ON MICRO $T_{\frac{1}{2}}$ SPACE

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Abstract: The aim of this paper is to introduce and study different properties of Micro generalized closed set in micro topological space. As applications to Micro generalized closed set, we introduce Micro $T_{\frac{1}{2}}$ space and obtain some of their basic properties. We analyze the behavior of Micro generalized closed set and Micro $T_{\frac{1}{2}}$ space under Micro-continuous and Micro-closed functions. Also, we introduce the notions of Micro difference sets and Micro kernel of sets and investigate some of their fundamental properties.

1. Introduction

The first step of generalizing closed sets was done by Levine [23] when he introduced the concept of generalized closed sets in topological spaces by comparing the closure of a subset with its open supersets. Rough set theory was proposed by Pawlak [24] to conceptualize, organize and analyze various types of data in data mining. The rough set method is especially useful for dealing with vagueness and granularity in information systems. It deals with the approximation of an arbitrary subset of a universe by two definable subsets which are referred to as the lower and upper approximations. By using the lower and upper ap-

proximations of decision classes, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The lower and upper approximation operators in the Pawlaks rough set model are induced by equivalence relations or partitions. However, the requirement of an equivalence relation or partition in the Pawlaks rough set model may limit the applications of the rough set model. The concept of nano topology was introduced by Thivagar et al [26, 27] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it. The concept of Micro open set in micro topological space was introduced and investigated by Chandrasekar [1]. In the past few years, different forms of generalized closed sets, Difference sets and kernel of sets have been studied [2, 3, 4, 18, 5, 19, 6, 7, 8, 9, 10, 20, 11, 12, 13, 14, 15, 21, 22, 16].

2. Preliminaries

Definition 1. ([25]) Let U be a nonempty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of U with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.

2. The upper approximation of U with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \phi\}$.

3. The boundary region of U with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = L_R(X) - L_R(X)$.

Definition 2. ([26, 27]) Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), L_R(X), B_R(X)\}$, where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$.

2. The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets. A subset F of U is nano closed if its complement is nano open.

Definition 3. ([1]) Let $(U, \tau_R(X))$ be a nano topological space. Then, $\mu_R(X) = \{N \cup (N' \cap \mu) : N, N' \in \tau_R(X) \text{ and } \mu \notin \tau_R(X)\}$ is called the micro topology on U with respect to X. The triplet $(U, \tau_R(X), \mu_R(X))$ is called micro topological space and the elements of $\mu_R(X)$ are called Micro open sets and the complement of a Micro open set is called a Micro closed set.

Definition 4. ([1]) The Micro closure of a set A is denoted by $Mic\text{-}cl(A)$ and is defined as $Mic\text{-}cl(A) = \cap\{B : B \text{ is Micro closed and } A \subseteq B\}$. The Micro interior of a set A is denoted by $Mic\text{-}int(A)$ and is defined as $Mic\text{-}int(A) = \cup\{B : B \text{ is Micro open and } A \supseteq B\}$.

Definition 5. ([1]) Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Let A and B be any two subsets of U. Then:

1. A is a Micro open set if and only if $Mic\text{-}int(A) = A$.

2. A is a Micro closed set if and only if $Mic\text{-}cl(A) = A$.

3. If $A \subseteq B$, then $Mic\text{-}cl(A) \subseteq Mic\text{-}cl(B)$.

4. $Mic\text{-}cl(Mic\text{-}cl(A)) = Mic\text{-}cl(A)$.

5. $Mic\text{-}cl(U \setminus A) = U \setminus Mic\text{-}int(A)$.

Definition 6. ([1]) Let $(U, \tau_R(X), \mu_R(X))$ and $(V, \tau_R(Y), \mu_R(Y))$ be two micro topological spaces. Then, a function $f : U \to V$ is said to be Micro-continuous if $f^{-1}(K)$ is Micro open in U, for every Micro open set K in V.

Theorem 7. ([1]) *A function $f : U \to V$ is Micro-continuous if and only if $f^{-1}(K)$ is Micro closed in U, for every Micro closed set K in V.*

Remark 1. ([17]) Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be any subset of U. Then:

1. $Mic\text{-}cl(A)$ is Micro closed.

2. $A \subseteq Mic\text{-}cl(A)$.

3. $x \in Mic\text{-}cl(A)$ if and only if for every Micro open subset L of U containing x, $A \cap L \neq \phi$.

3. Micro $T_{\frac{1}{2}}$ Space

Definition 8.   Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. A subset A of U is said to be a Micro generalized closed (briefly, Micro g.closed) if $Mic\text{-}cl(A) \subseteq L$ whenever $A \subseteq L$ and L is a Micro open set in U.

Remark 2.   It is clear that every Micro closed subset of U is also a Micro g.closed set. The following example shows that a Micro g.closed set need not be Micro closed.

Example 1.   Consider $U = \{p, q, s\}$ with $U/R = \{\{p, q, s\}\}$ and $X = \{p, s\}$. Then, $\tau_R(X) = \{U, \phi\}$. If $\mu = \{q\}$, then $\mu_R(X) = \{U, \phi, \{q\}\}$. Then, $\{q, s\}$ is Micro g.closed but it is not Micro closed.

Theorem 9.   Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. A subset A of U is Micro g.closed if and only if $Mic\text{-}cl(\{x\}) \cap A \neq \phi$ holds for every $x \in Mic\text{-}cl(A)$.

Proof. Let L be a Micro open set such that $A \subseteq L$. Let $x \in Mic\text{-}cl(A)$, then there exists an element $z \in Mic\text{-}cl(\{x\})$ and $z \in A \subseteq L$. It follows from Remark 1 (3), that $L \cap \{x\} \neq \phi$ and hence $x \in L$. This implies $Mic\text{-}cl(A) \subseteq L$. Thus, A is Micro g.closed set in U.

Conversely, let A be a Micro g.closed subset of U and $x \in Mic\text{-}cl(A)$ such that $Mic\text{-}cl(\{x\}) \cap A = \phi$. Since, $Mic\text{-}cl(\{x\})$ is Micro closed set in U. Thus, by Definition 3, $U \setminus (Mic\text{-}cl(\{x\}))$ is a Micro open set. Since $A \subseteq U \setminus (Mic\text{-}cl(\{x\}))$ and A is Micro g.closed implies that $Mic\text{-}cl(A) \subseteq U \setminus (Mic\text{-}cl(\{x\}))$ holds, and hence $x \notin Mic\text{-}cl(A)$. This is a contradiction. Thus, $Mic\text{-}cl(\{x\}) \cap A \neq \phi$. $\square$

Theorem 10.   If $Mic\text{-}cl(\{x\}) \cap A \neq \phi$ holds for every $x \in Mic\text{-}cl(A)$, then $Mic\text{-}cl(A) \setminus A$ does not contain a non empty Micro closed set.

Proof. Suppose there exists a non empty Micro closed set F such that $F \subseteq Mic\text{-}cl(A) \setminus A$. Let $x \in F$, then $x \in Mic\text{-}cl(A)$. It follows that $F \cap A = Mic\text{-}cl(F) \cap A \supseteq Mic\text{-}cl(\{x\}) \cap A \neq \phi$. Hence, $F \cap A \neq \phi$. This is a contradiction. Thus, $F = \phi$. $\square$

Corollary 11. *A is Micro g.closed if and only if $A = F \setminus N$, where F is Micro closed and N contains no non empty Micro closed subsets of U.*

Proof. Necessity follows from Theorems 9 and 10 with $F = Mic\text{-}cl(A)$ and $N = Mic\text{-}cl(A) \setminus A$.

Conversely, if $A = F \setminus N$ and $A \subseteq O$ with O is Micro open, then $F \cap (U \setminus O)$ is a Micro closed subset of N and thus is empty. Hence, $Mic\text{-}cl(A) \subseteq F \subseteq O$. $\square$

Theorem 12. *If a subset A of U is Micro g.closed and $A \subseteq B \subseteq Mic\text{-}cl(A)$, then B is a Micro g.closed set in U.*

Proof. Let A be a Micro g.closed set such that $A \subseteq B \subseteq Mic\text{-}cl(A)$. Let L be a Micro open subset of U such that $B \subseteq L$. Since A is Micro g.closed, then $Mic\text{-}cl(A) \subseteq L$. Now, $Mic\text{-}cl(A) \subseteq Mic\text{-}cl(B) \subseteq Mic\text{-}cl(Mic\text{-}cl(A)) = Mic\text{-}cl(A) \subseteq L$, that is $Mic\text{-}cl(B) \subseteq L$. Thus, B is a Micro g.closed set in U. $\square$

Theorem 13. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, for each $x \in U$, either $\{x\}$ is Micro closed or $U \setminus \{x\}$ is Micro g.closed.*

Proof. Suppose that $\{x\}$ is not Micro closed, then by Definition 3, $U \setminus \{x\}$ is not Micro open. Let L be any Micro open set such that $U \setminus \{x\} \subseteq L$, so $L = U$. Hence, $Mic\text{-}cl(U \setminus \{x\}) \subseteq L$. Thus, $U \setminus \{x\}$ is Micro g.closed. $\square$

Definition 14. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro symmetric if for x and y in U such that $x \in Mic\text{-}cl(\{y\})$ implies $y \in Mic\text{-}cl(\{x\})$.

Example 2. Consider $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}, \{a, b, d\}, \{b, d\}\}$. If $\mu = \{c\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. Then, U is Micro symmetric.

Theorem 15. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then,*

then the following statements are equivalent:

1. U *is a Micro symmetric.*

2. $\{x\}$ *is Micro g.closed, for each* $x \in U$.

Proof. $(1) \Rightarrow (2)$: Assume that $\{x\} \subseteq L \in \mu_R(X)$, but $Mic\text{-}cl(\{x\}) \nsubseteq L$. Then, $Mic\text{-}cl(\{x\}) \cap U \setminus L \neq \phi$. Now, we take $y \in Mic\text{-}cl(\{x\}) \cap U \setminus L$, then by hypothesis $x \in Mic\text{-}cl(\{y\}) \subseteq U \setminus L$ and $x \notin L$, which is a contradiction. Therefore, $\{x\}$ is Micro g.closed for each $x \in U$.

$(2) \Rightarrow (1)$: Assume that $x \in Mic\text{-}cl(\{y\})$, but $y \notin Mic\text{-}cl(\{x\})$. Then, $\{y\} \subseteq U \setminus Mic\text{-}cl(\{x\})$ and hence $Mic\text{-}cl(\{y\}) \subseteq U \setminus Mic\text{-}cl(\{x\})$. Therefore, $x \in U \setminus Mic\text{-}cl(\{x\})$ which is a contradiction and so $y \in Mic\text{-}cl(\{x\})$. $\qquad\square$

Definition 16. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro $T_{\frac{1}{2}}$ if every Micro g.closed in U is Micro closed.

Example 3. Consider $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}\}$. If $\mu = \{c\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{c\}, \{a, c\}\}$. Then, U is Micro $T_{\frac{1}{2}}$.

Theorem 17. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is a Micro $T_{\frac{1}{2}}$ if and only if $\{x\}$ is Micro closed or Micro open, for each $x \in U$.

Proof. Suppose $\{x\}$ is not Micro closed. Then it follows from assumption and Theorem 13, that $\{x\}$ is Micro open.

Conversely, let F be Micro g.closed set in U and x be any point in $Mic\text{-}cl(F)$, then $\{x\}$ is Micro open or Micro closed.

1. Suppose $\{x\}$ is Micro open. Then by Remark 1 (3), we have $\{x\} \cap F \neq \phi$ and hence $x \in F$. This implies $Mic\text{-}cl(F) \subseteq F$, therefore F is Micro closed.

2. Suppose $\{x\}$ is Micro closed. Assume $x \notin F$, then $x \in Mic\text{-}cl(F) \setminus F$. This is not possible by Theorem 10. Thus, we have $x \in F$. Therefore, $Mic\text{-}cl(F) = F$ and hence F is Micro closed.

$\qquad\square$

Theorem 18. *For a function* $f : U \to V$, *the following statements are equivalent:*

1. f is Micro-continuous.

2. $f(Mic\text{-}cl(A)) \subseteq Mic\text{-}cl(f(A))$, for each subset A of U.

3. $Mic\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(Mic\text{-}cl(B))$, for each subset B of V.

4. $f^{-1}(Mic\text{-}int(B)) \subseteq Mic\text{-}int(f^{-1}(B))$, for each subset B of V.

Proof. $(1) \Rightarrow (2)$: Let A be any subset of U. Then, $f(A) \subseteq Mic\text{-}cl(f(A))$ and $Mic\text{-}cl(f(A))$ is a Micro closed set in V. Hence, $A \subseteq f^{-1}(Mic\text{-}cl(f(A)))$. By (1), we have $f^{-1}(Mic\text{-}cl(f(A)))$ is a Micro closed set in U. Therefore, $Mic\text{-}cl(A) \subseteq f^{-1}(Mic\text{-}cl(f(A)))$. Hence, $f(Mic\text{-}cl(A)) \subseteq Mic\text{-}cl(f(A))$.

$(2) \Rightarrow (3)$: Let B be any subset of V. Then, $f^{-1}(B)$ is a subset of U. By (2), we have $f(Mic\text{-}cl(f^{-1}(B))) \subseteq Mic\text{-}cl(f(f^{-1}(B))) \subseteq Mic\text{-}cl(B)$. Hence, $Mic\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(Mic\text{-}cl(B))$.

$(3) \Leftrightarrow (4)$: Let B be any subset of V. Then, apply (3) to $V \setminus B$ we obtain $Mic\text{-}cl(f^{-1}(V \setminus B)) \subseteq f^{-1}(Mic\text{-}cl(V \setminus B)) \Leftrightarrow Mic\text{-}cl(U \setminus f^{-1}(B)) \subseteq f^{-1}(V \setminus Mic\text{-}int(B)) \Leftrightarrow U \setminus Mic\text{-}int(f^{-1}(B)) \subseteq U \setminus f^{-1}(Mic\text{-}int(B)) \Leftrightarrow f^{-1}(Mic\text{-}int(B)) \subseteq Mic\text{-}int(f^{-1}(B))$. Thus, $f^{-1}(Mic\text{-}int(B)) \subseteq Mic\text{-}int(f^{-1}(B))$.

$(4) \Rightarrow (1)$: Let $x \in U$ and K be any Micro open subset of V containing $f(x)$. By (4), we have $f^{-1}(Mic\text{-}int(K)) \subseteq Mic\text{-}int(f^{-1}(K))$ implies that $f^{-1}(K) \subseteq Mic\text{-}int(f^{-1}(K))$. Hence, $f^{-1}(K)$ is a Micro open set in U which contains x and clearly $f(f^{-1}(K)) \subseteq K$. Thus, f is Micro-continuous. $\square$

Definition 19. A function $f : U \to V$ is said to be Micro-closed if for any Micro closed subset A of U, $f(A)$ is Micro closed in V.

Example 4. Consider $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}\}$. If $\mu = \{b\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : U \to U$ as follows:

$$f(x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ c & \text{if } x = c \end{cases}$$

Then, f is Micro-closed.

Theorem 20. Let $f : U \to V$ be Micro-continuous and Micro-closed. Then:

1. $f(A)$ is Micro g.closed in V, for every Micro g.closed subset A of U.

2. $f^{-1}(B)$ is Micro g.closed in U, for every Micro g.closed subset B of V.

3. If f is injective and V is Micro $T_{\frac{1}{2}}$, then U is Micro $T_{\frac{1}{2}}$.

4. If f is surjective and U is Micro $T_{\frac{1}{2}}$, then V is Micro $T_{\frac{1}{2}}$.

   Proof.   1. Let K be any Micro open set in V such that $f(A) \subseteq K$. Since f is Micro-continuous, then $f^{-1}(K)$ is Micro open in U. Since A is Micro g.closed and $A \subseteq f^{-1}(K)$, then $Mic\text{-}cl(A) \subseteq f^{-1}(K)$, and hence $f(Mic\text{-}cl(A)) \subseteq K$. By Definition 19, $f(Mic\text{-}cl(A))$ is Micro closed in V and hence $Mic\text{-}cl(f(A)) \subseteq Mic\text{-}cl(f(Mic\text{-}cl(A))) = f(Mic\text{-}cl(A)) \subseteq K$. This implies $f(A)$ is Micro g.closed.

2. Let L be a Micro open subset of U such that $f^{-1}(B) \subseteq L$. Let $F = Mic\text{-}cl(f^{-1}(B)) \cap (U \setminus L)$, then F is Micro closed set in U. Since f is Micro-closed, this implies $f(F)$ is Micro closed in V. By Theorem 18, we have $f(F) \subseteq f(Mic\text{-}cl(f^{-1}(B))) \cap f(U \setminus L) \subseteq Mic\text{-}cl(f(f^{-1}(B))) \cap f(U \setminus f^{-1}(B)) \subseteq Mic\text{-}cl(B) \cap (V \setminus B)$ and by Theorems 9 and 10, $f(F) = \phi$ and hence $F = \phi$. Thus, $Mic\text{-}cl(f^{-1}(B)) \subseteq L$ and hence $f^{-1}(B)$ is Micro g.closed in U.

3. Let A be a Micro g.closed subset of U. By (1), $f(A)$ is Micro g.closed in V. Since V is Micro $T_{\frac{1}{2}}$, this implies that $f(A)$ is Micro closed and by Theorem 7, we have $A = f^{-1}(f(A))$ is Micro closed. Thus, U is Micro $T_{\frac{1}{2}}$.

4. Let B be a Micro g.closed subset of V. By (2), $f^{-1}(B)$ is Micro g.closed in U. Since U is Micro $T_{\frac{1}{2}}$, so $f^{-1}(B)$ is Micro closed and since f is both surjective and Micro-closed, so $f(f^{-1}(B)) = B$ is Micro closed. Thus, V is Micro $T_{\frac{1}{2}}$.

$\square$

   Definition 21.   Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, a subset A of U is called a Micro Difference set (briefly, MD-set) if there are $L, K \in \mu_R(X)$ such that $L \neq U$ and $A = L \setminus K$.

   It is true that every Micro open set L different from U is a MD-set if $A = L$ and $K = \phi$. So, we can observe the following.

Remark 3. Every proper Micro open set is a MD-set. But, the converse is not true in general as the next example shows.

Example 5. Consider $U = \{a, q, r, s, t\}$ with $U/R = \{\{a\},$ $\{q, r, s\}, \{t\}\}$ and $X = \{a, q\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}, \{a, q, r, s\},$ $\{q, r, s\}\}$. If $\mu = \{q, r, s, t\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{a, q, r, s\},$ $\{q, r, s\}, \{q, r, s, t\}\}$. If $L = \{q, r, s, t\} \neq U$ and $K = \{a, q, r, s\}$, then $\{t\} = L \setminus K = \{q, r, s, t\} \setminus \{a, q, r, s\} = \{t\}$. Thus, $A = \{t\}$ is a MD-set but it is not Micro open.

Theorem 22. If $f : U \to V$ is a Micro-continuous surjective function and A is a MD-set in V, then the inverse image of A is a MD-set in U.

Proof. Let A be a MD-set in V. Then, there are Micro open sets O_1 and O_2 in V such that $A = O_1 \setminus O_2$ and $O_1 \neq V$. By the Micro-continuous of f, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are Micro open in U. Since $O_1 \neq V$ and f is surjective, we have $f^{-1}(O_1) \neq U$. Hence, $f^{-1}(A) = f^{-1}(O_1) \setminus f^{-1}(O_2)$ is a MD-set. $\square$

Definition 23. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be a subset of U. Then, the Micro kernel of A denoted by $Mker(A)$ is defined to be the set

$$Mker(A) = \cap\{L \in \mu_R(X) : A \subseteq L\}.$$

Example 6. Consider $U = \{a, q, r, s, b\}$ with $U/R = \{\{a\},$ $\{q, r, s\}, \{b\}\}$ and $X = \{a, q\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}, \{a, q, r, s\},$ $\{q, r, s\}\}$. If $\mu = \{b\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{a, q, r, s\},$ $\{q, r, s\}, \{q, r, s, b\}\}$. Then, $Mker(\{q, r\}) = \{q, r, s\}$.

Theorem 24. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and $x \in U$. Then, $y \in Mker(\{x\})$ if and only if $x \in Mic\text{-}cl(\{y\})$.

Proof. Suppose that $y \notin Mker(\{x\})$. Then, there exists a Micro open set K containing x such that $y \notin K$. Therefore, we have $x \notin Mic\text{-}cl(\{y\})$. The proof of the converse case can be done similarly. $\square$

Theorem 25. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be a subset of U. Then, $Mker(A) = \{x \in U : Mic\text{-}cl(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in Mker(A)$ and suppose $Mic\text{-}cl(\{x\}) \cap A = \phi$. Hence, $x \notin U \setminus Mic\text{-}cl(\{x\})$ which is a Micro open set containing A. This is impossible, since $x \in Mker(A)$. Consequently, $Mic\text{-}cl(\{x\}) \cap A \neq \phi$. Next, let $x \in U$ such that $Mic\text{-}cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin Mker(A)$. Then, there exists a Micro open set K containing A and $x \notin K$. Let $y \in Mic\text{-}cl(\{x\}) \cap A$. Hence, K is a Micro open set containing y which does not contain x. By this contradiction $x \in Mker(A)$. $\square$

Theorem 26. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties hold for the subsets A, B of U:*

1. $A \subseteq Mker(A)$.

2. $A \subseteq B$ implies that $Mker(A) \subseteq Mker(B)$.

3. *If A is Micro open in U, then $A = Mker(A)$.*

4. $Mker(Mker(A)) = Mker(A)$.

Proof. (1), (2) and (3) are immediate consequences of Definition 23. To prove (4), first observe that by (1) and (2), we have $Mker(A) \subseteq Mker(Mker(A))$. If $x \notin Mker(A)$, then there exists $L \in \mu_R(X)$ such that $A \subseteq L$ and $x \notin L$. Hence, $Mker(A) \subseteq L$, and so we have $x \notin Mker(Mker(A))$. Thus $Mker(Mker(A)) = Mker(A)$. $\square$

Theorem 27. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, for any points x and y in U the following statements are equivalent:*

1. $Mker(\{x\}) \neq Mker(\{y\})$.

2. $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$.

Proof. $(1) \Rightarrow (2)$: Suppose that $Mker(\{x\}) \neq Mker(\{y\})$, then there exists a point z in U such that $z \in Mker(\{x\})$ and $z \notin Mker(\{y\})$. From $z \in Mker(\{x\})$ it follows that $\{x\} \cap Mic\text{-}cl(\{z\}) \neq \phi$ which implies $x \in Mic\text{-}cl(\{z\})$. By $z \notin Mker(\{y\})$, we have $\{y\} \cap Mic\text{-}cl(\{z\}) = \phi$. Since $x \in Mic\text{-}cl(\{z\})$, then $Mic\text{-}cl(\{x\}) \subseteq Mic\text{-}cl(\{z\})$ and $\{y\} \cap Mic\text{-}cl(\{x\}) = \phi$. Therefore, it follows that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Thus, $Mker(\{x\}) \neq Mker(\{y\})$ implies that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$.

$(2) \Rightarrow (1)$: Suppose that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Then, there exists a point z in U such that $z \in Mic\text{-}cl(\{x\})$ and $z \notin Mic\text{-}cl(\{y\})$. Then, there exists

a Micro open set containing z and therefore x but not y. Hence, $y \notin Mker(\{x\})$ and thus $Mker(\{x\}) \neq Mker(\{y\})$. □

Theorem 28. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, $\cap\{Mic\text{-}cl(\{x\}) : x \in U\} = \phi$ if and only if $Mker(\{x\}) \neq U$ for every $x \in U$.*

Proof. Necessity. Suppose that $\cap\{Mic\text{-}cl(\{x\}) : x \in U\} = \phi$. Assume that there is a point y in U such that $Mker(\{y\}) = U$. Let x be any point of U. Then, $x \in K$ for every Micro open set K containing y and hence $y \in Mic\text{-}cl(\{x\})$ for any $x \in U$. This implies that $y \in \cap\{Mic\text{-}cl(\{x\}) : x \in U\}$. But this is a contradiction.

Sufficiency. Assume that $Mker(\{x\}) \neq U$ for every $x \in U$. If there exists a point y in U such that $y \in \cap\{Mic\text{-}cl(\{x\}) : x \in U\}$, then every Micro open set containing y must contain every point of U. This implies that the space U is the unique Micro open set containing y. Hence, $Mker(\{y\}) = U$ which is a contradiction. Therefore, $\cap\{Mic\text{-}cl(\{x\}) : x \in U\} = \phi$. □

Theorem 29. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and U be Micro $T_{\frac{1}{2}}$. If $Mker(\{x\}) \neq U$ for a point $x \in U$, then $\{x\}$ is a MD-set in U.*

Proof. Let $Mker(\{x\}) \neq U$ for a point $x \in U$, then there exists a subset $L \in \mu_R(X)$ such that $\{x\} \subseteq L$ and $L \neq U$. Using Theorem 17, for the point x, we have $\{x\}$ is Micro open or Micro closed in U. When the singleton $\{x\}$ is Micro open, then $\{x\}$ is a MD-set in U. When the singleton $\{x\}$ is Micro closed, then $U \setminus \{x\}$ is Micro open in U. Put $L_1 = L$ and $L_2 = U \cap (U \setminus \{x\})$. Then, $\{x\} = L_1 \setminus L_2$ and $L_1 \neq U$. Thus, $\{x\}$ is a MD-set. □

Theorem 30. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If a singleton $\{x\}$ is a MD-set in U, then $Mker(\{x\}) \neq U$.*

Proof. Let $\{x\}$ be a MD-set in U, then there exist two subsets $L_1, L_2 \in \mu_R(X)$ such that $\{x\} = L_1 \setminus L_2$, $\{x\} \subseteq L_1$ and $L_1 \neq U$. Thus, we have that $Mker(\{x\}) \subseteq L_1 \neq U$ and so $Mker(\{x\}) \neq U$. □

Definition 31. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be:

1. Micro D_0 if for any pair of distinct points x and y of U there exists a MD-set of U containing x but not y or a MD-set of U containing y but

not x.

2. Micro D_1 if for any pair of distinct points x and y of U there exists a MD-set of U containing x but not y and a MD-set of U containing y but not x.

3. Micro D_2 if for any pair of distinct points x and y of U there exist disjoint MD-set G and E of U containing x and y, respectively.

Remark 4. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro D_k, then it is Micro D_{k-1}, for $k = 1, 2$.

Theorem 32. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro D_1 if and only if it is Micro D_2.

Proof. Necessity. Let $x, y \in U$, $x \neq y$. Then, there exist MD-sets G_1, G_2 in U such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = L_1 \setminus L_2$ and $G_2 = L_3 \setminus L_4$, where L_1, L_2, L_3 and L_4 are Micro open sets in U. From $x \notin G_2$, it follows that either $x \notin L_3$ or $x \in L_3$ and $x \in L_4$. We discuss the two cases separately.

(i) $x \notin L_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin L_1$. Since $x \in L_1 \setminus L_2$, it follows that $x \in L_1 \setminus (L_2 \cup L_3)$, and since $y \in L_3 \setminus L_4$ we have $y \in L_3 \setminus (L_1 \cup L_4)$. Therefore, $(L_1 \setminus (L_2 \cup L_3)) \cap (L_3 \setminus (L_1 \cup L_4)) = \phi$.

(b) $y \in L_1$ and $y \in L_2$. We have $x \in L_1 \setminus L_2$, and $y \in L_2$. Therefore, $(L_1 \setminus L_2) \cap L_2 = \phi$.

(ii) $x \in L_3$ and $x \in L_4$. We have $y \in L_3 \setminus L_4$ and $x \in L_4$. Hence, $(L_3 \setminus L_4) \cap L_4 = \phi$. Therefore, U is Micro D_2.

Sufficiency. Follows from Remark 4. □

Example 7. Consider $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}\}$. If $\mu = \{a, b\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{a, b\}\}$. Then, U is Micro D_0 but not Micro D_1 because there is no MD-set containing c but not b.

Definition 33. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and $x \in U$. Then, a subset N of U is said to be Micro neighbourhood of x, if there exists a Micro open set L in U such that $x \in L \subseteq N$.

Theorem 34. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then,

a subset A of U is Micro open if and only if it is a Micro neighbourhood of each of its points.

Proof. Let $A \subseteq U$ be a Micro open set, since for every $x \in A$, $x \in A \subseteq A$ and A is Micro open. This shows A is a Micro neighbourhood of each of its points.

Conversely, suppose that A is a Micro neighbourhood of each of its points. Then, for each $x \in A$, there exists $B_x \in \mu_R(X)$ such that $B_x \subseteq A$. Then, $A = \cup\{B_x : x \in A\}$. Since each B_x is Micro open. It follows that A is Micro open set. □

Remark 5. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Let A and B be any two subsets of U and $A \subseteq B$. If A is a Micro neighbourhood of a point $x \in U$, then B is also Micro neighbourhood of the same point x.

Definition 35. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. A point $x \in U$ which has only U as the Micro neighbourhood is called a Micro neat point.

Theorem 36. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro D_1, then U has no Micro neat point.

Proof. Let U be Micro D_1, then each point x of U is contained in a MD-set $A = L \setminus K$ and thus in U. By definition $L \neq U$. This implies that x is not a Micro neat point. □

Theorem 37. Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and U be a Micro $T_{\frac{1}{2}}$ with at least two points. Then, U is Micro D_1 if and only if $Mker(\{x\}) \neq U$ holds for every point $x \in U$.

Proof. Necessity. Let $x \in U$. For a point $y \neq x$, there exists a MD-set L such that $x \in L$ and $y \notin L$. Say $L = L_1 \setminus L_2$, where $L_i \in \mu_R(X)$ for each $i \in \{1, 2\}$ and $L_1 \neq U$. Thus, for the point x, we have a Micro open set L_1 such that $\{x\} \subseteq L_1$ and $L_1 \neq U$. Hence, $Mker(\{x\}) \neq U$.

Sufficiency. Let x and y be a pair of distinct points of U. We prove that there exist MD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Theorem 17, we can take the subsets A and B for the following four cases for two points x and y.

Case 1. $\{x\}$ is Micro open and $\{y\}$ is Micro closed in U. Since $Mker(\{y\}) \neq U$, then there exists a Micro open set K such that $y \in K$ and $K \neq U$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = K \setminus (U \setminus \{y\})$, then K is a Micro open set with $K \neq U$ and $U \setminus \{y\}$ is Micro open, and B is a required MD-set containing y such that $x \notin B$. Obviously, A is a required MD-set containing x such that $y \notin A$.

Case 2. $\{x\}$ is Micro closed and $\{y\}$ is Micro open in U. The proof is similar to Case 1.

Case 3. $\{x\}$ and $\{y\}$ are Micro open in U. Put $A = \{x\}$ and $B = \{y\}$.

Case 4. $\{x\}$ and $\{y\}$ are Micro closed in U. Put $A = U \setminus \{y\}$ and $B = U \setminus \{x\}$.

For each case of the above, the subsets A and B are the required MD-sets. Therefore, U is Micro D_1.   □

Proposition 1. *Let $f : U \to V$ be Micro-continuous bijective. If V is Micro D_1, then U is Micro D_1.—*

Proof. Suppose that V is a Micro D_1. Let x and y be any pair of distinct points in U. Since f is injective and V is Micro D_1, then there exist MD-sets O_x and O_y in V containing $f(x)$ and $f(y)$ respectively, such that $f(x) \notin O_y$ and $f(y) \notin O_x$. By Theorem 22, $f^{-1}(O_x)$ and $f^{-1}(O_y)$ are MD-sets in U containing x and y, respectively, such that $x \notin f^{-1}(O_y)$ and $y \notin f^{-1}(O_x)$. This implies that U is Micro D_1.   □

Proposition 2. *If for each pair of distinct points $x, y \in U$, there exists a Micro-continuous surjective function $f : U \to V$, where V is Micro D_1 such that $f(x)$ and $f(y)$ are distinct, then U is Micro D_1.*

Proof. Let x and y be any pair of distinct points in U. By hypothesis, there exists a Micro-continuous, surjective function f of a space U onto a Micro D_1 space V such that $f(x) \neq f(y)$. It follows from Theorem 32 that Micro D_1 = Micro D_2. Hence, there exist disjoint MD-sets O_x and O_y in V such that $f(x) \in O_x$ and $f(y) \in O_y$. Since f is Micro-continuous and surjective, by Theorem 22, $f^{-1}(O_x)$ and $f^{-1}(O_y)$ are disjoint MD-sets in U containing x and y, respectively. So, the space U is Micro D_1.   □

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