International Journal of Applied Mathematics

Volume 33 No. 4 2020, 621-633
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v33i4.7

PROBABILISTIC φ-CONTRACTION IN b-MENGER SPACES WITH FULLY CONVEX STRUCTURE

Abderrahim Mbarki[1 §], Rachid Oubrahim[2]

[1] ANO Laboratory
National School of Applied Sciences, P.O. Box 669
Mohammed First University Oujda, MOROCCO

[2] ANO Laboratory, Faculty of Sciences
Mohammed First University
60000 Oujda, MOROCCO

Abstract: We prove, in b-Menger spaces with fully convex structure the existence and uniqueness of fixed point for non-self mappings with nonlinear contractive conditions. We give also an example to validate our results.

AMS Subject Classification: 54E70, 54H25, 47S50, 34B15
Key Words: b-Menger space; fully convex structure; connectedness; fixed point

1. Introduction

A contraction is one of the main tools to prove the existence and uniqueness results on fixed point theory. So in 1972, Sehgal and Baharucha-Reid (see [17]) generalized the Banach Contraction Principle [1] to a complete Menger space (see [15] and [16]), which is a milestone in developing fixed point theorems in such space. They proved their result for mapping f such that for some constant $0 < \alpha < 1$, the probability that the distance between image points fx and fy is less than αt is at least as large as the probability that the distance between

Received: March 15, 2020

[§]Correspondence author

x and y is less than t. The contraction proved by Sehgal and Bharucha-Reid is known as Sehgal's contraction or B-contraction. The problem of obtaining fixed point theorems for probabilistic φ-contraction in Menger spaces has been studied by different authors (for example see [3]). In the former approaches to this topic the authors used conditions too much restrictive on the gauge function φ [2]. For example, Mbarki et al. proved a fixed point theorem for ϕ-contraction mapping in general probabilistic metric spaces (see [5], [8] and [9]). Later, Jachymski in [7] obtained a fixed point theorem for Menger spaces weakening the conditions on φ which improved the applicability of these types of theorems.

To extend the notion of probabilistic metric, Mbarki et al. [11] in 2017 proposed a new notion called probabilistic b-metric by generalizing the (probabilistic) triangle inequality axiom in the definition of standard probabilistic metric. They discussed some topological and geometrical properties of probabilistic b-metric spaces and they showed the fixed point and common fixed point property for a self mapping which is a nonlinear contractions in b-Menger spaces which are a particular spaces of probabilistic b-metric spaces (see [14]). Furthermore, they defined the notion of fully convex structure and established in fully convex b-Menger spaces the existence of common fixed point for nonexpansive mapping by using the normal structure property (see [12]). Also, they showed a fixed point theorem in b-Menger spaces using B-contraction with cyclical conditions (see [13]).

Furthermore, there exist in convex spaces the cases where the involved function is not necessarily a self mapping of a closed subset, then the condition $f(\partial C) \subset C$ plays a crucial role to guarantee the existence of fixed point for non self-mapping $f : C \to X$, where ∂C is boundary of set C.

In this present paper, we establish the existence and uniquenness of fixed point in b-Menger spaces for non-self mappings with nonlinear contractive conditions using the fully convex structure. An example will be given to support our results.

2. Preliminaries

We now recall some basic definitions and relevant lemmas in the theory of b-Menger spaces (see [11] and [12]).

A distance distribution function (briefly, a d.d.f.) is a nondecreasing function f defined on $\mathbb{R}^+ \cup \{\infty\}$ that satisfies $f(0) = 0$ and $f(\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f's will be denoted by Δ^+ and the set of all f in Δ^+ for which $\lim_{t\to\infty} f(t) = 1$ by D^+.

A simple example of distribution function is the Heaviside function in D^+

$$H(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is called a t-norm if and only if $T(a,\ 1) = a$ for all $a \in [0,\ 1]$. $T_M(a,\ b) = \text{Min}(a,\ b)$ is a typical example of continuous t-norm.

Definition 1. ([11]) A b-Menger space is a quadruple $(M,\ F,\ T,\ s)$ where M is a nonempty set, F is a function from $M \times M$ into Δ^+, T is a t-norm, $s \geq 1$ is a real number, and the following conditions are satisfied:

For all $a,\ b,\ c \in M$ and $x,\ y > 0$,

1. $F_{ab} = H \;\Leftrightarrow\; a = b$,
2. $F_{ab} = F_{ba}$,
3. $F_{ab}(s(x + y)) \geq T(F_{ac}(x), F_{cb}(y))$ (triangular inequality).

It should be clear that a Menger space is a b-Menger with $s = 1$.

Definition 2. Let $(M,\ F)$ be a probabilistic semimetric space (just (1) and (2) of Definition 1 are satisfied). For a in M and $t > 0$, the strong t-neighborhood of a is the set $N_a(t) = \{b \in M : \ F_{ab}(t) > 1 - t\}$. The strong neighborhood system at a is the collection $\wp_a = \{N_a(t) : \ t > 0\}$ and the strong neighborhood system for M is the union $\wp = \bigcup_{a \in M} \wp_a$.

If $(M,\ F,\ T,\ s)$ is a b-Menger space with T is continuous, then the family $\Im$ consisting of $\emptyset$ and all unions of elements of this strong neighborhood system for M determines a topology $\Im$ for M. Moreover, Mbarki et al. [11] showed that $(M,\ F,\ T,\ s)$ endowed with the topology $\Im$ is a Hausdorff space and the function F is in general not continuous.

In b-Menger space, the convergence of sequence is defined as follows.

Definition 3. Let $\{x_n\}$ be a sequence in a b-Menger space $(M,\ F,\ T,\ s)$.

1. A sequence $\{x_n\}$ is convergent to $x \in M$, if for every $\epsilon > 0$, there exists a positive integer N such that $F_{x_n x}(\epsilon) > 1 - \epsilon$ whenever $n \geq N$.

2. A sequence $\{x_n\}$ is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a positive integer N such that $n,\ m \geq N \Rightarrow F_{x_n x_m}(\epsilon) > 1 - \epsilon$.
3. $(M,\ F,\ T,\ s)$ is complete if every Cauchy sequence has a limit.

In what follows, we will be conserned by a b-Menger space $(M,\ F,\ T,\ s)$ such that T is a continuous t-norm and $RanF \subset D^+$. R. Egbert [4], in 1968, introduced a probabilistic generalization of the notion of diameter of nonempty set in metric space.

Definition 4. ([12]) Let $(M,\ F,\ T,\ s)$ be a b-Menger space and $A \subset M$. The probabilistic diameter of set A is given by $D_A(t) = \sup_{\varepsilon<t} \inf_{a,\ b\in A} F_{ab}(\varepsilon)$ and the diameter of the set A is defined by $D_A = \sup_{t>0} \sup_{\varepsilon<t} \inf_{a,\ b\in A} F_{ab}(\varepsilon)$. If $D_A = 1$ the set A will be called probabilistically bounded.

Lemma 2.1. ([16]) *For any* $a,\ b \in A$, $F_{ab} \geq D_A$ *and* $D_A = D_{\overline{A}}$.

Using the definition of $\sup A$ and $\inf A$ we have the following lemma.

Lemma 2.2. ([12]) *Let* $(M,\ F,\ T,\ s)$ *be a* b*-Menger space and* $A \subset M$. *A is probabilistically bounded if and only if for each* $\lambda \in (0,1)$ *there exists* $t > 0$ *such that* $F_{ab}(t) > 1 - t$ *for all* $a,\ b \in A$.

Let us recall the definition of the fully convex structure in b-Menger spaces introduced in [12].

Definition 5. ([12]) A convex b-Menger space $(M,\ F,\ T,\ s)$ with a convex structure $W : M \times M \times [0,\ 1] \to M$ will be called fully convex if, for arbitrary $a,\ b \in M$, $a \neq b$ there exists $\lambda \in (0,\ 1)$ such that $W(a,\ b,\ \lambda) \notin \{a,\ b\}$.

3. Main results

Definition 6. Let $(M,\ F,\ T,\ s)$ be a b-Menger space and A a nonempty subset of M. The set A is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, A is said to be connected.

Definition 7. Let $(M,\ F,\ T,\ s)$ be a b-Menger space and A a nonempty subset of M. The boundary of the set A is the intersection of the closure of A with the closure of its complement $\partial A = \overline{A} \cap \overline{(M \backslash A)}$.

Lemma 3.1. *Let $(M,\ F,\ T,\ s)$ be a b-Menger space, A and B two nonempty subsets of M. If A is connected, $A \cap \mathring{B} \neq \emptyset$ and $A \cap Ext(B) \neq \emptyset$, then $A \cap \partial B \neq \emptyset$.*

Proof. Suppose that $A \cap \partial B = \emptyset$, then $A \cap \mathring{B}$ and $A \cap Ext(B)$ are a partition of open sets of A, which contradict the connectdness of the set A. □

Definition 8. Let $(M,\ F,\ T,\ s)$ be a b-Menger space and $\{G_n\}_{n \in \mathbb{N}}$ a nested sequence of nonempty, closed subsets of M. We say that the sequence $\{G_n\}_{n \in \mathbb{N}}$ has probabilistic diameter zero if for each $\lambda \in (0,\ 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{ab}(t) > 1 - \lambda$ for all $a,\ b \in G_{n_0}$.

Since the proof of Theorem 1.4 in [18] and Theorem 2.1 in [10], the triangular inequality and the continuity of the probabilistic b-metric F play no role, we could claim the following results.

Lemma 3.2. *Let $(M,\ F,\ T,\ s)$ be a complete b-Menger space and $\{G_n\}_{n \in \mathbb{N}}$ a nested sequence of nonempty, closed subsets of M. The sequence $\{G_n\}_{n \in \mathbb{N}}$ has probabilistic diameter zero if and only if $D_{G_n} \to H$, for $n \to \infty$.*

Lemma 3.3. *Let $(M,\ F,\ T,\ s)$ be a complete b-Menger space and $\{G_n\}_{n \in \mathbb{N}}$ a nested sequence of nonempty, closed subsets of M such that $D_{G_n} \to H$, for $n \to \infty$. Then there is exactly one point $x_0 \in G_n$, for every $n \in \mathbb{N}$.*

Now, we will prove the existence and the uniqueness of fixed point in b-Menger space for mapping $f : M \to M$ satisfying some type of contraction used by Fang [6] called probabilistic φ-contraction.

Definition 9. Let $(M,\ F)$ be a probabilistic semi-metric space. A mapping $f : M \to M$ is called a probabilistic φ-contraction if $F_{fafb}(\varphi(t)) \geq F_{ab}(t)$ *for all* $t > 0$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a gauge function in the set $\Phi = \{\varphi / for\ each\ t > 0,\ there\ exists\ r \geq t\ such\ that\ \lim_{n \to \infty} \varphi^n(r) = 0\}$.

Definition 10. Let $\{x_n\}$ be a sequence which converges to x in the b-Menger space $(M,\ F,\ T,\ s)$. The mapping F verifies the condition (C_1) if for all $t > 0$, $F_{xy}(t) \geq \liminf_{n\to\infty} F_{x_n y}(t)$ for all $y \in M$.

Definition 11. Let $(M,\ F,\ T,\ s)$ be a b-Menger space. W satisfies the condition (C_2) if for every $\lambda \in (0,\ 1)$, $t > 0$ and $a,\ b,\ c \in M$, we have that $F_{W(a,\ b,\ \lambda)c}(t) \geq \min\{F_{ca}(t),\ F_{cb}(t)\}$.

We can easily show the following lemma.

Lemma 3.4. *Let $(M,\ F,\ T,\ s)$ be a b-Menger space and $a,\ b \in M$ such that $ImF \subseteq D^+$. If there exists a function $\varphi \in \Phi$, such that $F_{ab}(\varphi(t)) \geq F_{ab}(t)$ for every $t > 0$, then $a = b$.*

Now, we are ready to state and prove the main fixed point theorem of this work.

Theorem 12. *Let $(M,\ F,\ T,\ s)$ be a fully convex complete b-Menger space with convex structure $W : M \times M \times [0,\ 1] \to M$ satisfying the conditions (C_1) and (C_2). Let $f : A \to M$ be a non-self mapping satisfying $F_{fafb}(\varphi(t)) \geq F_{ab}(t)$ for all $a,\ b \in A$ and every $t > 0$, where $\varphi \in \Phi$ and A is a nonempty, closed and probabilistic bounded subset of M. Suppose also that for all $a,\ b \in M$, the set $\{W(a,\ b,\ \theta), \theta \in [0,\ 1]\}$ is connected and $f(\partial A) \subseteq \mathring{A}$. Then f has a unique fixed point in $\mathring{A}$.*

Proof. Step 1: Construction of sequence.

We select a sequence $\{p_n\}$ in A in the following way:

Let $p_0 \in \partial A$, then $fp_0 \in \mathring{A}$. Set $p_1 = fp_0$. If $fp_1 \in \mathring{A}$, set $p_2 = fp_1$. If $fp_1 \in \partial A$, set $p_2 = fp_1$ and we have $p_2 \in \mathring{A}$, and the last case is $fp_1 \notin A$, set $p_2 = W(p_1, fp_1, \theta) \in \partial A$, $\theta \in (0, 1)$. In this case, p_2 is well define because $W(p_1, fp_1, 1) = p_1 \in \mathring{A}$, $W(p_1, fp_1, 0) = fp_1 \notin A$ then $W(p_1, fp_1, 0) \in Ext(A)$ and since $\{W(p_1, fp_1; \theta), \theta \in [0, 1]\}$ is connected and the fact that (M, F, T, s) is fully convex, there exists $\theta' \in (0, 1)$ such that $W(p_1, fp_1, \theta') \notin \{p_1, fp_1\}$, by Lemma 3.1 we have

$$\{W(p_1, fp_1; \theta), \theta \in [0, 1]\} \cap \partial A \neq \emptyset.$$

By induction we may obtain sequence $\{p_n\}$ such that

$p_n = fp_{n-1}$ if $fp_{n-1} \in A$ and $p_n = W(p_{n-1}, fp_{n-1}; \theta)$, $\theta \in (0, 1)$ if $fp_{n-1} \notin A$. Additionally we have if $p_n = W(p_{n-1}, fp_{n-1}; \theta)$, $\theta \in (0, 1)$ then $p_{n+1} = fp_n$ and $p_{n-1} = fp_{n-2}$. We observe also that there exists a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$

such that

$$p_{n_k+1} = fp_{n_k}.$$

Now let

$$G_n = \{p_n, p_{n+1}, ...\};$$

$$P_n = \overline{G_n},\ n \in \mathbb{N}.$$

Observe that $\{P_n\}$ is nested sequence of nonempty closed sets.

Step 2: We prove that

$$D_{P_n}(\varphi(t)) \geq D_{P_{n-2}}(t) \tag{1}$$

holds for every $t > 0$.

Three cases must be considered:

Case 1.

$p_{n+i} = p_{n+i-1}$ and $p_{n+j} = p_{n+j-1}$ for arbitrary $i, j \in \mathbb{N}$.

In this case we have

$$\begin{aligned} F_{p_{n+i}p_{n+j}}(\varphi(t)) &= F_{fp_{n+i-1}fp_{n+j-1}}(\varphi(t)) \\ &\geq F_{p_{n+i-1}p_{n+j-1}}(t) \\ &\geq D_{P_{n-2}}(t). \end{aligned}$$

Case 2.

$p_{n+i} = p_{n+i-1}$ and $p_{n+j} = W(p_{n+j-1}, fp_{n+j-1}, \theta),\ \theta \in (0,1)$ for arbitrary $i, j \in \mathbb{N}$.

In this case we have

$$\begin{aligned} F_{p_{n+i}p_{n+j}}(\varphi(t)) &= F_{fp_{n+i-1}W(p_{n+j-1},fp_{n+j-1},\theta)}(\varphi(t)) \\ &\geq \min\left\{F_{fp_{n+i-1}p_{n+j-1}}\varphi(t), F_{fp_{n+i-1}fp_{n+j-1}}\varphi(t)\right\} \\ &= \min\left\{F_{fp_{n+i-1}fp_{n+j-2}}\varphi(t), F_{fp_{n+i-1}fp_{n+j-1}}\varphi(t)\right\} \\ &\geq \min\left\{F_{p_{n+i-1}p_{n+j-2}}(t), F_{p_{n+i-1}p_{n+j-1}}(t)\right\} \\ &\geq D_{P_{n-2}}(t). \end{aligned}$$

Case 3.

$$p_{n+i} = W(p_{n+i-1}, fp_{n+i-1}, \theta_1),\ \theta_1 \in (0,1)$$

and

$$p_{n+j} = W(p_{n+j-1}, fp_{n+j-1}, \theta_2),\ \theta_2 \in (0,1)$$

for arbitrary $i, j \in \mathbb{N}$.

In this case we have

$$\begin{aligned} F_{p_{n+i}p_{n+j}}(\varphi(t)) &= F_{W(p_{n+i-1},fp_{n+i-1},\theta_1)W(p_{n+j-1},fp_{n+j-1},\theta_2)}(\varphi(t)) \\ &\geq \min\{F_{p_{n+i-1}p_{n+j-1}}(\varphi(t)), F_{p_{n+i-1}fp_{n+j-1}}(\varphi(t)) \\ &\quad , F_{fp_{n+i-1}p_{n+j-1}}(\varphi(t)), F_{fp_{n+i-1}fp_{n+j-1}}(\varphi(t))\} \\ &= \min\{F_{fp_{n+i-2}fp_{n+j-2}}(\varphi(t)), F_{fp_{n+i-2}fp_{n+j-1}}(\varphi(t)) \\ &\quad , F_{fp_{n+i-1}fp_{n+j-2}}(\varphi(t)), F_{fp_{n+i-1}fp_{n+j-1}}(\varphi(t))\} \\ &\geq \min\{F_{p_{n+i-2}p_{n+j-2}}(t), F_{p_{n+i-2}p_{n+j-1}}(t) \\ &\quad , F_{p_{n+i-1}p_{n+j-2}}(t), F_{p_{n+i-1}p_{n+j-1}}(t)\} \\ &\geq D_{P_{n-2}}(t). \end{aligned}$$

By definition of the probabilistic diameter of P_n we have

$$D_{P_n}(\varphi(t)) = \sup_{\varepsilon<\varphi(t)} \inf_{i,j\in\mathbb{N}} F_{p_{n+i}p_{n+j}}(\varepsilon).$$

Since the function

$$\varepsilon \mapsto \inf_{i,j\in\mathbb{N}} F_{p_{n+i}p_{n+j}}(\varepsilon)$$

is nondecreasing, then

$$\sup_{\varepsilon<\varphi(t)} \inf_{i,j\in\mathbb{N}} F_{p_{n+i}p_{n+j}}(\varepsilon) \geq F_{p_{n+i}p_{n+j}}(\varphi(t)).$$

Then

$$D_{P_n}(\varphi(t)) \geq D_{P_{n-2}}(t).$$

Step 3:We prove that the family $\{P_n\}$ has probabilistic diameter zero. Let $\lambda \in (0,1)$ and $t > 0$. Since $G_k \subseteq A$ for all $k \in \mathbb{N}$ then G_k is probabilistically bounded set. By Lemma 2.2, there exists $t_0 > 0$ such that

$$F_{ab}(t_0) > 1 - \lambda,$$

for all $a, b \in G_k$, then

$$\inf_{a,b\in G_k} F_{ab}(t_0) \geq 1 - \lambda.$$

Hence

$$D_{G_k}(t_0) = \sup_{\varepsilon<t_0} \inf_{a,b\in G_k} F_{ab}(\varepsilon) \geq 1 - \lambda.$$

Since $\varphi \in \Phi$ then there exists $r \geq t_0$, such that

$$\lim_{n\to\infty} \varphi^n(r) = 0.$$

Thus, there exists $l \in \mathbb{N}$ such that

$$\varphi^l(r) < t.$$

From (1) we have

$$\begin{aligned} D_{G_n}(\varphi^l(r)) &\geq D_{G_{n-2}}(\varphi^{l-1}(r)) \\ &\geq D_{G_{n-4}}(\varphi^{l-2}(r)) \\ &\vdots \\ &\geq D_{G_{n-2l}}(r). \end{aligned}$$

Let $n_0 = 2l + k$ and $a, b \in G_{n_0}$, since $\varphi^l(r) < t$. Then

$$\begin{aligned} D_{G_{n_0}}(t) &\geq D_{G_{n_0}}(\varphi^l(r)) \\ &\geq D_{G_{n_0-2l}}(r) \\ &\geq D_{G_k}(r) \\ &\geq D_{G_k}(t_0) \\ &\geq 1 - \lambda. \end{aligned}$$

By Lemma 2.1, we get

$$D_{P_{n_0}}(t) \geq 1 - \lambda,$$

for all $a, b \in P_n$, then the sequence $\{P_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero.

By Lemma 3.2 and Lemma 3.3 we infer that there exists $q \in A$ such that

$$\bigcap_{n\in\mathbb{N}} P_n = q.$$

Step 4: We prove that q is a fixed point of f.

Since the sequence $\{P_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero, then for each $\lambda \in (0, 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$F_{p_n q} > 1 - \lambda.$$

Then

$$\liminf_{n\to\infty} F_{p_n q} \geq 1 - \lambda,$$

and by the condition (C_1) we get

$$\lim_{n\to\infty} p_n = q.$$

Hence

$$\lim_{n\to\infty} p_{n_k+1} = q \quad \text{and} \quad \lim_{n\to\infty} fp_{n_k} = q.$$

Since $\varphi \in \Phi$, then there exists $u \geq t$, such that $\lim_{n\to\infty} \varphi^n(u) = 0$, hence there exists $l \in \mathbb{N}$ such that $\varphi^l(u) < t$. Then we have

$$\begin{aligned} F_{fp_{n_k}fq}(t) &\geq F_{fp_{n_k}fq}(\varphi^l(u)) \\ &\geq F_{p_{n_k}q}(\varphi^{l-1}(u)). \end{aligned}$$

Then

$$\liminf_{n\to\infty} F_{fp_{n_k}fq}(t) \geq \liminf_{n\to\infty} F_{p_{n_k}q}(\varphi^{l-1}(u)).$$

Since $\lim_{n\to\infty} p_{n_k} = q$ and $\lim_{n\to\infty} fp_{n_k} = q$ and by the condition (C_1) we get

$$F_{qfq}(t) \geq F_{qq}(\varphi^{l-1}(u)).$$

Hence

$$F_{qfq}(t) \geq 1 \quad \text{for} \quad \text{every} \quad t > 0.$$

Therefore

$$fq = q,$$

and q is a fixed point of f.

Suppose that $q \in \partial A$, then $fq \in \mathring{A}$, hence $q \in \mathring{A}$ a contradiction because $\mathring{A} \cap \partial A = \emptyset$, therefore $q \in \mathring{A}$.

Step 5: We prove the uniqueness of the fixed point.

Assume that there exists $w \in \mathring{A}$ such that $fw = w$. We have

$$F_{fqfw}(\varphi(t)) \geq F_{qw}(t)$$

for every $t > 0$. So

$$F_{qw}(\varphi(t)) \geq F_{qw}(t)$$

for every $t > 0$.

By Lemma 3.4 we get $q = w$. □

The following example illustrates our results.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{3}{5} - \frac{x^2}{2}$ and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\varphi(t) = \begin{cases} \frac{t}{t+1}, & t \in (0,1), \\ -\frac{t}{3} + \frac{4}{3}, & t \in [1, \frac{4}{3}], \\ t - \frac{7}{12}, & t \in (\frac{4}{3}, \infty). \end{cases}$$

$(\mathbb{R}, F, T_M, 2^{n-1})$ is a fully convex b-Menger space with the convex structure $W(a,b;\lambda) = \lambda a + (1-\lambda)b$ satisfying the condition (C_2) and since F is continuous then the convex structure verifies the condition (C_1) (see [12]).

We take $A = [-\frac{1}{2}, \frac{1}{2}]$, we have $f(\partial A) \subseteq \mathring{A}$. It is clear that the set A is closed and bounded, then A is probabilistically bounded. Also we have that the set $\{W(a,b;\lambda), \lambda \in [0,1]\}$ is an interval for all $a, b \in \mathbb{R}$ then $\{W(a,b;\lambda), \lambda \in [0,1]\}$ is a connected set.

Now we prove that f is probabilistic φ-contraction. We have

$$\begin{aligned} F_{fafb}(\varphi(t)) &= H(\varphi(t) - |fa - fb|^n) \\ &= H(\varphi(t) - \frac{1}{2^n}|a^2 - b^2|^n). \end{aligned}$$

In other hand it is clear that $\varphi(t) > \frac{t}{4}$, then $\varphi(t) \geq \frac{t}{2^n}$ for all $n \geq 2$ and we have since $a, b \in [-\frac{1}{2}, \frac{1}{2}]$, $|a^2 - b^2| \leq |a - b|$, then

$$\begin{aligned} F_{fafb}(\varphi(t)) &= H(\varphi(t) - \frac{1}{2^n}|a^2 - b^2|^n) \\ &\geq H(\frac{t}{2^n} - \frac{1}{2^n}|a - b|^n) \\ &= H(t - |a - b|^n) \\ &= F_{ab}(t). \end{aligned}$$

We have that $\varphi \in \Phi$. Therefore all conditions of Theorem 12 are satisfied. Then f has unique fixed point $\frac{-5+\sqrt{55}}{5} \in \mathring{A}$.

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