x - \cosh 2y} = \frac{2y(\mu - x^2 - y^2)}{(\mu + x^2 - y^2)^2 + 4x^2y^2}. \quad (3)$$

Dividing Equation (2) by Equation (3), we have

$$\frac{\sin 2x/2x}{\sinh 2y/2y} = \frac{x^2 + y^2 + \mu}{x^2 + y^2 - \mu}. \quad (4)$$

For $\mu > 0$, it is observed that the left side less than 1 and the right hand side greater than 1 for $x \neq 0$ and $y \neq 0$ which is contradiction. Therefore, Equation (1) has no solution for $x \neq 0$ and $y \neq 0$.

(ii) If $x = 0$, then, by Equation (1),

$$\coth y = -\frac{2y}{\mu - y^2}.$$

For $\mu > 0$, the roots of this equation are purely imaginary. Hence, Equation (1) has purely imaginary solutions for $x = 0$. It can be justified from Figure 1 that $\coth y$ and $\frac{-2y}{\mu - y^2}$ have only two intersections for nonzero y . It gives that Equation (1) has only two solutions on imaginary axis.

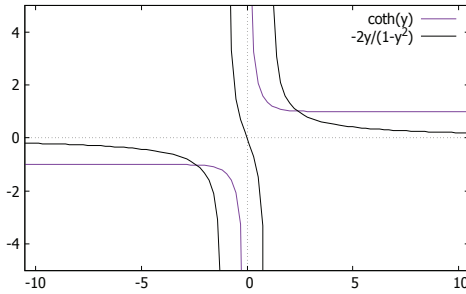


Figure 1: Two intersections in graphs of $\coth y$ and $-\frac{2y}{\mu-y^2}$ for $\mu = 1$

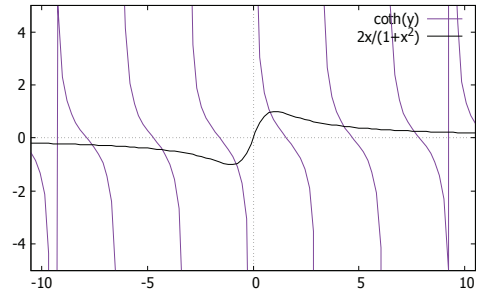


Figure 2: Infinitely many intersections in graphs of $\cot x$ and $\frac{2x}{\mu+x^2}$ for $\mu = 1$

(iii) If $y = 0$, then, by Equation (1),

$$\cot x = \frac{2x}{x^2 + \mu}.$$

For $\mu > 0$, the roots of this equation are purely real. Therefore, Equation (1) has purely real solutions for $y = 0$. From Figure 2, it is observed that $\cot x$ and $\frac{2x}{\mu+x^2}$ have infinitely many intersections. It shows that Equation (1) has infinitely many solutions on the real axis.

Let $\{x_k\}_{k=-\infty}^{\infty}$ be real critical points. Then, the critical values $\frac{\sin x_k}{x_k^2 + \mu}$ are distinct for different x_k , $k = 0, \pm 1, \pm 2, \dots$. It provides that the critical values of $f_\mu \in \mathcal{S}$ are infinite in number.

Since $|\sin x_k| \leq 1$ for real critical points $\{x_k\}_{k=-\infty}^{\infty}$, then

$$|f_\mu(x_k)| = \frac{|\sin x_k|}{|x_k^2 + \mu|} \leq \frac{1}{|x_k^2 + \mu|}.$$

It shows that all the real critical values $f_\mu \in \mathcal{S}$ are bounded.

Similarly, for two imaginary critical points $\pm iy_c$,

$$|f_\mu(\pm iy_c)| = \frac{|\sin(\pm iy_c)|}{|(\pm iy_c)^2 + \mu|} = \frac{|\sinh y_c|}{|\mu - y_c^2|}.$$

Since y_c is a finite value, it gives that the critical values $f_\mu(\pm iy_c)$ are bounded.

The asymptotic value of $f_\mu \in \mathcal{S}$ is 0 since $f_\mu(z)$ tends to 0 along positive and negative axes.

Thus, it proves that $f_\mu \in \mathcal{S}$ possesses infinitely many bounded singular values for $\mu > 0$. \square

In the following theorem, the singular values of function $g_\eta \in \mathcal{C}$ are discussed.

Theorem 2. *Let $g_\eta \in \mathcal{C}$. Then, the function $g_\eta(z)$ has infinitely many bounded singular values.*

Proof. The proof of this result is similar as Theorem 1. For critical points of $g_\eta(z)$, $g'_\eta(z) = -\frac{(z^2+\eta)\sin z+2z\cos z}{(z^2+\eta)^2} = 0$. This implies

$$\tan z = -\frac{2z}{z^2 + \eta}. \quad (5)$$

(i) If $x \neq 0$ and $y \neq 0$, then, from Equation (5),

$$\frac{\sin(x+iy)}{\cos(x+iy)} = -\frac{2(x+iy)}{(\eta+x^2-y^2)+2ixy}.$$

Using real and imaginary parts, we have

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \frac{-2x(\eta+x^2+y^2)}{(\eta+x^2-y^2)^2+4x^2y^2}, \quad (6)$$

$$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \frac{-2y(\eta-x^2-y^2)}{(\eta+x^2-y^2)^2+4x^2y^2}. \quad (7)$$

After dividing Equation (6) by Equation (7), we get

$$\frac{\sin 2x/2x}{\sinh 2y/2y} = \frac{\eta+x^2+y^2}{\eta-x^2-y^2}. \quad (8)$$

For $\eta > 0$, it is found that the left side less than 1 and the right hand side greater than 1 for $x \neq 0$ and $y \neq 0$ which is contradiction. Therefore, Equation (5) has no solution for $x \neq 0$ and $y \neq 0$.

(ii) For $x = 0$, by Equation (5),

$$\tanh y = -\frac{2y}{\eta-y^2}.$$

For $\eta > 0$, the roots of this equation are purely imaginary. It follows that Equation (5) has purely imaginary solutions for $x = 0$. Moreover, from Figure 3, it is seen that $\tanh y$ and $\frac{-2y}{\eta - y^2}$ have only two intersections for nonzero y . Hence, Equation (5) has only two solutions on imaginary axis.

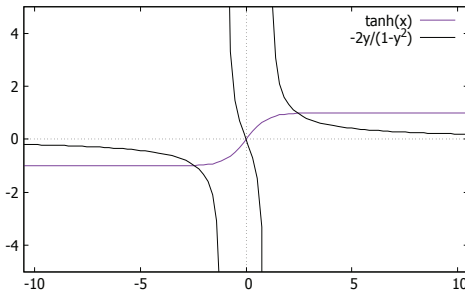


Figure 3: Two intersections in graphs of $\tanh y$ and $-\frac{2y}{\eta - y^2}$ for $\eta = 1$

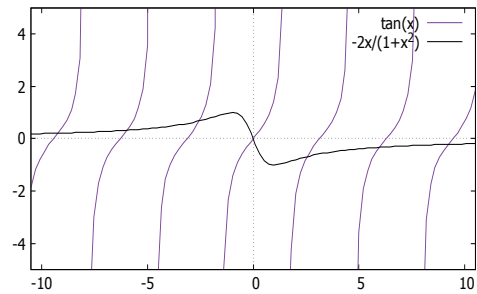


Figure 4: Infinitely many intersections in graphs of $\tan x$ and $-\frac{2x}{x^2 + \eta}$ for $\eta = 1$

(iii) If $y = 0$, then, by Equation (5),

$$\tan x = -\frac{2x}{x^2 + \eta}.$$

For $\eta > 0$, the roots of this equation are purely real. Hence, Equation (5) has purely real solutions for $y = 0$. Moreover, From Figure 4, it is observed that $\tan x$ and $-\frac{2x}{\eta + x^2}$ have infinitely many intersections. It gives that Equation (5) has infinitely many solutions on the real axis.

(iv) If $x = 0$ and $y = 0$, then the trivial solution is $z=0$.

The remaining proof is similar lines as Theorem 1. So we omit it here.

Therefore, the proof is completed. \square

The following theorem explains that the function $h_\kappa \in \mathcal{T}$ has infinitely many singular values.

Theorem 3. *Let $h_\kappa \in \mathcal{T}$. Then, the function $h_\kappa(z)$ has infinitely many unbounded singular values.*

Proof. The proof of this result is similar as Theorem 1. For critical points of $h_\kappa(z)$, $h'_\kappa(z) = \frac{(z^2+\kappa)\sec^2 z - 2z \tan z}{(z^2+\kappa)^2} = 0$. This gives us $(z^2 + \kappa) - z \sin(2z) = 0$. It can be written as

$$\sin(2z) = \frac{z^2 + \kappa}{z}. \quad (9)$$

- (I) If $x \neq 0$ and $y \neq 0$, then, from Equation (9), it has infinitely many solutions for $\kappa > 0$. Let $\{z_k\}_{k=-\infty}^\infty$ be critical points. Then, the critical values $\frac{\tan z_k}{z_k^2 + \kappa}$ are distinct for different z_k . Hence, the critical values of $f_\kappa \in \mathcal{T}$ are infinite in number.

Since

$$|f_\kappa(z_k)| = \frac{|\tan z_k|}{|z_k^2 + \kappa|}.$$

This quantity is not bounded when $z_k \rightarrow \infty$. It shows that the critical values $f_\kappa \in \mathcal{T}$ are unbounded.

- (II) For $x = 0$, by Equation (9), we have

$$\sinh(2y) = \frac{y^2 - \kappa}{y}.$$

From Figure 5, it is observed that it has no any solution for $\kappa > 0$.

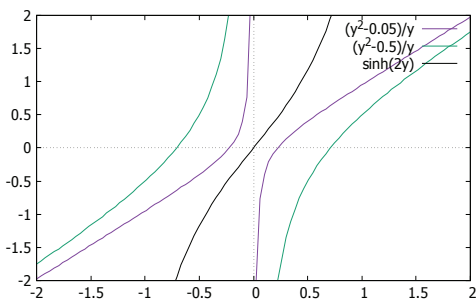


Figure 5: No intersections in graphs of $\sinh(2y)$ and $\frac{y^2 - \kappa}{y}$ for $\kappa = 0.05, 0.5$

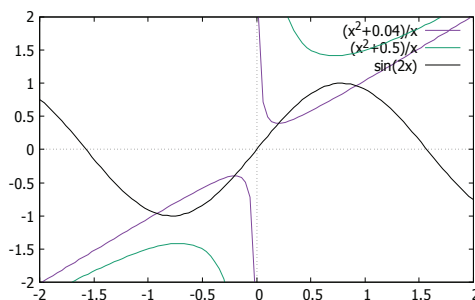


Figure 6: Two or no intersections in graphs of $\sin(2x)$ and $\frac{x^2 + \kappa}{x}$ for $\kappa = 0.04, 0.5$

- (III) If $y = 0$, then, by Equation (9), then

$$\sin(2x) = \frac{x^2 + \kappa}{x}. \quad (10)$$

For $\kappa > 0$, from Figure 6, it is seen that this equation has two solutions for $0 < \kappa < \kappa^*$ while no any solution for $\kappa > \kappa^*$.

The proof of theorem is completed. \square

3. Real fixed points of $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$

In this section, the real fixed points of $f_\mu \in \mathcal{S}$, $g_\eta \in \mathcal{C}$ and $h_\kappa \in \mathcal{T}$ are described.

3.1. Real fixed points of $f_\mu \in \mathcal{S}$

We need the following lemma to determine the fixed points of $f_\mu \in \mathcal{S}$.

Lemma 4. Let $\psi(x) = \frac{\sin x}{x} - x^2$ and $\psi(0) = 1$. Then,

- (i) $\psi(x)$ is continuous in \mathbb{R} .
- (ii) $\psi(x) < 0$, $x \in (-\infty, -\alpha) \cup (\alpha, \infty)$ and $\psi(x) > 0$, $x \in (-\alpha, \alpha)$, where $\alpha (\approx 0.92863)$ is a solution of the equation $\sin x - x^3 = 0$.
- (iii) $\psi(x) \rightarrow -\infty$ for $x \rightarrow -\infty$ and $x \rightarrow \infty$.
- (iv) $\psi'(x)$ is continuous in \mathbb{R} .
- (v) $\psi'(x)$ has a unique zero at $x = 0$.
- (vi) $\psi(x)$ increases strictly in $(-\infty, 0)$, decreases strictly in $(0, \infty)$ and attains maximum at $x = 0$.

Proof. We can easily proof from (i) to (v). To prove (vi), we have

$$\psi'(x) = \frac{x \cos x - \sin x}{x^2} - 2x,$$

$$\psi''(x) = -\frac{(x^2 - 2) \sin x + 2x \cos x}{x^3} - 2.$$

It follows that $\psi''(0) = -\frac{7}{3} < 0$. Therefore, $\psi(x)$ has maxima at $x = 0$.

By (iii), it is easily seen that $\psi(x)$ increases strictly in $(-\infty, 0)$ and decreases strictly in $(0, \infty)$. \square

Theorem 5. *Let $f_\mu \in \mathcal{S}$. Then, the point 0 is a fixed point for all $\mu > 0$ and the nonzero real fixed points of $f_\mu(x)$ are given by the following:*

- (a) *For $\mu > 1$, $f_\mu(x)$ has no nonzero fixed points.*
- (b) *For $0 < \mu < 1$, $f_\mu(x)$ has exactly two nonzero fixed points.*

Proof. Since $f_\mu(0) = 0$, then the point 0 is a fixed point for all μ . For the nonzero fixed points, we have $f_\mu(x) = x$. Then, the nonzero fixed points are solutions of the following equation

$$\mu = \frac{\sin x}{x} - x^2 = \psi(x),$$

where, $\psi(x)$ is given in Lemma 4.

- (a) By (vi) of Lemma 4, $\psi(x)$ has a maxima at $x = 0$. Since $\psi(0) = 1$, then there is no intersection between the line $u = \mu$ and the graph of $u = \psi(x)$. Hence, there is no any solution of $\mu = \psi(x)$. It follows that $f_\mu(x)$ has no any nonzero fixed points for $\mu > 1$.
- (b) Since $\psi(0) = 1$ and $0 < \mu < 1$, by (i), (iii) and (vi) of Lemma 4, the line $u = \mu$ and the graph of $u = \psi(x)$ intersect at two points. Since $\psi(-x) = \psi(x)$, then one nonzero solution of $\mu = \psi(x)$ lies in $(-\infty, 0)$ and other lies in $(0, \infty)$. Therefore, for $0 < \mu < 1$, $f_\mu(x)$ has exactly two nonzero fixed points.

Thus, the proof is completed. □

3.2. Real fixed points of $g_\eta \in \mathcal{C}$

We need the following lemma to investigate the fixed points of $g_\eta \in \mathcal{C}$.

Lemma 6. *Let $\phi(x) = \frac{\cos x}{x} - x^2$ for $x \in \mathbb{R} \setminus \{0\}$. Then,*

- (i) *$\phi(x)$ is continuous in $(-\infty, 0) \cup (0, \infty)$.*
- (ii) *$\phi(x) < 0$ for $x \in (-\infty, 0) \cup (\alpha^*, \infty)$ and $\phi(x) > 0$ for $x \in (0, \alpha^*)$, where $\alpha^* (\approx 0.86547)$ is a solution of the equation $\cos x - x^3 = 0$.*
- (iii) *$\phi(x) \rightarrow +\infty$ for $x \rightarrow 0^+$ and $\phi(x) \rightarrow -\infty$ for $x \rightarrow 0^-$, $x \rightarrow -\infty$ and $x \rightarrow \infty$.*
- (iv) *$\phi'(x)$ is continuous in $(-\infty, 0) \cup (0, \infty)$.*

- (v) $\phi'(x)$ has a unique zero at $x = \alpha^{**}$, where α^{**} is a real negative zero of the equation $x \sin x + \cos x + 2x^3 = 0$.
- (vi) $\phi(x)$ increases strictly in $(-\infty, \alpha^{**})$, decreases strictly in $(\alpha^{**}, 0) \cup (0, \infty)$ and attains maximum at $x = \alpha^{**}$.

Theorem 7. Let $g_\eta \in \mathcal{C}$. Then, the function $g_\eta(x)$ has only one nonzero real fixed point for all $\eta > 0$.

Proof. For the nonzero fixed points, we have $g_\eta(x) = x$. Then, the nonzero fixed points are solutions of the following equation

$$\eta = \frac{\cos x}{x} - x^2 = \phi(x),$$

where, $\phi(x)$ is defined in Lemma 6.

Since $u = \phi(x)$ and $u = \eta$ have only one intersection for $\eta > 0$. Hence, $\phi(x) = \eta$ has only one nonzero real solution. It follows that $g_\eta(x)$ has only one nonzero real fixed points. \square

3.3. Real fixed points of $h_\kappa \in \mathcal{T}$

We need the following lemma to obtain the fixed points of $h_\kappa \in \mathcal{T}$.

Lemma 8. Let $\xi(x) = \frac{\tan x}{x} - x^2$ and $\xi(0) = 1$. Then,

- (i) $\xi(x)$ is piecewise continuous in \mathbb{R} and has discontinuities at $x = 2m\pi \pm \frac{\pi}{2}$, where m is an integer.
- (ii) $\xi(x) < 0$ for $x \in (-\infty, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \infty)$ and $\xi(x) > 0$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- (iii) $\xi(x) \rightarrow -\infty$ for $x \rightarrow -\frac{\pi}{2}^-$ and $x \rightarrow \frac{\pi}{2}^+$; and $\xi(x) \rightarrow +\infty$ for $x \rightarrow -\frac{\pi}{2}^+$ and $x \rightarrow \frac{\pi}{2}^-$.
- (iv) $\xi'(x)$ is piecewise continuous in \mathbb{R} and has discontinuities at $x = 2m\pi \pm \frac{\pi}{2}$, where m is an integer.
- (v) $\xi'(x)$ has infinitely many zeroes including zero at $x = 0$.
- (vi) $\xi(x)$ increases strictly in some intervals, decreases strictly in intervals and attains maximum at many points as well as minimum at two points.

Theorem 9. *Let $h_\kappa \in \mathcal{T}$. Then, the point 0 is a fixed point for all $\kappa > 0$ and the nonzero real fixed points of $f_\kappa(x)$ are infinitely many in number:*

Proof. Since $h_\kappa(0) = 0$, then the point 0 is a fixed point for all $\kappa > 0$. For the nonzero fixed points, we have $h_\kappa(x) = x$. Then, the nonzero fixed points are solutions of the following equation

$$\kappa = \frac{\tan x}{x} - x^2 = \xi(x),$$

where, $\xi(x)$ is given in Lemma 8.

Since $u = \xi(x)$ and $u = \kappa$ have infinitely many intersections for $\kappa > 0$. Hence, $\xi(x) = \kappa$ has infinitely many nonzero real solutions. It shows that $h_\kappa(x)$ has infinitely many nonzero real fixed points. \square

Remark 10. *For $\mu = 0$, $\eta = 0$ and $\kappa = 0$, the research work has been found in the literature. For $\mu < 0$, $\eta < 0$ and $\kappa < 0$, it is left for forthcoming work.*

Remark 11. *In the present families of functions, the occurrence of periodic points of period two or more may possible which is left for forthcoming work.*

4. Conclusion

In this paper, the author has been investigated the singular values and the real fixed points of three one-parameter families of transcendental meromorphic functions associated with fundamental trigonometric functions $\sin z$, $\cos z$ and $\tan z$. It is found that the family of functions associated with $\sin z$ and $\cos z$ have infinite number of bounded singular values while the family of functions associated with $\tan z$ has infinite number of unbounded singular values. Moreover, it is seen that the family of functions associated with $\sin z$ and $\cos z$ have finite number of real fixed points although the family of functions associated with $\tan z$ has infinitely many real fixed points.

References

- [1] T.K. Chakra and T. Nayak, Iteration of the translated tangent, *Bull. Malays. Math. Sci. Soc.*, **42**, No 5 (2019), 1993-2008; doi: 10.1007/s40840-017-0588-3.
- [2] R.E. de Carvalho and E.D. Leonel, Squared sine logistic map, *Physica A*, **463** (2016), 37-44; doi: 10.1016/j.physa.2016.07.008.
- [3] P. Domnguez, J. Vazquez and M.A.M. de Oca, A study of the dynamics of the family $f_{\lambda,\mu} = \lambda \sin z + \frac{\mu}{z-k\mu}$ where $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{Z} \setminus \{0\}$, *Discontinuity, Nonlinearity, and Complexity*, **5**, No 4 (2016), 407-414; doi: 10.5890/DNC.2016.12.006.
- [4] N. Fagella, X. Jarque and J. Taixs, On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points I, *Proc. Lond. Math. Soc.*, **97**, No 3 (2008), 599-622; doi: 10.1112/plms/pdn012.
- [5] M. Kisaka, On topological properties of Fatou sets and Julia sets of transcendental entire functions, *Sugaku Expositions*, **30**, No 2 (2017), 235-273; doi: 10.1090/suga/425.
- [6] D. Lim, Fixed points and dynamics on generating function of Genocchi numbers, *J. Nonlinear Sci. Appl.*, **9**, No 3 (2016), 933-939; doi: 10.22436/jnsa.009.03.22.
- [7] T. Nayak and M.G.P. Prasad, Julia sets of Joukowski-Exponential maps, *Complex Anal. Oper. Theory*, **8**, No 5 (2014), 1061-1076; doi: 10.1007/s11785-013-0335-1.
- [8] M.G.P. Prasad, Chaotic burst in the dynamics of $f_{\lambda}(z) = \lambda \frac{\sinh(z)}{z}$, *Regular and Chaotic Dynamics*, **10**, No 1 (2005), 71-80; doi: 10.1070/RD2005v010n01ABEH000301.
- [9] M. Sajid, Some dynamical properties of the cosine-meromorphic family, *Int. J. Math. Sci. & Engg. Appls.*, **8**, No 1 (2014), 91-99.
- [10] M. Sajid, Singular values and fixed points of family of function $ze^z/(e^z - 1)$, *Int. J. Appl. Math.*, **27**, No 2 (2014), 147-154; doi: 10.12732/ijam.v27i2.4.
- [11] M. Sajid, Singular values and fixed points of family of generating function of Bernoulli's numbers, *J. Nonlinear Sci. Appl.*, **8**, No 1 (2015), 17-22; doi: 10.22436/jnsa.008.01.03.

- [12] M. Sajid, On real fixed points of one parameter family of function $\frac{x}{b^x-1}$, *Tamkang J. Math.*, **46**, No 1 (2015), 61-65; doi: 10.5556/j.tkjm.46.2015.1577.
- [13] M. Sajid, Real fixed points and dynamics of one parameter family of function $(b^x - 1)/x$, *J. of the Association of Arab Universities for Basic and Applied Sci.*, **21**, No 1 (2016), 92-95; doi: 10.1016/j.jaubas.2015.10.001.
- [14] M. Sajid and G.P. Kapoor, Dynamics of a family of non-critically finite even transcendental meromorphic functions, *Regular and Chaotic Dynamics*, **9**, No 2 (2004), 143-162; doi: 10.1070/RD2004v009n02ABEH000272.
- [15] S. Sharifi, M. Salimi, S. Siegmund and T. Lotfi, A new class of optimal four-point methods with convergence order 16 for solving nonlinear equations, *Math. Comput. Simulation*, **119** (2016), 69-90; doi: 10.1016/j.matcom.2015.08.011.
- [16] J.H. Zheng, On fixed-points and singular values of transcendental meromorphic functions, *Science China Mathematics*, **53**, No 3 (2010), 887-894; doi: 10.1007/s11425-010-0036-4.

