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ON THE MULTI-INDEX MITTAG-LEFFLER FUNCTIONS AND THEIR MELLIN TRANSFORMS

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Abstract: In this survey paper we consider the classes of the multi-index Mittag-Leffler functions, introduced and studied by the authors as extensions of the classical Mittag-Leffler functions $E_{\alpha,\beta}$ and of the Prabhakar function $E^{\gamma}_{\alpha,\beta}$, by means of replacing the 2 parameters α, β, respectively the 3 parameters α, β, γ, by $2m$-, resp. $3m$- sets of parameters, $m = 1, 2, 3, ...$:

$$\alpha \to (\alpha_1, \alpha_2, ..., \alpha_m),\ \beta \to (\beta_1, \beta_2, ..., \beta_m), \gamma \to (\gamma_1, \gamma_2, ..., \gamma_m).$$

Some of their basic properties are discussed, such as the order and type of these entire functions, their place among the special functions of fractional calculus and previously known classical special functions, especially their representations as Wright's generalized hypergeometric functions and Fox's H-functions. A very long list of interesting and useful special functions that appear as particular cases is provided.

The importance of the Mellin integral transform is well known as a tool for development of the theories of the special functions and fractional calculus, in many problems for fractional order differential equations and systems whose solutions are usually presented in terms of Mittag-Leffler type functions, and in treating various mathematical models in stochastics, control theory, financial mathematics, etc., that are also widely exploring this kind of special functions. Therefore, in this survey we emphasize on the results for the Mellin-Barnes type

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contour integral representation of the multi-index Mittag-Leffler functions, and thus on their Mellin transform images.

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1. Introduction

The special functions, defined in the whole complex plane $\mathbb{C}$ by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{1}$$

with α, $\beta \in \mathbb{C}$, $Re(\alpha) > 0$, are known as Mittag-Leffler (M-L) functions ([4, Vol.3, Sect.18.1]). The first one was introduced by Mittag-Leffler, who investigated some of its properties in a series of articles published mainly in 'Acta Mathematica' (1899-1905), while the latter appeared first in a paper of Wiman (1905). For a long time in previous century, the M-L functions have been almost ignored in the common handbooks on special functions and existing tables of Laplace and Mellin transforms, although a description of their properties has appeared yet in Vol. 3 of the Bateman Project (Eds. Erdélyi et al.), in a chapter devoted to 'miscellaneous functions' ([4, Sect.18.1]). The modern aspects and a deep study on the M-L functions are presented in the book of Dzherbashyan [3]: asymptotic formulae in different parts of the complex plane, distribution of the zeros, kernel functions of inverse Borel type integral transforms, various relations and representations. The detailed properties of these functions can be found in the monographs of Podlubny [32], Kilbas-Srivastava-Trujillo [12], Gorenflo-Kilbas-Mainardi-Rogosin [5], see also Kiryakova [13], [8], [35], etc.

Recently the interest to M-L type functions and their generalizations has grown up in view of their important role in fractional calculus and related integral and differential equations of fractional order (as their solutions) and applications [35], for example in modelling some evolution problems [7], fractional diffusion processes [24], nonlinear waves, etc. The M-L function enjoys also applications in the stochastic processes, statistical distributions and conditional expectations as the so-called M-L (probability) density, see for example Mathai, Haubold et al. [27], [8], [9], etc.

Prabhakar [33] extended (1) to 3 parameters, by introducing the function (also known as Prabhakar function)

$$E^{\gamma}_{\alpha,\,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad \alpha,\ \beta,\ \gamma\in\mathbb{C},\quad Re(\alpha)>0, \tag{2}$$

where $(\gamma)_k$ is the Pochhammer symbol ([4, Sect.2.1.1])

$$(\gamma)_0=1,\quad (\gamma)_k=\gamma(\gamma+1)\dots(\gamma+k-1). \tag{3}$$

For $\gamma=1$ this function coincides with $E_{\alpha,\,\beta}$, and for $\gamma=\beta=1$ with E_{α}, i.e.:

$$E^{1}_{\alpha,\,\beta}(z)=E_{\alpha,\,\beta}(z),\qquad E^{1}_{\alpha,\,1}(z)=E_{\alpha}(z). \tag{4}$$

Prabhakar introduced this function for γ with a positive real part, and in this case it is an entire function of z of order $\rho=1/Re(\alpha)$ (as mentioned in [12], [26], [5]) and of type $\sigma=1$.

Prabhakar studied some properties of the three parametric Mittag-Leffler type function (2) and of an integral operator containing this function in the kernel, and applied his results to prove the existence and uniqueness of the solution for the corresponding integral equation. Further, some properties of $E^{\gamma}_{\alpha,\beta}(z)$ including differentiation and integration relations of integer and fractional order are proved by Kilbas, Saigo and Saxena [11], see more in the recent book by Gorenflo-Kilbas-Mainardi-Rogosin [5]. For a recent use of the Prabhakar (3-parametric) M-L type function (2) in the friction memory kernel and in the exact solutions of the fractional generalized Langevin equation, see for example Sandev and Tomovski [36].

At the end of 20th century a class of special functions of Mittag-Lefler type that are multi-index (or vector index) analogues of $E_{\alpha,\beta}(z)$ has been introduced and studied. The indices α, β are replaced by two sets of multi-indices

$$\alpha\to(\alpha_1,\alpha_2,...,\alpha_m)\text{ and }\beta\to(\beta_1,\beta_2,...,\beta_m).$$

Definition 1. Let $m>1$ be an integer, $\alpha_1,...,\alpha_m>0$, $\beta_1,...,\beta_m$ be arbitrary real (complex) numbers. By means of these 'multi-indices', the multi-index ($2m$-parametric) Mittag-Lefler functions are defined as:

$$E_{(\alpha_i),(\beta_i)}(z)=E^{m}_{(\alpha_i),(\beta_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\alpha_1k+\beta_1\right)\ ...\ \Gamma\left(\alpha_mk+\beta_m\right)}. \tag{5}$$

The class of functions (5) has been introduced first by Luchko et al. (see e.g. Yakubovich and Luchko [39], [21], [20]), and studied in details by Kiryakova (see e.g. [15], [16], [17]). A more general case of the multi-index M-L functions (5) allowing the indices $\alpha_1,\alpha_2,...,\alpha_m$ to be arbitrary real (not obligatory positive)

was introduced and studied by Kilbas et al. (see for example, Kilbas-Koroleva-Rogosin [10]). For an expectedly long list of their particular cases, see Kiryakova [16], [17], and for various applications, including as scale-invariant solutions of the diffusion-wave equation, as in [21], [6], in fractional modelling [35], see also [19]. Representations of many Special Functions of Fractional Calculus (SF of FC) in terms of the multi-index Mittag-Leffler functions (5) are also discussed. Details can be seen in [16], [17], [18], also in the monograph [29] and in the papers [30], [31].

As proved in Kiryakova [15], [16], the multi-index Mittag-Leffler functions (5) (with $2m$ parameters) are entire functions of order ρ and type σ, given by the formulas:

$$\frac{1}{\rho} = \alpha_1 + \cdots + \alpha_m, \qquad \frac{1}{\sigma} = (\rho\alpha_1)^{\rho\alpha_1} \cdots (\rho\alpha_m)^{\rho\alpha_m}, \tag{6}$$

and have a prescribed asymptotic behaviour for sufficiently large of variable $|z|$.

Some other properties of these functions, such as asymptotic formulae for 'large' values of the parameters and series in systems of such kind of functions in the complex plane $\mathbb{C}$ are studied by Paneva-Konovska [29], also the domains of convergence of such series are found, and their behaviour on the boundaries of these domains are studied.

The next level of extension is the class of the $3m$-parametric multi-index Mittag-Leffler functions, defined for complex parameters (for details see [28] and [29], generalizing both the functions (2) and (5). In this survey we mention some properties of these functions, discuss their place among the previously known special functions, and provide important integral representations and various special cases.

The emphasize is given to the Mellin transform images of the considered M-L type functions, via their Mellin-Barnes type integral representations. It is well known that the Mellin transform plays important role in the theory of special functions, see for example: [23] (for the contributions of S. Pincherle to Mellin-Barnes integrals); the book by Marichev [25] (devoted to methods of evaluation of integrals of special functions via the Mellin transform); also in fractional calculus - see works by Butzer-Kilbas-Trujillo [1], Luchko-Kiryakova [22], in probability and statistics [27], [8], [9], etc.

2. The multi-index ($3m$- and $2m$-parametric) Mittag-Leffler functions

The $3m$-parametric multi-index Mittag-Leffler functions, as further extention of the ($2m$-) multi-index M-L functions (5), are introduced by Paneva-Konovska

[28] as follows.

Definition 2. Let $m \geq 1$ be an integer and consider the parameters $\alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{C},\quad Re(\alpha_i) > 0$, for $i = 1, 2, \dots m$. By means of the multi-indices $(\alpha_i), (\beta_i), (\gamma_i)$ the so-called $3m$-parametric multi-index Mittag-Leffler ($3m$-, multi-M-L) functions are introduced as

$$E^{(\gamma_i),\, m}_{(\alpha_i),\,(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \dots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \dots \Gamma(\alpha_m k + \beta_m)}\ \frac{z^k}{(k!)^m}, \tag{7}$$

with $(\gamma_i)_k$ as in (3).

Naturally, the basic properties of (7) depend on the parameters. Starting with the case when among the parameters γ_i-s there are no negative integers or zero, we have the following result.

Theorem 3. *If each of the parameters $\gamma_1, \dots, \gamma_m$ is neither a negative integer nor zero, then the multi-index Mittag-Leffler function* (7) *is an entire function of order ρ and type σ with*

$$\frac{1}{\rho} = Re(\alpha_1) + \dots + Re(\alpha_m), \tag{8}$$

respectively

$$\frac{1}{\sigma} = |(\rho\alpha_1)^{\rho\alpha_1}| \dots |(\rho\alpha_m)^{\rho\alpha_m}|\,. \tag{9}$$

Moreover, for each positive ε the asymptotic estimate

$$|E^{(\gamma_i),\, m}_{(\alpha_i),\,(\beta_i)}(z)| < \exp((\sigma + \varepsilon)|z|^{\rho}), \quad |z| \geq r_0 > 0, \tag{10}$$

holds with ρ, σ like in (8), (9), *for $|z| \geq r_0(\varepsilon)$, $r_0(\varepsilon)$ sufficiently large.*

For the proof, see Paneva-Konovska [28, Th.2.1] and [29]. Let us note that for $\alpha_i > 0$ and all $\gamma_i = 1$, $i = 1, ..., m$, the formulae (8), (9), (10) reduce to (6) and corresponding asymptotic formula, obtained by Kiryakova in [15, Th.1], [16], etc.

In the case when there is a parameter of $\gamma_1, \dots, \gamma_m$ which is a negative integer or zero, the corresponding result is given with the next theorem.

Theorem 4. *If at least one of the parameters $\gamma_1, \dots, \gamma_m$ is a non-positive integer, then the multi-index Mittag-Leffler function* (7) *reduces to a finite sum as follows*:

$$E^{(\gamma_i),m}_{(\alpha_i),\,(\beta_i)}(z) = \sum_{k=0}^{M} \frac{(\gamma_1)_k \dots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \dots \Gamma(\alpha_m k + \beta_m)} \, \frac{z^k}{(k!)^m} \, . \tag{11}$$

3. The multi-index Mittag-Leffler functions as Fox's and Wright's functions

Most of the (classical) special functions of mathematical physics are special cases of the generalized hypergeometric functions ${}_pF_q$ (Kiryakova [14]), and thus, of the more general Meijer's G-functions, see in Kiryakova [13, Appendix], and also in [4, Ch.5], [34], [25]. However, the Mittag-Leffler function serves as an example of special function that could not be included in the scheme of Meijer's G-functions, being a case of a more general Fox's H-function, and only for rational $\alpha = p/q$, (1) reduces to a G-function. Same is the case with the Wright generalized hypergeometric functions ${}_p\Psi_q$.

Therefore, it is important to emphasize the place that the multi-index M-L functions (5) and (7) occupy among the previously known special functions, especially in the scheme of Wright generalized hypergeometric functions ${}_p\Psi_q$ and Fox's H-functions (see e.g. [34], [13, Appendix]), and mainly - the important role in FC and as solutions of fractional order differential and integral equations and systems, as extensions of the 'Queen function of FC' , the M-L function (cf. [7]).

Definition 5. Under the Fox H-function we mean a generalized hypergeometric function defined by means of the Mellin-Barnes-type contour integral ([34], [26], [13], etc.)

$$H^{m,\,n}_{p,\,q}(\sigma) = H^{m,\,n}_{p,\,q}\left[\sigma \,\middle|\, \begin{matrix}(a_1,\ A_1)\dots(a_p,\ A_p)\\ (b_1,\ B_1)\dots(b_q,\ B_q)\end{matrix}\right]$$

$$= H^{m,\,n}_{p,\,q}\left[\sigma \,\middle|\, \begin{matrix}(a_k,\ A_k)_1^p\\ (b_k,\ B_k)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{\mathcal{L}'} \mathcal{H}^{m,\,n}_{p,\,q}(s)\sigma^s ds, \tag{12}$$

with the integrand of the form

$$\mathcal{H}^{m,\,n}_{p,\,q}(s) = \frac{\prod\limits_{i=1}^{m}\Gamma(b_i - sB_i)\prod\limits_{j=1}^{n}\Gamma(1-a_j+sA_j)}{\prod\limits_{i=m+1}^{q}\Gamma(1-b_i+sB_i)\prod\limits_{j=n+1}^{p}\Gamma(a_j-sA_j)} \, . \tag{13}$$

(Practically, expression (13) but with $s \mapsto -s$, i.e. $\mathcal{H}^{m,\,n}_{p,\,q}(-s)$, is the Mellin transform (30) of (12).) The curve $\mathcal{L}'$ is a suitable contour in $\mathbb{C}$, $m,\ n,\ p,\ q$ are integers $0 \le m \le q$, $0 \le n \le p$, the parameters $a_j, b_i \in \mathbb{C}$, $A_j, B_i > 0$, $j = 1, \ldots, p$, $i = 1, \ldots, q$ and $A_j(b_i + l) \neq B_i(a_j - l' - 1)$, $l,\ l' = 0, 1, 2, \ldots$. For various type of contours and conditions for existence and analyticity of function (12) in disks $\subset \mathbb{C}$ with radii $\rho = \prod_{j=1}^{p} {A_j}^{-A_j} \prod_{i=1}^{q} {B_i}^{B_i}$, one can see in [34], [13, Appendix], etc.

For $A_1 = \cdots = A_p = 1$, $B_1 = \cdots = B_q = 1$, the H-function (12) turns into the more popular Meijer's G-function (see e.g. [4, Vol.1, Ch.5],[34],[13]). Since the G- and H- functions encompass almost all elementary and special functions ([14], [18]), this makes the knowledge on them very useful. Note that the generalized hypergeometric functions ${}_pF_q$ are special cases of the G-function, namely:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ \ldots, b_q; \sigma) = B\, G^{1,\,p}_{p,\,q+1}\left[-\sigma \,\middle|\, \begin{matrix} 1 - a_1, \ldots, 1 - a_p \\ 0, 1 - b_1, \ \ldots, 1 - b_q \end{matrix} \right], \qquad (14) $$

with the coefficient B as

$$ B = \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)}, $$

while the Mittag-Leffler functions (1) with irrational parameters $\alpha > 0$ and the Wright generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j, B_i > 0$, give examples of H-functions, not reducible to G-functions, namely:

$$ {}_p\Psi_q\left[\begin{matrix} (a_1,\ A_1) \ldots (a_p,\ A_p) \\ (b_1,\ B_1) \ldots (b_q,\ B_q) \end{matrix} \,\middle|\, \sigma \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1) \ldots \Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1) \ldots \Gamma(b_q + kB_q)} \frac{\sigma^k}{k!} $$

$$ = H^{1,\,p}_{p,\,q+1}\left[-\sigma \,\middle|\, \begin{matrix} (1 - a_1,\ A_1), \ldots, (1 - a_p,\ A_p) \\ (0, 1), (1 - b_1,\ B_1), \ldots, (1 - b_q,\ B_q) \end{matrix} \right], \qquad (15) $$

and specially for the M-L function (see [15], etc.),

$$ E_{\alpha,\beta}(\sigma) = {}_1\Psi_1\left[\begin{matrix} (1, 1) \\ (\beta, \alpha) \end{matrix} \,\middle|\, \sigma \right] = H^{1,\,1}_{1,\,2}\left[-\sigma \,\middle|\, \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right]. \qquad (16) $$

For $A_1 = \cdots = A_p = 1$, $B_1 = \cdots = B_q = 1$, cf. (14),

$$ {}_p\Psi_q\left[\left.\begin{matrix}(a_1,\ A_1)\dots(a_p,\ A_p)\\ (b_1,\ B_1)\dots(b_q,\ B_q)\end{matrix}\right|\sigma\right] = C\ {}_pF_q(a_1,\dots,a_p;b_1,\ \dots,b_q;\sigma)$$

$$= G^{1,\,p}_{p,\,q+1}\left[-\sigma\left|\begin{matrix}1-a_1,\dots,1-a_p\\ 0,1-b_1,\ \dots,1-b_q\end{matrix}\right.\right],\quad C=\frac{\prod\limits_{i=1}^{p}\Gamma(a_i)}{\prod\limits_{i=1}^{q}\Gamma(b_i)}. \tag{17}$$

In what follows we need the two numerical sets S_l and S_r, defined as follows:

$$S_l=\{s: s=-k\quad (k\in\mathbb{N}_0)\},$$

and

$$S_r=\{s: s=l+\gamma_i,\quad (l\in\mathbb{N}_0,\gamma_i\in\mathbb{C}, Re(\gamma_i)>0;\ i=1,\dots,m)\}.$$

Remark 6. *The intersection of the sets S_l and S_r is empty, i.e. $S_l\cap S_r=\varnothing$. Moreover, if*

$$\widetilde{\gamma}=\min_{i=1\div m} Re(\gamma_i), \tag{18}$$

then the set S_l lies on the left hand side of the strip

$$S=\{s: s\in\mathbb{C},\quad 0<Re(s)<\widetilde{\gamma}\}, \tag{19}$$

while the set S_r lies on its right.

Theorem 7. *Let $\alpha_i>0$, $\beta_i,\gamma_i\in\mathbb{C}$, $Re(\gamma_i)>0$ for $i=1,\dots,m$. Then the multi-index Mittag-Leffler functions* (7) *are expressed by Wright's generalized hypergeometric functions ${}_p\Psi_q$,* (15)*, as well as Fox's H-function* (12) *as follows:*

$$E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z)=A\ {}_m\Psi_{2m-1}\left[\left.\begin{matrix}(\gamma_1,\ 1),\dots,(\gamma_m,\ 1)\\ (\beta_1,\ \alpha_1),\dots,(\beta_m,\ \alpha_m),(1,1),\dots,(1,1)\end{matrix}\right|z\right]$$

$$= A\ H^{1,\,m}_{m,\,2m}\left[-z\left|\begin{matrix}(1-\gamma_1,\ 1),\dots,(1-\gamma_m,\ 1)\\ [(0,1),(1-\beta_i,\ \alpha_i)]_1^m\end{matrix}\right.\right],\ A=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}. \tag{20}$$

They have the following Mellin-Barnes-type contour integral representation:

$$E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z)=\frac{A}{2\pi i}\int\limits_{\mathcal{L}'}\mathcal{H}^{1,\,m}_{m,\,2m}(s)(-z)^s ds$$

$$=\frac{A}{2\pi i}\int\limits_{\mathcal{L}}\mathcal{H}^{1,\,m}_{m,\,2m}(-s)(-z)^{-s}ds,\quad |\arg(-z)|<\pi, \tag{21}$$

where the constant A is as in (20)*, the integrand is*

$$\mathcal{H}^{1,\,m}_{m,\,2m}(s) = \frac{\Gamma(-s)\prod\limits_{i=1}^{m}\Gamma(\gamma_i+s)}{[\Gamma(1+s)]^{m-1}\prod\limits_{i=1}^{m}\Gamma(\beta_i+s\alpha_i)}, \tag{22}$$

and $\mathcal{L}$ is an arbitrary contour in $\mathbb{C}$ running from $-i\infty$ to $+i\infty$ in a way that the poles $s=-k$ $(k\in\mathbb{N}_0)$ of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles $s=l+\gamma_i$ $(l\in\mathbb{N}_0)$ of $\Gamma(\gamma_i-s)$ $(i=1,\dots,m)$ to the right of it.

Proof. According to Remark 6, none of the poles of $\Gamma(s)$ and $\Gamma(\gamma_i-s)$ are in the strip S, given by (19). Moreover, the poles $s=-k$ $(k\in\mathbb{N}_0)$ of $\Gamma(s)$ lie to the left of this strip, and the poles $s=l+\gamma_i$ $(l\in\mathbb{N}_0)$ of $\Gamma(\gamma_i-s)$ $(i=1,\dots,m)$ to its right.

Let us consider the right hand side of (21) and introduce the notation

$$I(z)=\frac{A}{2\pi i}\int\limits_{\mathcal{L}}\mathcal{H}^{1,\,m}_{m,\,2m}(-s)(-z)^{-s}ds,\ \text{ with }\ A=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}. \tag{23}$$

Calculating the residues of the integrand of (23) at the simple poles $s_k=-k$, $k=0,1,2,\dots$, and taking into account the asymptotic formula (see e.g. [4, Vol.1, 1.1.(8)])

$$\Gamma(s)=\frac{(-1)^k}{k!(s+k)}[1+O(s+k)]\quad (s\to -k;\ k=0,1,2,\dots),$$

we have

$$I(z)=A\sum_{k=0}^{\infty}\mathrm{Res}_{s=-k}\left\{\mathcal{H}^{1,\,m}_{m,\,2m}(-s)(-z)^{-s}\right\}$$

$$=A\sum_{k=0}^{\infty}\mathrm{Res}_{s=-k}\left\{\frac{\Gamma(s)\prod\limits_{i=1}^{m}\Gamma(\gamma_i-s)}{[\Gamma(1-s)]^{m-1}\prod\limits_{i=1}^{m}\Gamma(\beta_i-s\alpha_i)}(-z)^{-s}\right\}$$

$$=A\sum_{k=0}^{\infty}\left\{\frac{(-1)^k\prod\limits_{i=1}^{m}\Gamma(\gamma_i+k)}{k![\Gamma(1+k)]^{m-1}\prod\limits_{i=1}^{m}\Gamma(\beta_i+k\alpha_i)}(-z)^{k}\right\}$$

$$=A\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{m}\Gamma(\gamma_i+k)}{\prod\limits_{i=1}^{m}\Gamma(\beta_i+k\alpha_i)}\frac{z^k}{(k!)^m}=E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z),$$

that proves (21).

Furthermore, writing (7) in its explicit form, we have

$$E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z) = A\,\sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma_1)\ldots\Gamma(k+\gamma_m)}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)(\Gamma(k+1))^{m-1}}\,\frac{z^k}{k!},$$

with A as in (23), and comparing with (15), we obtain (20) that completes the proof of the theorem. □

If all the $\gamma_i = 1$, then Theorem 7 gives the corresponding result for the $2m$-multi-index M-L functions (5), proved in Kiryakova [15, Lemma 1], namely:

Corollary 8. *Let $\alpha_i > 0$, $\beta_i, \in \mathbb{C}$, for $i = 1, ..., m$. Then the multi-index Mittag-Leffler functions* (5) *are expressed by Wright's generalized hypergeometric functions as well as Fox's H-function in the form*

$$E^{m}_{(\alpha_i),\,(\beta_i)}(z) = {}_1\Psi_m\left[\left.\begin{matrix}(1,\ 1)\\ (\beta_1,\ \alpha_1),\ldots,(\beta_m,\ \alpha_m)\end{matrix}\right| z\right]$$

$$= H^{1,\,1}_{1,\,m+1}\left[-z\left|\begin{matrix}(0,\ 1)\\ (0,1),\ [(1-\beta_i,\ \alpha_i)]_1^m\end{matrix}\right.\right]. \tag{24}$$

They have the following Mellin-Barnes type contour integral representation:

$$E^{m}_{(\alpha_i),\,(\beta_i)}(z) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}'} \mathcal{H}^{1,\,1}_{1,\,m+1}(s)(-z)^s ds$$

$$= \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \mathcal{H}^{1,\,1}_{1,\,m+1}(-s)(-z)^{-s} ds, \quad |\arg(-z)| < \pi, \tag{25}$$

where

$$\mathcal{H}^{1,\,1}_{1,\,m+1}(s) = \frac{\Gamma(-s)\Gamma(1+s)}{\prod\limits_{i=1}^{m}\Gamma(\beta_i+s\alpha_i)}, \tag{26}$$

and $\mathcal{L}$ is an arbitrary contour in $\mathbb{C}$ running from $-i\infty$ to $+i\infty$ in a way that the poles $s = -k$ $(k \in \mathbb{N}_0)$ of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles $s = l+1$ $(l \in \mathbb{N}_0)$ of $\Gamma(1-s)$ $(i = 1, \ldots, m)$ to the right of it.

Proof. Here we only give the idea of the proof. Taking all the $\gamma_i = 1$ we observe that m pairs of parameters of the H-function in the formula (20) become the same. Further we consider the following 3 sets of these pairs of parameters:

$$A_1 = \{(0,\ 1)|_1^m\},\ \ B_1 = \{(0,1)\},\ B_i = \{(0,\ 1)|_1^{m-1}(1-\beta_i,\ \alpha_i)|_1^m\}.$$

Since $m-1$ pairs in the sets A_1 and B_i are the same and due the symmetric property of the H-function and the reduction formula (for details see e.g. Kiryakova [13, Appendix]), the corollary follows. □

Corollary 9. *Let $\alpha > 0$, $\beta, \gamma \in \mathbb{C}$, $Re(\gamma) > 0$. Then the 3-parametric Mittag-Leffler (Prabhakar) function* (2) *can be expressed by Wright's generalized hypergeometric function as well as a Fox's H-function in the form*

$$E^{\gamma}_{\alpha,\,\beta}(z) = [\Gamma(\gamma)]^{-1}\ {}_1\Psi_1\left[\left.\begin{matrix}(\gamma,\ 1)\\ (\beta,\ \alpha)\end{matrix}\right| z\right]$$

$$= [\Gamma(\gamma)]^{-1}\, H^{1,\,1}_{1,\,2}\left[-z\,\middle|\,\begin{matrix}(1-\gamma,\ 1)\\ (0,1),(1-\beta,\ \alpha)\end{matrix}\right]. \tag{27}$$

And the Prabhakar function has the following Mellin-Barnes-type contour integral representation:

$$E^{\gamma}_{\alpha,\,\beta}(z) = \frac{[\Gamma(\gamma)]^{-1}}{2\pi i}\int\limits_{\mathcal{L}'} \mathcal{H}^{1,\,1}_{1,\,2}(s)(-z)^s ds$$

$$= \frac{[\Gamma(\gamma)]^{-1}}{2\pi i}\int\limits_{\mathcal{L}} \mathcal{H}^{1,\,1}_{1,\,2}(-s)(-z)^{-s} ds, \quad |\arg(-z)| < \pi, \tag{28}$$

where

$$\mathcal{H}^{1,\,1}_{1,\,2}(s) = \frac{\Gamma(-s)\Gamma(\gamma+s)}{\Gamma(\beta+\alpha s)} \tag{29}$$

and $\mathcal{L}$ is an arbitrary contour in $\mathbb{C}$ running from $-i\infty$ to $+i\infty$ in a way that the poles $s = -k$ $(k \in \mathbb{N}_0)$ of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles $s = l+\gamma$ $(l \in \mathbb{N}_0)$ of $\Gamma(\gamma - s)$ to its right.

For $\gamma = 1$, one gets the corresponding representations and Mellin-Barnes integrals for the classical M-L function as in (16)*, see also the extended survey by Haubold-Mathai-Saxena [8].*

Remark 10. The relation, given with the formula (28), was mentioned in Kilbas-Koroleva-Rogozin [10] but it has been known since from Kilbas-Saigo-Saxena [11]. Unfortunately, there is a unpleasant mistake in [11]. The condition $\gamma \neq 0$, imposed there, is not sufficient for separating the poles, used in the given proof.

Theorem 7 and its above mentioned special cases allow us to describe the asymptotic behaviour of the multi-index M-L functions as $z \to 0; z \to \infty$ also via the theory of H- and ${}_p\Psi_q$-functions. Note that in the case of the classical

M-L function (1) ($m = 1$), Dzrbashjan [2], [3] established various asymptotic formulas for $|z| \to \infty$, valid in different parts of the complex plain and under different conditions on ρ and μ. For example, if $\rho > 1/2$ and inside angle domains, (1) is $\approx \rho z^{\rho(1-\mu)} \exp(z^\rho)$. An asymptotic estimate for multi-index M-L functions (5) in the case $m > 1$ is given in [17] similar to (10), and in more detailed situations, the asymptotic of the multi-index M-L functions could be found from their interpretation as ${}_m\Psi_{2m-1}$-, and respectively as ${}_1\Psi_m$−functions.

4. Mellin integral transform for the multi-index Mittag-Leffler functions

According to the theory of the Mellin transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$, it is defined by

$$(\mathcal{M}f)(s) = \mathcal{M}[f(t)](s) = F(s) = \int_0^\infty f(t)t^{s-1}dt \quad (s \in S \subset \mathbb{C}), \tag{30}$$

S is a suitable vertical strip (see, for example, the book by Titchmarsh [37]), and the inverse Mellin transform is given for $t \in \mathbb{R}^+$ by the formula ($\gamma = Re(s)$):

$$(\mathcal{M}^{-1}F)(t) = \mathcal{M}^{-1}[F(s)](t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)t^{-s}ds, \quad 0 < t < \infty, \tag{31}$$

where the integral is understood in the sense of the Cauchy principal value.

For more detailed information on the Mellin integral transform, its properties and applications, we refer the reader to the classical books with tables of integral transforms, and also to works as [1], [25], [24], [22], etc.

In what follows we use the results of Theorem 7 and its corollaries. By setting $-z = t$ and having in mind that $0 < t < \infty$, we see that $|\arg(-z)| = |\arg(t)| = 0 < \pi$. That is why the representation (21) holds true with $-z = t$.

The corresponding results, referring to the functions (5) and (2), will follow merely as corollaries from the main result below, and as modifications of Corollaries 8 and 9.

In this way, we can formulate the results for the Mellin transform image of the multi-index Mittag-Leffler functions.

Theorem 11. *Let the parameters $\alpha_i > 0$, $\beta_i, \gamma_i \in \mathbb{C}$, and $Re(\gamma_i) > 0$ for $i = 1, ..., m$ ($m \in \mathbb{N}$). Then the Mellin transform of the $3m$-multi-index Mittag-Leffler function is expressed as*

$$\mathcal{M}[E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(-t)](s)=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}\mathcal{H}^{1,\,m}_{m,\,2m}(-s)\quad(0<Re(s)<\widetilde{\gamma}),\tag{32}$$

with $\widetilde{\gamma}$ *and* $\mathcal{H}^{1,\,m}_{m,\,2m}$ *as defined in* (18), *resp.* (22), *and* $t>0$.

Proof. In particular, if $\mathcal{L}$ is the straight line $Res=\gamma,\ 0<\gamma<\widetilde{\gamma}$ and taking $t\in(0,\infty)$, then the relation (21) with $-z=t$ leads to

$$E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(-t)=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{H}^{1,\,m}_{m,\,2m}(-s)t^{-s}ds.\tag{33}$$

The relation (33) means that the function $E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(-t)$ is the inverse Mellin transform of the function $\left[\prod\limits_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}\mathcal{H}^{1,\,m}_{m,\,2m}(-s)$, cf. with (31). Therefore, the direct Mellin transform of the $3m$-parametric function is given by the relation

$$\mathcal{M}[E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(-t)](s)=\left[\prod_{i=1}^{m}\Gamma(\gamma_i)\right]^{-1}\mathcal{H}^{1,\,m}_{m,\,2m}(-s),$$

that should be proved. □

Taking all the $\gamma_i=1$, or respectively $m=1$, and using the relations (25) and (28) in Corollaries 8 and 9, the result of Theorem 11 reduces to Mellin transform images of the functions (5), respectively (2). The corresponding corollaries are formulated below.

Corollary 12. *Let the parameters* $\alpha_i>0$ *and* $\beta_i\in\mathbb{C}$ *for* $i=1,...,m$. *Then the Mellin transform of the* $2m$*-multi-index Mittag-Leffler function* (5) *is expressed by the following formula*

$$\mathcal{M}[E^{m}_{(\alpha_i),\,(\beta_i)}(-t)](s)=\mathcal{H}^{1,\,1}_{1,\,m+1}(-s)\quad(0<Re(s)<1),\tag{34}$$

with $\mathcal{H}^{1,\,1}_{1,\,m+1}$ *like in* (26), *and* $t>0$.

Corollary 13. *Let the parameters* $\alpha>0$, $\beta,\gamma\in\mathbb{C}$ *and* $Re(\gamma)>0$. *Then the Mellin transform of the 3-parametric Mittag-Leffler (Prabhakar) function*

(2) *is expressed as follows (a known result, see for example in the book [5, eq.(5.1.27)]):*

$$\mathcal{M}[E^{\gamma}_{\alpha,\,\beta}(-t)](s) = [\Gamma(\gamma)]^{-1}\,\mathcal{H}^{1,\,1}_{1,\,2}(-s) = \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-s\alpha)}\,, \tag{35}$$

for $0 < Re(s) < Re(\gamma)$, $t > 0$ *and with* $\mathcal{H}^{1,\,1}_{1,\,2}$ *as in* (29).

Remark 14. Let us note that there is an idea for the Mellin transform of the 3-parametric Mittag-Leffler function (2) in the paper [11]. Unfortunately, the result is not completely correct, since it is not taken into account what the argument of the function $E^{\gamma}_{\alpha,\,\beta}$ should be, and the constant $[\Gamma(\gamma)]^{-1}$ is missing as well.

5. Special cases of the multi-index Mittag-Leffler functions

In this section we consider a number of interesting special cases of the $3m$- and $2m$-multi-index Mittag-Leffler functions (7), (5), depending on the particular choice of parameters. When all the parameters α_i are positive, i.e.

$$\alpha_i > 0, \quad \text{for} \quad i = 1, \ldots, m,$$

their relations with the Wright ${}_p\Psi_q$-functions and Fox H-functions are given. For our purpose we consider some specific cases of the parameters.

Case 1. If $\underline{m = 1}$, the formula (7) gives the three parametric Mittag-Leffler function (2), known also as the Prabhakar function, i.e.

$$E^{\gamma}_{\alpha,\,\beta}(z) = E^{(\gamma),\,1}_{(\alpha),\,(\beta)}(z).$$

Its representations by the Wright ${}_1\Psi_1$-function and Fox H-function are given by the formula (27). In addition, if $\gamma = 1$ and in view of (5), the function (2) reduces to the two parametric M-L function (1), namely

$$E_{\alpha,\,\beta}(z) = E^{1}_{\alpha,\,\beta}(z) = E^{1}_{(\alpha),\,(\beta)}(z) = E^{(1),\,1}_{(\alpha),\,(\beta)}(z),$$

and then each of both formulae (24) and (27) produces

$$E_{\alpha,\,\beta}(z) = {}_1\Psi_1\left[\left.\begin{matrix}(1,\ 1)\\(\beta,\ \alpha)\end{matrix}\right| z\right] = H^{1,\,1}_{1,\,2}\left[-z\left|\begin{matrix}(0,\ 1)\\(0,\ 1),(1-\beta,\ \alpha)\end{matrix}\right.\right]. \tag{36}$$

Some more particular examples are considered below.

For $\beta = 1$, or respectively $\alpha = \beta = 1$, we have:

$$E_{\alpha}(z) = E^{(1),\,1}_{(\alpha),\,(1)}(z) = {}_1\Psi_1\left[\left.\begin{matrix}(1,\ 1)\\(1,\ \alpha)\end{matrix}\right| z\right] = H^{1,\,1}_{1,\,2}\left[-z\left|\begin{matrix}(0,\ 1)\\(0,\ 1),(0,\ \alpha)\end{matrix}\right.\right], \tag{37}$$

and

$$E_1(z)=\exp(z)=E^{(1),\,1}_{(1),\,(1)}(z)={}_0\Psi_0\left[\begin{matrix}--\\--\end{matrix}\,\middle|\,z\right]=H^{1,\,0}_{0,\,1}\left[-z\,\middle|\,\begin{matrix}--\\(0,\ 1)\end{matrix}\right], \tag{38}$$

while for $\alpha=\beta$, we get the Rabotnov function $E_{\alpha,\alpha}$, see [32].

Case 2. If $\underline{\gamma_1=\cdots=\gamma_m=1}$, the definition of the $3m$-parametric M-L function (7) reduces to the multi-index Mittag-Leffler function (5) with $2m$ parameters:

$$E_{(\alpha_i),(\beta_i)}(z)=E^{(1),\,m}_{(\alpha_i),\,(\beta_i)}(z)=E^{(1,\ldots,1),\,m}_{(\alpha_i),\,(\beta_i)}(z), \tag{39}$$

and its representation by the Wright ${}_1\Psi_m$-function and Fox H-function is shown by the formula (24). For $\alpha_i>0$, this is the function studied in details by Kiryakova in a series of publications, and for arbitrary real α_i - considered in Kilbas et al., [10].

Further, in this case (see details in Kiryakova [16], [18], etc.), for $\underline{m=2}$, $\alpha_{1,\,2}:=1/\rho_{1,\,2}$ and $\beta_{1,\,2}:=\mu_{1,\,2}$, the function (5), (39) is Dzrbashjan's M-L type function from [2]. In view of (24), it can be presented as follows:

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\,\mu_2)}(z)=E^{(1,1),\,2}_{(1/\rho_1,1/\rho_2),(\mu_1,\,\mu_2)}(z)={}_1\Psi_2\left[\begin{matrix}(1,\ 1)\\(\mu_i,\ 1/\rho_i)\ |_1^2\end{matrix}\,\middle|\,z\right]$$

$$=H^{1,\,1}_{1,\,3}\left[-z\,\middle|\,\begin{matrix}(0,\ 1)\\(0,\ 1),(1-\mu_i,\ 1/\rho_i)\ |_1^2\end{matrix}\right]. \tag{40}$$

The so-called Wright function (classical, to distinguish from the notion 'generalized Wright function') has arisen in the studies of Fox (1928), Wright (1933), Humbert and Agarwal (1953), and it is also referred to in [4, Vol.3]. Initially, Wright defined it only for $\alpha>0$, then prolonged its definition for $\alpha>-1$. We see that it appears now as a case of multi-M-L function with $m=2$:

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\,\frac{z^k}{k!}={}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\,\middle|\,z\right]=E^{(2)}_{(\alpha,1),(\beta,1)}(z). \tag{41}$$

This function plays important role in the solutions of linear partial fractional differential equations as the fractional diffusion-wave equation studied by Nigmatullin (1984-1986, to describe the diffusion process in media with fractal geometry, $0<\alpha<1$) and by Mainardi et al. (1994 and next, for propagation of mechanical diffusive waves in viscoelastic media, $1<\alpha<2$). In the form $M(z;\beta)=\phi(-\beta,1-\beta;-z)$, $\beta:=\alpha/2$, it is recently called also as the Mainardi function, see [32, Ch.1]. In our denotations, it appears as $M(z;\beta)=E^{(2)}_{(-\beta,1),(1-\beta)}(-z)$, and has its examples like:

$M(z;1/2) = 1/\sqrt{\pi}\,\exp(-z^2/4)$, and the Airy function:
$M(z;1/3) = 3^{2/3}\,Ai(z/3^{1/3})$.

In this case ($m = 2$, $\gamma_1 = \gamma_2 = 1$) we have also the Lommel, Struve and classical Bessel functions (details in [16], [18], etc.)

Case 3. Again, let $\underline{\gamma_1 = \cdots = \gamma_m = 1}$, take $\underline{m \geq 2}$, and see details in [16], [18]. One special case then is the generalized Lommel–Wright function with 4 indices ($\mu > 0, q \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$), introduced by de Oteiza, Kalla and Conde (details in [18]):

$$J^{\mu,\,q}_{\nu,\,\lambda}(z) = (z/2)^{\nu+2\lambda}\;\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

with $\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}$ denoting the entire function

$$\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\,\Gamma(\nu+k\mu+\lambda+1)}. \tag{42}$$

Now, again by (24)

$$\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}(z) = E^{(1,1,\ldots,1),\;q+1}_{(\mu,1,\ldots,1),\;(\nu+\lambda+1,\;\lambda+1,\ldots,\;\lambda+1)}\left(-(z/2)^2\right) \tag{43}$$

$$= {}_1\Psi_{q+1}\left[\left.\begin{matrix}(1,\;1)\\ (\lambda+1,\;1)^q_1, (\lambda+\nu+1,\;\mu)\end{matrix}\right| -(z/2)^2\right]$$

$$= H^{1,\,1}_{1,\,q+2}\left[(z/2)^2\left|\begin{matrix}(0,\;1)\\ (0,\;1), (-\lambda,\;1)^q_1, (-\lambda-\nu,\;\mu)\end{matrix}\right.\right].$$

This is an interesting example of a multi-index M-L function with arbitrary $m = q + 1$.

Some other interesting cases are given below.

Obviously for $q = 1$, the special function (42) turns into the generalization of the Bessel function $J_\nu(z)$, introduced by Pathak (for details, see again [18]):

$$J^{\mu}_{\nu,\,\lambda}(z) = (z/2)^{\nu+2\lambda}\;\widetilde{J}^{\mu}_{\nu,\,\lambda}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\widetilde{J}^{\mu}_{\nu,\,\lambda}$ denotes the entire function

$$\widetilde{J}^{\mu}_{\nu,\,\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)} \tag{44}$$

and

$$\widetilde{J}^{\mu}_{\nu,\,\lambda}(z) = E^{(1,1),\;2}_{(\mu,1),\;(\nu+\lambda+1,\;\lambda+1)}\left(-(z/2)^2\right) \tag{45}$$

$$= {}_1\Psi_2\left[\left.\begin{matrix}(1,\;1)\\ (\lambda+1,\;1), (\lambda+\nu+1,\;\mu)\end{matrix}\right| -(z/2)^2\right]$$

$$= H^{1,\,1}_{1,\,3}\left[(z/2)^2\left|\begin{matrix}(0,\;1)\\ (0,\;1), (-\lambda,\;1), (-\lambda-\nu,\;\mu)\end{matrix}\right.\right],$$

that is obtained by setting $q = 1$ in (43).

For particular choices of the other parameters λ and μ we obtain results for more particular cases as follows.

Let $\lambda = 0$, then the special function (44) produces the generalization of the Bessel–Clifford function $C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z})$, introduced by Wright [38], and called Bessel–Wright or Bessel–Maitland function, as an alternative denotation for the already discussed Wright function (41) with $m = 2$. See e.g. [18], [38],

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu + \mu k + 1)} = E^{(1,1),\,2}_{(\mu,1),(\nu+1,1)}(-z) \tag{46}$$

and, along with (45), we have

$$J_\nu^\mu(z) = {}_0\Psi_1 \left[\begin{matrix} -- \\ (\nu+1,\ \mu) \end{matrix} \middle| -z \right] = H^{1,\,0}_{0,\,2} \left[z \middle| \begin{matrix} -- \\ (0,\ 1), (-\nu,\ \mu) \end{matrix} \right]. \tag{47}$$

As mentioned before, initially, Wright (Sir Edward Maitland Wright) defined (46) only for $\mu > 0$, and on a later stage he extended its definition to $\mu > -1$.

Additionally, if $\mu = 1$, then (44) becomes the classical Bessel function of the first kind

$$J_\nu(z) = (z/2)^\nu \widetilde{J}_\nu(z), \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\widetilde{J}_\nu$ is the entire function

$$\widetilde{J}_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\nu+1)} = E^{(1,1),\,2}_{(1,1),\,(\nu+1,\,1)}(-(z/2)^2) \tag{48}$$

$$= {}_0\Psi_1 \left[\begin{matrix} -- \\ (\nu+1,\ 1) \end{matrix} \middle| -(z/2)^2 \right] = H^{1,\,0}_{0,\,2} \left[(z/2)^2 \middle| \begin{matrix} -- \\ (0,\ 1), (-\nu,\ 1) \end{matrix} \right].$$

The latest relation follows by setting $\mu = 1$ in (46), or $\lambda = 0$ and $\mu = 1$ in (45).

Case 4. We emphasize here on the more specific case of the so-called hyper-Bessel functions. Again, let $\underline{\gamma_1 = \cdots = \gamma_m = 1}$, take $\underline{m \geq 2}$, and see details in Kiryakova [13, Ch.3], and papers like [16], [18], etc. If additionally, we let $\forall \alpha_i = 1$, $i = 1, \ldots, m$, then:

$$E^{(m)}_{(1,1,\ldots,1),(\beta_i)}(z) = {}_1\Psi_m \left[\begin{matrix} (1,1) \\ (\beta_i, 1)_1^m \end{matrix} \middle| z \right]$$

$$= \Big[\prod_{i=1}^{m} \Gamma(\beta_i)\Big]^{-1} {}_1F_m\,(1; \beta_1, \beta_2, \ldots, \beta_m; z)$$

reduces to ${}_1F_m$- and to a Meijer's $G^{1,1}_{1,m+1}$-function. Denote $\beta_i = \gamma_i + 1, i = 1, \ldots, m$, and let additionally one of the β_i to be 1, say: $\beta_m = 1$, i.e. $\gamma_m = 0$. Then the multi-index M-L function becomes a hyper-Bessel function, in the sense of Delerue (1953); see Kiryakova [13], App.,(D.30) and Ch.3:

$$J^{(m-1)}_{\gamma_i,\ldots,\gamma_{m-1}}(z) = \Big(\frac{z}{m}\Big)^{\sum\limits_{i=1}^{m-1}\gamma_i} E^{(m)}_{(1,1,\ldots,1),(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1,1)}\Big(-(\frac{z}{m})^m\Big) \qquad (49)$$

$$= \left[\prod_{i=1}^{m-1}\Gamma(\gamma_i+1)\right]^{-1}\Big(\frac{z}{m}\Big)^{\sum\limits_{i=1}^{m-1}\gamma_i} {}_0F_{m-1}\Big(\gamma_1+1,\gamma_2+1,..,\gamma_{m-1}+1;-(\frac{z}{m})^m\Big).$$

In view of the above relation, the multi-index M-L functions with $2m$-parameters and with arbitrary $(\alpha_1,\ldots,\alpha_m) \neq (1,\ldots,1)$ can be seen as fractional-indices analogues of the hyper-Bessel functions (49), which themselves are multi-index analogues of the Bessel function. The functions (49) are closely related to the hyper-Bessel operators, introduced by Dimovski (1966), see details in Kiryakova [13, Ch.3], and to the normalized hyper-Bessel functions, which we can call as Bessel-Clifford functions of m-th order:

$$C_{\nu_1,\ldots,\nu_m}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(\nu_1+k+1)\ldots\Gamma(\nu_m+k+1)\,k!}$$
$$= E^{(m+1)}_{(1,\ldots,1),(\nu_1+1,\ldots,\nu_m+1,1)}(-z).$$

Remark 15. Let us consider the exponential function (38), Bessel function (47) and Bessel–Maitland function (48). Let us pay attention to the formula (21). Taking into account that the numerators of their integrands, respectively

$$\mathcal{H}^{1,\,0}_{0,\,1}(-s) = \Gamma(s),\ \mathcal{H}^{1,\,0}_{0,\,2}(-s) = \frac{\Gamma(s)}{\Gamma(1+\nu-s)},$$
$$\mathcal{H}^{1,\,0}_{0,\,2}(-s) = \frac{\Gamma(s)}{\Gamma(1+\nu-\mu s)},$$

have no poles for $Re(s) > 0$, then the contour $\mathcal{L}$ may lie down in the whole right half-plane. Thus the strip (19) turns into the right half-plane $Re(s) > 0$.

6. Mellin transforms for the special cases of the multi-index Mittag-Leffler functions

Using the results in the previous sections, we give the Mellin transform images of the functions discussed in Section 5. Most of them are also special cases of the $2m$-functions (5). That is why we mainly use the relation (34), supposing that the variable t is positive and the parameters $\alpha_i > 0$ for all the $i = 1, ..., m$,

$m \in \mathbb{N}$. We categorize the results in several groups depending on their kinds and representations.

Starting with the case $0 < Re(s) < 1$, we firstly consider Dzrbashjan's M-L type functions (40), with $\alpha_1 = 1/\rho_1 > 0$ and $\alpha_2 = 1/\rho_2 > 0$. Then, due to (34) for $m = 2$, the following Mellin image is obtained

$$\mathcal{M}[E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(-t)](s) = \mathcal{H}^{1,\,1}_{1,\,3}(-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\mu_1 - s/\rho_1)\Gamma(\mu_2 - s/\rho_2)}. \quad (50)$$

Using (36) and (37), the relation (34) taken with $m = 1$ and $\alpha > 0$ gives the result for the two-parametric Mittag-Leffler function $E_{\alpha,\beta}$:

$$\mathcal{M}[E_{\alpha,\beta}(-t)](s) = \mathcal{H}^{1,\,1}_{1,\,2}(-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)}, \quad (51)$$

and if additionally $\beta = 1$, it reduces to the Mellin image for E_α,

$$\mathcal{M}[E_\alpha(-t)](s) = \mathcal{H}^{1,\,1}_{1,\,2}(-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1 - \alpha s)}. \quad (52)$$

Further, taking $m = q + 1$ and $\mu > 0$, (34) along with (43) give the Mellin transform of the function $\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}$, related to the Lommel–Wright function, namely

$$\mathcal{M}[\widetilde{J}^{\mu,\,q}_{\nu,\,\lambda}(2t^{1/2})](s) = \mathcal{H}^{1,\,1}_{1,\,q+2}(-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+\lambda+\nu-\mu s)\prod\limits_{i=1}^{q}\Gamma(1+\lambda-s)}. \quad (53)$$

If additionally $q = 1$ and in view of (45), the result refers to the generalized Bessel–Wright function, i.e.

$$\mathcal{M}[\widetilde{J}^{\mu}_{\nu,\,\lambda}(2t^{1/2})](s) = \mathcal{H}^{1,\,1}_{1,\,3}(-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+\lambda-s)\Gamma(1+\lambda+\nu-\mu s)}. \quad (54)$$

Now we continue with the case $Re(s) > 0$. According to (47) and Remark 15, and taking $\mu > 0$, we obtain the Mellin transform image of the Bessel-Wright (Bessel-Maitland) function (46)

$$\mathcal{M}[\widetilde{J}^{\mu}_{\nu}(t)](s) = \mathcal{H}^{1,\,0}_{0,\,2}(-s) = \frac{\Gamma(s)}{\Gamma(1+\nu-\mu s)}, \quad (55)$$

or in the denotation (41), for the Wright function:

$$\mathcal{M}[\phi(\alpha,\beta;t)](s) = \frac{\Gamma(s)}{\Gamma(\beta - \alpha s)}, \;\; Re(s) > 0. \quad (56)$$

Further, taking $\mu = 1$, the formula (55) becomes the image of the entire function (48) related to the Bessel function, namely

$$\mathcal{M}[\widetilde{J}_\nu(2t^{1/2})](s) = \mathcal{H}^{1,\,0}_{0,\,2}(-s) = \frac{\Gamma(s)}{\Gamma(1+\nu-s)}. \tag{57}$$

Finally, in view of (38), the Mellin transform of the exponential function is

$$\mathcal{M}[\exp(-t)](s) = \mathcal{H}^{1,\,0}_{0,\,1}(-s) = \Gamma(s). \tag{58}$$

Many more Mellin transform images of the mentioned particular cases of the multi-index M-L functions can be easily derived by suitable choice of parameters.

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