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RAINBOW CONNECTION NUMBER OF FLOWER SNARK GRAPH

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Abstract: Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$ of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it. The minimum k for which there exists such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$.

In this paper we find the rainbow connection number of flower snark graph and their criticalness with respect to rainbow coloring.

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Key Words: diameter; edge-coloring; rainbow path; rainbow connection number; criticality

1. Introduction

We use [1] for terminology and notation not defined here and consider finite

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and simple graphs only.

Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, \cdots, k\},\ k \in N$, where adjacent edges may be colored the same. An edge-colored graph G is called rainbow-connected if any two vertices are connected by a path whose edges have different colors. The minimum k for which there exist a rainbow $k-$coloring of G is called the rainbow connection number of G, denoted by $rc(G)$.

The concept of rainbow connection in graphs was introduced by Chartrand et al. in [2]. In [2] the authors determined $rc(G)$ and $src(G)$ of the cycle, path, tree and wheel graphs. In [8] and [9] Srinivasa Rao and Murali, studied rainbow connection number $rc(G)$ and the strong rainbow connection number $src(G)$ of some classes of graphs like the stacked book graph, the grid graph, the prism graph etc. and discussed the critical property of these graphs with respect to rainbow coloring. The rainbow connection number has been studied for brick product graphs in [10] and [11]. Other results on the rainbow connection number of a graph can be found in [12]. In [6], Nabila et al. determined the rainbow connection number of Origami graphs and Pizza graphs. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et al. in [5].

The flower snarks (for odd n) form an infinite family of snarks introduced by Isaacs [3]. In this paper we determine the rainbow connection number $rc(G)$ of flower snark graphs and also critical property of G with respect to rainbow connection number.

2. Definitions

Definition 1. The flower snarks are a connected, bridgeless cubic graphs. The flower snarks are non-Hamiltonian and non-planar. The flower snark J_n can be constructed with the following process:

- Build n copies of the star $K_{1,3}$. Denote the central vertex of each star by b_i and the outer vertices a_i, c_i and d_i. This results in a disconnected graph on $4n$ vertices with $3n$ edges ($b_i a_i, b_i c_i\ and\ b_i d_i\ for\ 1 \le i \le n$).
- Construct the $n-$cycle $(a_1, a_2, \ldots, a_n a_1)$. This adds n edges.
- Finally construct the $2n-$cycle $(c_1 c_2, \cdots, c_n d_1 d_2, \cdots, d_n c_1)$. This adds $2n$ edges.

The flower snark graph J_5 and J_7 are shown in Figure 1.

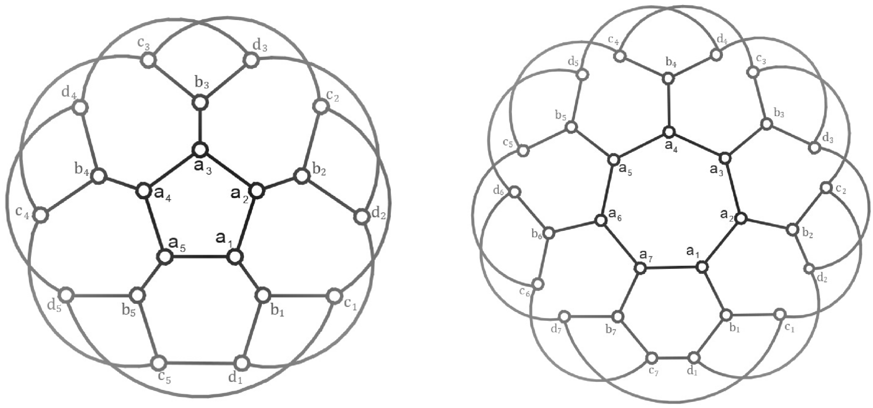

Figure 1: Flower snark graph J_5 and J_7

The rainbow criticalness was first studied by Srinivasa Rao and Murali in [8].

Definition 2. A graph G is said to be *rainbow critical* if the removal of any edge from G increases the rainbow connection number of G, i.e. if $rc(G) = k$ for some positive integer k, then $rc(G - e) > k$ for any edge e in G.

Theorem 3. *Let G be a connected graph on $n \geq 3$ vertices. Then, $rc(G) \geq max\{diam(G), n_i(G)\}$, where $n_i(G)$ denotes the number of vertices of G which have degree i for $1 \leq i \leq n-1$.*

In the next section, we determine rainbow connection number $rc(G)$ for the flower snark graph J_n for $n \geq 5$ and odd.

3. Main results

Theorem 4. *Let $G = J_n$. Then for odd $n \geq 5$,*

$$rc(G) = \lceil \tfrac{n}{2} \rceil + 3.$$

Proof. Let us define the vertex set V and the edge set E of G as $V(G) = \{a_1, a_2, \cdots, a_n\} \cup \{b_1, b_2, \cdots, b_n\} \cup \{c_1, c_2, \cdots, c_n\} \cup \{d_1, d_2, \cdots, d_n\}$, where b_i denotes the central vertex of star graph and a_i, c_i and d_i denotes outer vertices and $E(G) = \{e_i\} \cup \{a_i b_i\} \cup \{b_i c_i\} \cup \{b_i d_i\} \cup \{e_i^{'}\} \cup \{e_i^{''}\} \cup \{c_n d_1\} \cup \{d_n c_1\}$ where

e_i is the edge a_ia_{i+1}, $e_i^{'}$ is the edge c_ic_{i+1} and $e_i^{''}$ is the edge d_id_{i+1}, where $1 \leq i \leq n-1$.

Since $diam(G) = \lceil \frac{n}{2} \rceil + 1$, it follows that $rc(G) \geq \lceil \frac{n}{2} \rceil + 1$. By the definition of snark graph, G contains n copies of star graph $K_{1,3}$ with the vertices a_i, b_i, c_i and d_i where $1 \leq i \leq n$. Vertices a_i forms inner cycle and from [2] $rc(C_n) = \lceil \frac{n}{2} \rceil$ so we have to assign $\lceil \frac{n}{2} \rceil$ colors to the edges of inner cycle. Since $K_{1,3}$ is a subgraph of G, again by the definition we have to assign three different colors to the edges other than $\lceil \frac{n}{2} \rceil$ colors already assigned to the edges of inner cycle. Contrary, let us assign any one of $\lceil \frac{n}{2} \rceil$ color to the edges a_ib_i, then we fail to get rainbow path between the vertices $a_{\frac{n+3}{2}} b_{\frac{n+1}{2}}$, $\forall n$ and this is true for other pair of vertices also. So assign $\lceil \frac{n}{2} \rceil + 1$ color to a_ib_i. Similarly, for the edges b_ic_i and b_id_i we should assign two different colors other than $\lceil \frac{n}{2} \rceil + 1$ colors. If not, this coloring will not give rainbow $c_{\frac{n+1}{2}} d_n$ path $\forall n$.

Combining all combinations of assignment of colors discussed above and also from Theorem 3,

$$rc(G) \geq diam(G) + 2, \quad \text{i.e.} \quad rc(G) \geq \lceil \frac{n}{2} \rceil + 3.$$

To show $rc(G) \leq \lceil \frac{n}{2} \rceil + 3$, we define a coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil + 3\}$ and assign the colors to the edges of G as,

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ i - \lceil \frac{n}{2} \rceil & \text{if} \quad \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

$$c(a_ib_i) = \lceil \tfrac{n}{2} \rceil + 1 \quad \text{for} \quad 1 \leq i \leq n$$

$$c(b_ic_i) = \lceil \tfrac{n}{2} \rceil + 2 \quad \text{for} \quad 1 \leq i \leq n$$

$$c(b_id_i) = \lceil \tfrac{n}{2} \rceil + 3 \quad \text{for} \quad 1 \leq i \leq n$$

$$c(e_i^{'}) = c(e_i^{''}) = \begin{cases} i & \text{if} \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ i - \lceil \frac{n}{2} \rceil & \text{if} \quad \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-1 \end{cases}$$

$$c(c_nd_1) = c(d_nc_1) = n - \lceil \tfrac{n}{2} \rceil.$$

From the above assignment, it is clear that for every two distinct vertices $x, y \in V(G)$, there exists an $x - y$ rainbow path.

Hence $rc(G) \leq \lceil \frac{n}{2} \rceil + 3$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil + 3$. □

The critical nature of the flower snark graph in Theorem 4 has been observed. This is illustrated in our next result.

Theorem 5. *Let $G = J_n$, for odd $n \geq 5$. Then G is rainbow critical, i.e.*

$$rc(G - e) = 2n \quad \textit{for} \quad n \geq 5 \ \textit{and odd.}$$

Proof. We consider vertex set $V(G)$ and edge set $E(G)$ as defined in Theorem 4. Let E_1, E_2 and E_3 are the subset of $E(G)$, i.e.,

$$E(G) = E_1 \cup E_2 \cup E_3,$$

where

$$E_1 = \{a_i a_{i+1}\} \quad - \quad \text{edges of inner cycle,}$$

$$E_2 = \{a_i b_i,\, b_i c_i\, b_i d_i\} \quad - \quad \text{edges of star } K_{1,3} \text{ and}$$

$$E_3 = \{c_i c_{i+1},\, d_i d_{i+1},\, c_n d_1,\, d_n c_1\} \quad - \quad \text{edges of outer cycle.}$$

Consider the graph G and let $e \in E(G)$ be the any edge in G. Deletion of the edge e in G does not yield a rainbow path between the end vertices of e in $G - e$. So, we prove this result in following cases.

Case 1:

Let $e \in E_1$. Deletion of edge e in G, leads to path of length $n - 1$ and for any arbitrary edge $e \in E_1$, it requires n colors such that $G - e$ is rainbow connected.

Case 2:

Now consider the edge $e \in E_2$. According to the assignment of colors to the edges of star $K_{1,3}$ in the Theorem 4, $\lceil \frac{n}{2} \rceil + 3$ colors is not sufficient in $G - e$. i.e deletion of e will not get rainbow path between the end vertices. So assign same n colors already assigned to the edges of E_1 in different combination.

Particularly, if $e = b_i c_i$ or $b_i d_i$ if we assign same n colors in any combination, the graph $G - e$ will not get rainbow path between the vertices a_i to c_i or between the vertices a_i to d_i. Also the vertex sets $\{a_i, a_{i+1}, b_i, b_{i+1}, c_i, c_{i+1}\}$ and $\{a_i, a_{i+1}, b_i, b_{i+1}, d_i, d_{i+1}\}$ form a cycle of length 6. Deletion of any edge in the cycle leads to the path of length 5, which will be the shortest path. It holds up to n vertices. So to get rainbow path in $G - e$ where $e = b_i c_i$ or $e = b_i d_i$, it requires different n colors other than the colors already assigned to the edges $a_i a_{i+1}$ and $a_i b_i$.

Case 3:

Finally let $e \in E_3$. In this case also we fail to get rainbow path between the end vertices in $G - e$. Since the edges of E_3 forms outer cycle and also we show that in above 2 cases, $2n$ colors requires to get rainbow path in $G - e$, implies assignment of same colors to the edges in different combinations leads $G - e$ is rainbow connected.

Considering the above 3 cases, it is clear that $rc(G-e) \geq 2n$.

To prove $rc(G-e) \leq 2n$, we construct a rainbow coloring $c : E(G) \to \{1, 2, \cdots, 2n\}$ to the edges of G as,

$$c(a_i a_{i+1}) = i \text{ if } 1 \leq i \leq n$$

$$c(a_i b_i) = \begin{cases} i + (\frac{n-1}{2}) & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ i - (\frac{n+1}{2}) & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

$$c(b_i c_i) = c(b_i d_i) = i + n \text{ if } 1 \leq i \leq n-1$$

$$c(c_i c_{i+1}) = c(d_i d_{i+1}) = i + 1 \text{ if } 1 \leq i \leq n-1$$

$$c(c_n d_1) = c(d_n c_1) = 1.$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x-y$ path in $G-e$ and hence $rc(G-e) \leq n$.

This proves $rc(G-e) = 2n$.

Hence the proof is done. □

4. Conclusion

In this paper, we have determined the rainbow connection number of flower snark graph J_n. The critical property of flower snark graphs with respect to rainbow coloring is also investigated.

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