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SOLUTIONS FOR A FRACTIONAL-ORDER DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS OF THIRD ORDER

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Abstract: This paper is concerned with a fractional-order boundary value problem involving the Riemann-Liouville fractional derivative of order $\alpha \in (3, 4]$. Existence and uniqueness results of solutions are established by the Banach fixed point theorem.

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1. Introduction

Let $3 < \alpha \leq 4$. In this paper, we consider the existence and uniqueness of solutions for the fractional-order boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases} \tag{1}$$

where, f is a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$, g is a function of $\mathbb{R}$ into itself, and D_{0+}^{α} denotes the Riemann-Liouville fractional derivative of order α

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which is defined in Section 2. A function $u \in C[0,1]$, where $C[0,1]$ is the set of all continuous functions of $[0,1]$ into $\mathbb{R}$, is called a solution of the problem (1) if $D^{\alpha}_{0+}u \in C[0,1]$ and u satisfies (1).

Problem (1) contains the fourth-order boundary value problem

$$\begin{cases} u''''(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)). \end{cases} \tag{2}$$

Ma and da Silva [4] studied the existence and iterative schemes to solve problem (2).

On the other hand, Toyoda and Watanabe [5] considered the fractional-order boundary value problem:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{3}$$

Problem (3) represents the case of the problem (1) in which $g \equiv 0$. Due to the restriction of g, results in [5] cannot deal with problem (2).

The purpose of this paper is to extend results of the fourth-order problem (2) to results of the fractional-order problem (1).

2. Preliminaries

In this section, we introduce preliminary facts. In particular, we construct the function, $G(s,t)$, for the boundary value problem (1), and discuss some properties of that function.

We begin with the definition of the Riemann-Liouville fractional integral and fractional derivative. Let $\alpha > 0$ and u be a continuous function of $[0,1]$ into $\mathbb{R}$. The Riemann-Liouville fractional integral of order α of u, denoted $I^{\alpha}_{0+}u$, is defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

and the Riemann-Liouville fractional derivative of order α of u, denoted $D^{\alpha}_{0+}u$, is defined by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds,$$

where n denotes a positive integer such that $n - 1 < \alpha \leq n$. For $\alpha \geq 0$ and $\beta > -1$, we have

$$D_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha},$$

see [3].

Let us remind that in this paper we consider the problem (1) in the case $3 < \alpha \leq 4$ and $0 \leq t \leq 1$.

Let $3 < \alpha \leq 4$, and $\gamma \in \mathbb{R}$. A function, u, is then a solution of the boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = h(t), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = \gamma, \end{cases}$$

if and only if u is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\gamma t^{\alpha-1}}{(\alpha-1)(\alpha-2)} - \frac{\gamma t^{\alpha-2}}{(\alpha-2)(\alpha-3)} \tag{4}$$

for $0 \leq t \leq 1$, where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \big((t-s)^{\alpha-1} + t^{\alpha-1}(1-s)^{\alpha-4}((4-\alpha)s-1) \\ \quad +(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s\big) \quad \text{for } 0 \leq s < t \leq 1, \\ \frac{1}{\Gamma(\alpha)} \big(t^{\alpha-1}(1-s)^{\alpha-4}((4-\alpha)s-1) \\ \quad +(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s\big) \quad \text{for } 0 \leq t \leq s < 1. \end{cases}$$

A solution u of the problem has the form

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4}$$

for $0 \leq t \leq 1$, where C_1, C_2, C_3 and C_4 are constants; see [1]. The conditions $u(0) = 0$ and $u'(0) = 0$ imply $C_3 = 0$ and $C_4 = 0$, respectively. Since

$$u'(t) = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2}h(s)ds + (\alpha-1)C_1 t^{\alpha-2} + (\alpha-2)C_2 t^{\alpha-3}$$

for $0 \leq t \leq 1$, we have

$$u''(t) = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-3}h(s)ds + (\alpha-1)(\alpha-2)C_1 t^{\alpha-3} + (\alpha-2)(\alpha-3)C_2 t^{\alpha-4}$$

for $0 < t \le 1$. Moreover, since $u''(1) = 0$, we have

$$\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} h(s)ds + (\alpha-1)(\alpha-2)C_1 + (\alpha-2)(\alpha-3)C_2 = 0.$$

Furthermore, we have

$$u'''(t) = \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-4} h(s)ds + (\alpha-1)(\alpha-2)(\alpha-3)C_1 t^{\alpha-4} + (\alpha-2)(\alpha-3)(\alpha-4)C_2 t^{\alpha-5}$$

for $0 < t \le 1$. Finally, since $u'''(1) = \gamma$, we have

$$\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-4} h(s)ds + (\alpha-1)(\alpha-2)(\alpha-3)C_1 + (\alpha-2)(\alpha-3)(\alpha-4)C_2 = \gamma.$$

Then, we obtain

$$\begin{aligned} C_1 &= \frac{\gamma}{(\alpha-1)(\alpha-2)} + \frac{1}{\Gamma(\alpha)} \int_0^1 \left((\alpha-4)(1-s)^{\alpha-3} - (\alpha-3)(1-s)^{\alpha-4}\right) h(s)ds \\ &= \frac{\gamma}{(\alpha-1)(\alpha-2)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-4}((4-\alpha)s-1)h(s)ds, \end{aligned}$$

and

$$\begin{aligned} C_2 &= -\frac{\gamma}{(\alpha-2)(\alpha-3)} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 \left((1-s)^{\alpha-4} - (1-s)^{\alpha-3}\right) h(s)ds \\ &= -\frac{\gamma}{(\alpha-2)(\alpha-3)} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-4} s h(s)ds. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \\ &= \int_0^1 G(t,s)h(s)ds + \frac{\gamma t^{\alpha-1}}{(\alpha-1)(\alpha-2)} - \frac{\gamma t^{\alpha-2}}{(\alpha-2)(\alpha-3)}. \end{aligned}$$

This is equation (4).

Remark 1. Let $3 < \alpha \le 4$. The function $G(t,s)$ satisfies

$$l(t,s) \le G(t,s) \le m(t,s)$$

for $0 \le t \le 1$ and $0 \le s < 1$, where

$$l(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4}(2s+st-t) & (s<t), \\ \dfrac{\alpha-2}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4}s & (t \le s), \end{cases}$$

and

$$m(t,s) = \begin{cases} \dfrac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4}s & (s<t), \\ \dfrac{3}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4}s & (t \le s). \end{cases}$$

In fact, when $s < t$, we have

$$\begin{aligned}
&\Gamma(\alpha)G(t,s) \\
&= t^{\alpha-2}(1-s)^{\alpha-4}\left((t-s)^{\alpha-1}(1-s)^{4-\alpha}t^{2-\alpha}\right. \\
&\quad \left.+((4-\alpha)s-1)t+(\alpha-1)s\right) \\
&\ge t^{\alpha-2}(1-s)^{\alpha-4}\left((\alpha-1)s(1-t)+3st-t\right) \\
&\ge t^{\alpha-2}(1-s)^{\alpha-4}(2s(1-t)+3st-t) \\
&= t^{\alpha-2}(1-s)^{\alpha-4}(2s+st-t)
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma(\alpha)G(t,s) \\
&= t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1} + t^{\alpha-1}(1-s)^{\alpha-4}\left((4-\alpha)s-1\right) \\
&\quad + (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s \\
&\le t^{\alpha-1}(1-s)^{\alpha-1} + t^{\alpha-1}(1-s)^{\alpha-4}\left((4-\alpha)s-1\right) \\
&\quad + (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s \\
&= t^{\alpha-2}(1-s)^{\alpha-4}\left(t(1-s)^3 + \left((4-\alpha)s-1\right)t + (\alpha-1)s\right) \\
&\le t^{\alpha-2}(1-s)^{\alpha-4}\left(t(1-s) + \left((4-\alpha)s-1\right)t + (\alpha-1)s\right) \\
&\le t^{\alpha-2}(1-s)^{\alpha-4}\left(t(1-s) + (s-1)t + (\alpha-1)s\right) \\
&= (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s.
\end{aligned}$$

When $t \leq s$, we have

$$\begin{aligned}\Gamma(\alpha)G(t,s) &= t^{\alpha-2}(1-s)^{\alpha-4}((4-\alpha)st + (\alpha-2)s + (s-t)) \\ &\geq (\alpha-2)t^{\alpha-2}(1-s)^{\alpha-4}s\end{aligned}$$

and

$$\begin{aligned}\Gamma(\alpha)G(t,s) &= t^{\alpha-2}(1-s)^{\alpha-4}((4-\alpha)st - t + (\alpha-1)s) \\ &\leq t^{\alpha-2}(1-s)^{\alpha-4}s((4-\alpha) + (\alpha-1)) \\ &= 3t^{\alpha-2}(1-s)^{\alpha-4}s.\end{aligned}$$

Notice that the function $G(t,s)$ is not bounded on $[0,1] \times [0,1)$. However, since the function $G(t,s)$ satisfies

$$\int_0^1 |G(t,s)|ds \leq \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{\alpha}{\alpha-3}\right) \tag{5}$$

for all $0 \leq t \leq 1$, $\sup_{0\leq t\leq 1}\int_0^1 |G(t,s)|ds$ is finite. In fact, since $|(4-\alpha)s-1| \leq 1$ for all $0 \leq s < 1$, we have

$$\begin{aligned}&\int_0^1 |G(t,s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 |\,((4-\alpha)s-1)\,|(1-s)^{\alpha-4}ds \\ &\quad + \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-4}ds \\ &\leq \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-4}ds + \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-4}ds \\ &= \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{t^{\alpha-1}}{(\alpha-3)\Gamma(\alpha)} + \frac{(\alpha-1)t^{\alpha-2}}{(\alpha-3)\Gamma(\alpha)} \leq \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{\alpha}{\alpha-3}\right)\end{aligned}$$

for $0 \leq t \leq 1$. By (5), $\sup_{0\leq t\leq 1}\int_0^1 |G(t,s)|ds \leq \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{\alpha}{\alpha-3}\right)$.

3. Main results

In this section, we consider the boundary value problem (1). By the Banach fixed point theorem, we obtain a sufficient condition for the uniqueness and existence of solutions of the problem.

Theorem 2. *Let $3 < \alpha \leq 4$, f be a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$, and g be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant L. Assume that there exists a nonnegative constant λ with*

$$\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} < 1,$$

such that for any $0 \leq t \leq 1$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,u_1) - f(t,u_2)| \leq \lambda |u_1 - u_2|,$$

where Λ is the constant

$$\Lambda = \sup_{0 \leq t \leq 1} \int_0^1 |G(t,s)| ds.$$

Then, the boundary value problem (1) has a unique solution.

Proof. Note that by (5), Λ is finite. By (4), we see that the solution of the boundary value problem can be written as

$$u(t) = \int_0^1 G(t,s) f(s,u(s)) ds + \frac{g(u(1)) t^{\alpha-1}}{(\alpha-1)(\alpha-2)} - \frac{g(u(1)) t^{\alpha-2}}{(\alpha-2)(\alpha-3)},$$

for $0 \leq t \leq 1$. The set $C[0,1]$ is a Banach space with the supremum norm $\|u\| = \sup_{0 \leq t \leq 1} |u(t)|$ for $u \in C[0,1]$. Let T be mappings of X into itself defined by:

$$\begin{aligned} &T(u)(t) \\ &= \int_0^1 G(t,s) f(s,u(s)) ds + \frac{g(u(1)) t^{\alpha-1}}{(\alpha-1)(\alpha-2)} - \frac{g(u(1)) t^{\alpha-2}}{(\alpha-2)(\alpha-3)}, \end{aligned}$$

for $u \in C[0,1]$ and $0 \leq t \leq 1$. Then, a fixed point of T is a solution of the boundary value problem.

Let $u_1, u_2 \in C[0,1]$ and $0 \leq t \leq 1$. Then we have

$$\begin{aligned} &|T(u_1)(t) - T(u_2)(t)| \\ &\leq \left| \int_0^1 G(t,s) \left(f(s,u_1(s)) - f(s,u_2(s)) \right) ds \right| \\ &\quad + \left(\frac{1}{(\alpha-2)(\alpha-3)} - \frac{1}{(\alpha-1)(\alpha-2)} \right) |g(u_1(1)) - g(u_2(1))| \end{aligned}$$

$$\leq \int_0^1 |G(t,s)||f(s,u_1(s)) - f(s,u_2(s))|ds$$
$$+ L\left(\frac{1}{(\alpha-2)(\alpha-3)} - \frac{1}{(\alpha-1)(\alpha-2)}\right)\|u_1 - u_2\|$$
$$\leq \lambda\Lambda\|u_1 - u_2\| + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)}\|u_1 - u_2\|.$$

Therefore we have

$$\|Tu_1 - Tu_2\| \leq \left(\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)}\right)\|u_1 - u_2\|.$$

Since $\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} < 1$, T is a contraction. By the Banach fixed point theorem, T has a unique fixed point. This fixed point is a solution of the problem. □

For the case where $\alpha = 4$ in Theorem 2, we have the following; see Theorem 1 in [4].

Corollary 3. *Let f be a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$ with a bounded partial derivative with respect to the second variable. Let g be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant L. Let*

$$\lambda = \max_{(t,u)\in[0,1]\times\mathbb{R}} \left|\frac{\partial f}{\partial u}(t,u)\right|.$$

If

$$\frac{\lambda}{8} + \frac{L}{3} < 1,$$

then the boundary value problem (2) has a unique solution.

Proof. By the mean value theorem, we have for any $0 \leq t \leq 1$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,u_1) - f(t,u_2)| \leq \lambda|u_1 - u_2|.$$

In the case that $\alpha = 4$, the function $G(t,s)$ reduces to

$$G(t,s) = \begin{cases} \frac{1}{6}s^2(3t-s) & (s < t), \\ \frac{1}{6}t^2(3s-t) & (t \leq s). \end{cases}$$

Since

$$\Lambda = \sup_{0\le t\le 1}\int_0^1 |G(t,s)|ds = \frac{1}{8},$$

we obtain the conclusion by Theorem 2. □

To conclude the paper, we present an example demonstrating an application of Theorem 2.

Example 4. Let us consider the boundary value problem

$$\begin{cases} D_{0+}^{3.1}u(t) = \frac{3}{(54e^t+1)(1+|u(t)|)}, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases} \tag{6}$$

where

$$g(t) = \frac{1}{100}\sin t.$$

By (5), the constant Λ in Theorem 2 satisfies

$$\Lambda \le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{\alpha}{\alpha-3}\right) = 14.2530\cdots < 15.$$

Moreover, for any $0 \le t \le 1$ and $u_1, u_2 \in \mathbb{R}$, we have,

$$|f(t,u_1) - f(t,u_2)| \le \frac{3}{55}|u_1 - u_2|,$$

where

$$f(t,u) = \frac{3}{(54e^t+1)(1+|u|)}$$

for $0 \le t \le 1$ and $u \in \mathbb{R}$; see Section 4 in [2]. Since the constants $\lambda = \frac{3}{55}$ and $L = \frac{1}{100}$ in Theorem 2, we have

$$\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} \le \frac{3}{55}\times 15 + \frac{2\times\frac{1}{100}}{2.1\times 1.1\times 0.1} = 0.\dot{9}0476\dot{1}.$$

It follows from Theorem 2 that problem (6) has a unique solution.

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