International Journal of Applied Mathematics

Volume 33 No. 3 2020, 385-392
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v33i3.2

THE INTEGER RECURRENCE $P(n) = a + P(n - \phi(a))$

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Abstract: We prove that for a positive integer a the integer sequence $P(n)$ satisfying for all $n, -\infty < n < \infty$, the recurrence $P(n) = a + P(n - \phi(a))$, $\phi(a)$ the Euler function, generates in increasing order all integers $P(n)$ coprime to a. The finite Fourier expansion of $P(n)$ is given in terms of a, n, and the $\phi(a)$-th roots of unity. Properties of the sequence are derived.

AMS Subject Classification: 11B37
Key Words: recurrence sequences; Euler function; Fourier expansion; radical of an integer

1. Introduction

For a positive integer $a = \prod_{i=1}^{\omega} p_i^{e_i}$, we set $R(a) = \prod_{i=1}^{\omega} p_i$, $Q(a) = \prod_{i=1}^{\omega} (p_i - 1)$, $\phi(a) = \frac{a}{R(a)} Q(a)$, which, if a is fixed, we write R, Q, ϕ, respectively. Note that $Q|\phi$ and that $\phi(R(a)) = Q(R(a))$.

We show that the integer sequence $P(n)$ defined for all n running from $-\infty$ to ∞ , by the inhomogeneous recurrence of order $\phi(a)$

$$P(n) = a + P(n - \phi(a))$$

and certain $\phi(a) + 1$ initial conditions, to be specified, gives in increasing order, $P(n) < P(n + 1)$, all integers coprime to a.

The minimal integer recurrence satisfied by $P(n)$ is

$$P(n) = R(a) + P(n - Q(a)).$$

Using known facts from the theory of linear integer recurrences, Graham et

Received: December 12, 2019

al. [3], $P(n)$ is explicitly expressed as a finite Fourier expansion, Zygmund [7], involving a, n, and the $\phi(a)$-th roots of unity as studied in [4]. Similarly, $P(n)$ can be expressed as a finite Fourier expansion, Ahlfors [1], involving $R(a)$, n, and the $Q(a)$-th roots of unity, Derbyshire [2].

Properties of the function $P(n)$ are established, such as f.ex. $\lim_{n\to\infty} \frac{P(n)}{n} = \frac{R(a)}{Q(a)}$.

The infinite sequence $P(n)$, with index n suitably normalized, can be regarded as the natural extension to $\mathcal{Z}$ of the "Euler" set of a, namely the $\phi(a)$ positive integers smaller than a and coprime to a, arranged in increasing order, Sloane [6].

2. The sequence $P(n)$

We first prove the minimal integer recurrence satisfied by $P(n)$, from which, afterwards, we deduce the integer recurrence $P(n) = a + P(n - \phi(a))$.

Theorem 1. *For a fixed integer $a \geq 1$, let $a_1(= 1) < a_2 < \ldots < a_Q(= R - 1)$ denote the Q positive integers smaller than R and coprime to R. The integer sequence $P(n)$, normalized for $n = 1$ to give $P(1) = R + 1$, which satisfies for all n running from $-\infty$ to ∞, the integer recurrence*

$$P(n) = R + P(n - Q),$$

and the $Q + 1$ initial conditions

$$\begin{gathered} P(-Q+1) = a_1 \ \ (= 1) \\ P(-Q+2) = a_2 \\ \ldots \qquad \ldots \qquad \ldots \\ P(0) = a_Q \ \ (= R-1), \\ P(1) = a_{Q+1} \ \ (= R+1), \end{gathered} \tag{1}$$

generates in monotonically increasing order, $P(n) < P(n + 1)$, all integers coprime to a.

Proof. We first show that $P(n)$ is coprime to a.

For $n = 0$ this is obvious since $P(0) = R - 1$.

For $n > 0$, we have by definition

$$P(n) = R + P(n - Q),$$
$$P(n - Q) = R + P(n - 2Q),$$
$$\dots \qquad \dots \qquad \dots$$
$$P(n - (k-1)Q) = R + P(n - kQ),$$

which, when added, give for any integer $k \geq 0$.

$$P(n) = kR + P(n - kQ). \tag{2}$$

Setting $k = \left[\frac{n+Q-1}{Q}\right]$, where $[x]$ the greatest integer $\leq x$, we have

$$\frac{n-1}{Q} < \left[\frac{n+Q-1}{Q}\right] \leq \frac{n+Q-1}{Q},$$

$$n - 1 < \left[\frac{n+Q-1}{Q}\right]Q \leq n + Q - 1,$$

$$-Q + 1 \leq n - \left[\frac{n+Q-1}{Q}\right]Q < 1.$$

Hence from (1) we get

$$1 \leq P\Big(n - \left[\frac{n+Q-1}{Q}\right]Q\Big) < R + 1.$$

Since $P\big(n - \left[\frac{n+Q-1}{Q}\right]Q\big)$ is coprime to a because of (1), it follows from (2) that $P(n)$ is also coprime to a.

For $n < 0$ it can be shown, by a similar argument, that

$$P(-n) = -kR + P(-n + kQ),$$

and by choosing $k = \left[\frac{n+1}{Q}\right]$ we again derive that $P(-n)$ is coprime to a.

It results that for any $n \in \mathcal{Z}$ and any $k \in \mathcal{Z}$, we have

$$P(n) = kR + P(n - kQ). \tag{3}$$

We now show that $P(n) < P(n+1)$. Substituting in (3) n by $n+1$ and subtracting we get

$$P(n+1) - P(n) = P(n + 1 - kQ) - P(n - kQ).$$

Choosing, as above, for k either $\left[\frac{n+Q-1}{Q}\right]$ or $\left[\frac{n+1}{Q}\right]$, we infer from the initial conditions (1), that $P(n) < P(n+1)$. □

From Theorem 1 we obtain the following theorem.

Theorem 2. *For a fixed integer $a \geq 1$, let $a_1(=1) < a_2 < \ldots < a_\phi(= a-1)$ denote the ϕ positive integers smaller than a and coprime to a. The integer sequence $P(n)$, normalized for $n = 1$ to give $P(1) = a+1$, which satisfies for all n running from $-\infty$ to ∞, the integer recurrence*

$$P(n) = a + P(n-\phi),$$

and the $\phi+1$ initial conditions

$$P(-\phi+1) = a_1 \quad (=1),$$

$$P(-\phi+2) = a_2,$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$P(0) = a_\phi \ \ (= a-1),$$

$$P(1) = a_{\phi+1} \ \ (= a+1),$$

generates in monotonicaly increasing order $P(n) < P(n+1)$, all integers coprime to a.

Proof. Setting in (3) $k = \frac{a}{R}$, we have

$$P(n) = a + P(n-\phi).$$

□

3. The Fourier expansion of $P(n)$

The integer recurrence $P(n) = R + P(n-Q)$ is inhomogeneous. Subtracting $P(n+1) = R + P(n+1-Q)$ we get the homogeneous recurrence

$$P(n+1) - P(n) - P(n+1-Q) + P(n-Q) = 0.$$

Its characteristic polynomial is

$$x^{Q+1} - x^Q - x + 1 = (x-1)^2(x - e^{2\pi i \frac{1}{Q}}) \ldots (x - e^{2\pi i \frac{Q-1}{Q}}).$$

Using known facts from the theory of integer recurrences Graham [3], we therefore have

$$P(n) = c_0 n + c_1 + \sum_{\nu=1}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q} n}$$

$$= c_0 n + \sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q} n}. \tag{4}$$

The $Q+1$ coefficients c_ν can be determined by solving the system of $Q+1$ linear equations, resulting from following $Q+1$ initial conditions

$$c_0(-Q+1) + \sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(-Q+1)} = a_1,$$

$$c_0(-Q+2) + \sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(-Q+2)} = a_2,$$

$$\dots \qquad \dots \qquad \dots$$

$$c_0(0) + \sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(0)} = a_Q,$$

$$c_0(1) + \sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}}(1) = a_{Q+1}.$$

Another way is to take only the first Q equations of above system, transfer the terms $c_0(-Q+1), \dots, c_0(0)$ to the right side, and find the value of c_0 afterwards. Accordingly we write

$$\sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(-Q+1)} = a_1 - c_0(-Q+1),$$

$$\sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(-Q+2)} = a_2 - c_0(-Q+2),$$

$$\dots \qquad \dots \qquad \dots$$

$$\sum_{\nu=0}^{Q-1} c_{\nu+1} e^{2\pi i \frac{\nu}{Q}(0)} = a_Q - c_0(0).$$

Putting (Vandermonde)

$$D_a = |e^{2\pi i \frac{k}{Q} \ell}|, \quad 0 \le k \le Q-1, \quad -Q+1 \le \ell \le 0,$$

we get for the coefficients

$$c_\nu = \frac{1}{D_a} \sum_{\mu=1}^{Q} (-1)^{\mu-1} (a_\mu - c_0(-Q+\mu))\, D_{\nu,\mu}, \tag{5}$$

where $D_{\nu,\mu}$ are the $(Q-1)\times(Q-1)$ minors of D_a, obtained by replacing the μ-th column with $a_\mu - c_0(-Q+\mu)$.

In order to find the value of c_0 we substitute in (4) n by $n+Q$. This gives

$$P(n+Q) = c_0(n+Q) + \sum_{\nu=0}^{Q-1} c_\nu e^{2\pi i \frac{\nu}{Q}(n+Q)}$$

$$= c_0 Q + c_0 n + \sum_{\nu=0}^{Q-1} c_\nu e^{2\pi i \frac{n}{Q} n} = c_0 Q + P(n).$$

But $P(n+Q) = R + P(n)$, as can be seen from (3), if we substitute n by $n+kQ$ and put $k=1$. Hence,

$$c_0 Q = R,$$

$$c_0 = \frac{R}{Q}.$$

Summarizing, we have proved the following theorem.

Theorem 3. *$P(n)$, $-\infty < n < \infty$, can be expressed by the Fourier expansion*

$$P(n) = \frac{R}{Q} n + \sum_{\nu=0}^{Q-1} c_\nu e^{2\pi i \frac{\nu}{Q} n},$$

where the coefficients c_ν are given by (5).

Exactly the same procedure as above, applied to the recurrence

$$P(n) = a + P(n-\phi)$$

gives the following theorem.

Theorem 4. *$P(n)$, $-\infty < n < \infty$, can be expressed by the Fourier expansion*

$$P(n) = \frac{R}{Q} n + \sum_{\nu=0}^{\phi-1} d_\nu e^{2\pi i \frac{\nu}{\phi} n},$$

where the coefficients d_ν, depending on ϕ, are given by a formula similar to (4).

Dividing by n, we get

Corollary 5.

$$\lim_{n\to\infty} \frac{P(n)}{n} = \frac{R}{Q}.$$

The following transformation formulas are an immediate consequence of above theorems.

Corollary 6. *If $R(a) = R(b)$, then for all n:*

$$P(n - Q(a)) = P(n - Q(b)),$$

$$P(n - \phi(a)) = P(n - \phi(b)).$$

Note. In a next communication, we will examine $P(n)$ from the angle of generating functions. To this end, we introduce the GF of the sequence $P(n)$ for a fixed integer $a \geq 1$,

$$G(t) = \sum_{n=1}^{\infty} P(n)t^n.$$

The recurrence $P(n) = a + P(n - \phi)$ promptly gives

$$G(t) = a\frac{t}{(t-1)(t^{\phi}-1)} - \frac{1}{t^{\phi}-1}\sum_{\nu=1}^{\phi} a_{\nu}t^{\nu}.$$

The coefficients are then expressed by Ahlfors [1], Rudin [5], as

$$P(n) = \frac{a}{2\pi i}\int_c \frac{t}{(t-1)(t^{\phi}-1)}\,\frac{1}{t^{n+1}}dt - \sum_{\nu=1}^{\phi}\frac{a_{\nu}}{2\pi i}\int_c \frac{t^{\nu}}{t^{\phi}-1}\,\frac{1}{t^{n+1}}dt.$$

Evaluation of the integrals or expansion in series of the GF yields expressions of $P(n)$ in terms of a, n, $\phi(a)$ and the "Euler" set $\{a_{\nu}\}$. Comparison with the resp. Fourier expansions results in identities involving the parameters.

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