

EXTREMUM PROBLEM IN CONVOLUTIONS WITH ADDITIONAL CONDITIONS ON THE AXIS

Yurii A. Hryhoriev¹§, Andrii Yu. Grygoriev²

^{1,2}Odessa National Maritime University

Odessa - 65029, UKRAINE

Abstract: The research paper deals with a partially overspecified problem with a Noetherian operator in a complex Hilbert space. On the basis of the performed analysis, it was found that the solution of the extremum problem in convolutions in a complex Hilbert space with an additional condition on the axis is urgent, since no complete solution has been provided so far. In this connection, the necessary and sufficient conditions for the solvability of the posed extremum problem are determined, which are obtained by the factorization method and are formulated in terms of the basis of the kernel of the adjoint operator. As a result, it is illustrated as the example that the Wiener-Hopf equation with the sought and given space functions is solvable at an arbitrarily positive interval. In case of a negative operator index, the determination problem is solvable in quadratures and has a unique solution. As a result of formulation and solution of an extremum problem in convolutions with an additional condition on the axis, within the scope of complex Hilbert spaces, its solution is obtained in solvable in quadratures.

AMS Subject Classification: 45E10, 47Axx

Key Words: extremum problem; Noetherian operator; complex Hilbert space; adjoint operator; Wiener-Hopf equation

1. Introduction

The research objective is to formulate and solve an extremum problem in convolutions with an additional condition on the axis.

Received: December 27, 2019

© 2020 Academic Publications

§Correspondence author

The research topicality is determined by the fact that there is no solution to the extremum problem in convolutions with an additional condition on the axis within the scope of complex Hilbert spaces.

The reason for the study is the lack of solutions to the extremum problem within the scope of complex Hilbert spaces in contractions with an additional condition on the axis.

The scientific value of the study is the formulation of the extremum problem in convolutions with an additional condition on the axis, within the scope of complex Hilbert spaces and its solution obtained for the first time. Necessary and sufficient conditions for its solvability are obtained by the factorization method and formulated in terms of the basis of the kernel of the adjoint operator.

As a result of the formulation of the extremum problem in convolutions with an additional condition on the axis, within the scope of complex Hilbert spaces we obtain its solution, which will open up the opportunity for its use in an unlimited range.

2. Theoretical background

In [4], for general curves of twistor spaces, it was shown that the Hilbert scheme of real cohomologically stable curves of a fixed genus and degree P^3 that do not intersect a fixed real line carries a natural pseudo-hyperkähler structure. When the real line is removed from P^3 , the result is unsatisfactory; therefore, in [3] the results are given for the case of describing the differential geometry of real and stable projective spatial curves with a fixed genus and degree. It also indicates the possibility of opening subsets with non-trivial geometry only for a very limited range.

In [25], finite-dimensional subsets of any separable Hilbert space described, for which the concept of C-hypercyclicity coincides with the concept of hypercyclicity, where the operator of the set T , which has a dense distribution in a topological vector space X , is hypercyclic.

If we consider a linear mapping within the scope of Hilbert spaces [6, 14], its pattern space will form the natural reproducing core of the Hilbert space, which is a functional space.

For any positively definite quadratic function $K(p, q)$ on the set E , there is a uniquely defined reproducing kernel of a Hilbert space admitting the kernel $K(p, q)$. In [23], the relations generated between positively definite functions of a quadratic form are considered. The problems caused by the constraint,

sum and product of positively definite functions of a quadratic form are also considered.

Really reproducing kernels are used as input physical data and observations of outputs [17] and calculate the reciprocal values of linear systems [2].

From this relation, we see that the square of the Szeg kernel is an analytic differential, it extends to a double according to Danalite, which is a closed Riemann surface, as in the Bergman kernel. In this sense, the Szeg kernel is a half-differential on a closed Riemann surface, therefore its properties are very important for many-valuedness on a closed Riemann surface.

The properties of Szeg kernels consist in the existence of a half differential on a closed Riemann surface [19, 8]. In this case, Szeg kernels are Cauchy-type kernels on a Riemann surface, which is important for the reproducing kernel.

In this section, we present some general applications of the general theory of reproducing kernels in Section 3, (1): Integral Transformations ([48]).

Using the basic theorems of reproducing kernels, many particular results can be obtained: for typical integral transformations [26], with the formulation and solution of inverse problems [18], integral transformations for some smooth functions [16], identification of nonlinear transformations on Hilbert spaces [20, 27] and much more.

In [24, 22], stationary properties were studied under optimal control in Hilbert spaces, which are mainly due to the hyperbolic property of the Hamiltonian system, which follows from the Pontryagin's maximum principle. It was shown that for a long time the optimal state, the control and the adjoint vector remain close to the optimally robust most of the time. The results were obtained on the basis of a dichotomous transformation, which is based on solutions of the algebraic equations of Riccati and Lyapunov.

Studies using the Wiener-Hopf equation are given in [13, 21], which are the starting points for this work. At the same time, the possibility of establishing a connection between the integral equation theory and the calculus of variations with the variational derivative of a quadratic functional is noted.

The Lagrange multiplier method is applied in physics, economics, mathematics, mechanics, engineering, etc. [15, 7]. In economics and technology, there are many problems that require optimization of a certain volume of data under certain constraints, and therefore the Lagrange multiplier method is applied. The Lagrange multiplier method will also be applied in this paper.

Let us also mention earlier works, which deal with solving the extremum problem in convolutions in a Hilbert space [12, 5, 1, 11, 9, 10].

Summing up, it can be argued that the solution of the extremum problem in convolutions in a Hilbert space with an additional condition on the axis is

topical, since the complete solution has not yet been provided [4, 25, 6].

3. Results and discussion

Let a complex Hilbert space be represented as the orthogonal sum of two half-spaces: $H = H_+ \oplus H_-$; P_{\pm} – orthogonal projectors operating from H in H_{\pm} . In case when, for a given element $g \in H$ and for a given linear operator $K : H \rightarrow H$, the equation

$$Ku = g$$

is unsolvable, we can consider the problem of determining $u \in H$ by conditions

$$\begin{cases} P_+Ku = P_+g, \\ \|P_-(Ku - g)\|^2 \rightarrow \min, \end{cases}$$

[12], or – an extremum problem of a more general form:

$$P_+Ku = P_+g, \quad (1)$$

$$Re(AP_-(Ku - g), BP_-(Ku - f)) \rightarrow \min \quad (2)$$

[5], where f is the element given in H , A and B are bounded linear operators acting from H in H .

Necessary and sufficient conditions for the solvability of these extremum problems are formulated in [12, 5]. In the particular case considered below for the problem (1), (2), we apply the factorization method and these conditions take on a simpler form.

Theorem 1. *If the operator $D = A * B + B * A$ is positively definite, that is, if $(Dh, h) \geq \gamma \|h\|^2$ for any $h \in H$ and some $\gamma > 0$, and the operator $K \in L(H)$ is Noetherian, the satisfaction of the following condition is necessary and sufficient for the solvability of the problem (1), (2):*

$$\{w_- \in H_- : (\psi_j, w_-) = -(\psi_j, P_+g), \quad j = 1, 2, \dots, m\} \neq \emptyset, \quad (3)$$

where ψ_1, \dots, ψ_m – kernel basis of the operator K^* . The solution of the problem is not unique and contains $\dim \text{Ker } K$ of arbitrary complex constants.

Proof. Let us denote through $\mathfrak{M} \subset H_-$ the set of all elements $w_- \in \mathfrak{M}$ for which the problem of finding $u \in H$ is solvable by the conditions

$$\begin{cases} P_+Ku = P_+g, \\ P_-Ku = w_-, \end{cases}$$

or

$$Ku = P_+Ku + P_-Ku = P_+g + w_-. \quad (4)$$

Due to the Noetherian property of the operator K

$$w_- \in \mathfrak{M} \Leftrightarrow (\psi_j, w_-) = -(\psi_j, P_+g), \quad j = 1, \dots, m.$$

According to (4), the problem (1), (2) reduces to solving two problems: a solvable Noetherian equation (4) and an extremum problem of determination of $w_- \in H_-$ by the conditions:

$$Re(A(w_- - P_-g), B(w_- - P_-f)) \rightarrow \min, \quad (5)$$

$$(\psi_j, w_-) = -(\psi_j, P_+g), \quad j = 1, 2, \dots, n. \quad (6)$$

Without loss of generality, we assume that the elements $\psi_{j-} = P_- \psi_j$, $j = 1, \dots, n$; $n \leq m$ are linearly independent. Indeed, if we assume, for example, that the element ψ_{1-} is a linear combination of the others, the first condition (6) follows from the remaining conditions or contradicts them. In the first case, it can be excluded from (6), and in the second – the conditions (6) are incompatible.

Let us assume that the condition (3) is not satisfied. Then the conditions (6) are incompatible and the extremum problems (5), (6) and (1), (2) are unsolvable. If the condition (3) is satisfied, we show that the extremum problem (5), (6) has a unique solution \hat{w}_- . This will mean that the original problem (1), (2) is equivalent to equation (4) for $w_- = \hat{w}_-$. The solution of the equation (4) (and hence the problem (1), (2)) is not unique and contains $\dim Ker K$ of arbitrary complex constants. \square

Proof. Let us solve the extremum problem (5), (6) under the assumption that the elements $\psi_{1-}, \dots, \psi_{n-}$ are linearly independent. The method of Lagrange multipliers (see, for example, [1]) leads to the equation

$$Dw_- - A^*BP_-f - B^*AP_-g + \sum_{j=1}^n \lambda_j \psi_{j-} = w_+ \quad (7)$$

with the sought elements $w_{\pm} \in H_{\pm}$ and unknown Lagrange multipliers $\lambda_1, \dots, \lambda_n$.

Let us denote through

$$h = A^*BP_-f + B^*AP_-g,$$

and represent the operator D as

$$D = X_+^{-1}X_-,$$

where $X_+^{-1} = X_-^*$. The introduced operators $X_+, X_- \in L(H)$ one-to-one reflect half-spaces H_+ and H_- , respectively (such a decomposition of a positively definite operator was also used in [11]). From (7) we find

$$w_- = X_-^{-1}P_-X_+h - \sum_{j=1}^n \lambda_j X_-^{-1}P_-X_+\psi_{j-}. \quad (8)$$

Substituting (8) into (6), we arrive at a linear system of algebraic equations

$$\sum_{j=1}^n \lambda_j (P_-X_+\psi_{l-}, P_-X_+\psi_{j-}) = (X_+\psi_{l-}, P_-X_+h) + (\psi_{l-}, P_+g), \quad l = 1, \dots, n \quad (9)$$

to determine the Lagrange multipliers. Let us note that the main determinant of this system is the Gram determinant, composed of elements $P_-X_+\psi_{j-}$, $j = 1, \dots, n$. We prove the linear independence of these elements from the contrary. Let us assume that not all zero complex numbers α_j are such that

$$\sum_{j=1}^n \alpha_j P_-X_+\psi_{j-} = 0.$$

Hence,

$$\sum_{j=1}^n \alpha_j P_-X_+^{-1}P_-X_+\psi_{j-} = 0. \quad (10)$$

For any $h \in H$ we will have

$$\begin{aligned} (P_-X_+^{-1}P_-X_+\psi_{j-}, h) &= (X_+^{-1}P_-X_+\psi_{j-}, P_-h) \\ &= (P_-X_+\psi_{j-}, X_+^{-1*}P_-h) \\ &= (X_+\psi_{j-}, X_+^{-1*}P_-h) = (\psi_{j-}, P_-h) = (\psi_{j-}, h). \end{aligned}$$

Consequently,

$$P_-X_+^{-1}P_-X_+\psi_{j-} = \psi_{j-}.$$

Now, a linear dependence of the elements $\psi_{1-}, \dots, \psi_{n-}$ follows from (10), which contradicts our assumption.

Thus, the system of equations (9) has a unique solution $\hat{\lambda}_1, \dots, \hat{\lambda}_n$, since its determinant is the Gram determinant, composed of linearly independent elements. Formula (8) for $\lambda_j = \hat{\lambda}_j$ gives a unique solution \hat{w}_- to the extremum problem (5), (6). We will find all solutions to the original problem (1), (2) by solving equation (4) for $w_- = \hat{w}_-$. The theorem is proved. \square

Example 1. Find the function $u(x) \in L_2(0, \infty)$ by the conditions

$$(Wu)(x) = \int_0^\infty k(x-s)u(s)ds = g(x), \quad x \in R_+ \setminus \mu, \quad (11)$$

$$\int_\mu \rho(x) \left| \int_0^\infty k(x-s)u(s)ds - g(x) \right|^2 dx \rightarrow \min. \quad (12)$$

All other values are given: μ - a set of positive measure given on the right semi-axis $R_+ = [0, \infty)$; $g \in L_2(R_+)$; the generalized function $k(x)$ has a continuous, bounded and non-zero Fourier transform on the closed axis \dot{R}

$$K(x) = (Fk)(x) = \int_{-\infty}^\infty k(t)e^{itx}dt;$$

the weight function $\rho(x)$ is bounded above and below by positive constants.

Let us note that the operator W is Noetherian, its index is

$$\chi = -\text{ind}K(x) = \frac{-1}{2\pi} [\arg K(x)]_{-\infty}^\infty$$

(see, for example [9]). For $\chi \geq 0$, the Wiener-Hopf equation

$$(Wu)(x) = g(x), \quad x \in R_+ \quad (11')$$

is solvable; if $\chi > 0$, the solution is not unique and is expressed through χ of arbitrary complex constants. For $\chi < 0$ the last (11') equation is solvable if and only if its right side is orthogonal to the basis of the kernel of the operator W^* :

$$\psi_j(x) = F^{-1}\Psi_j(x) = F^{-1} \frac{\overline{X^-(x)}}{(x+i)^j}, \quad j = 1, \dots, |\chi|. \quad (13)$$

Here, the inverse Fourier transform operator is denoted through F^{-1} ,

$$X^-(x) = \exp \left\{ -(x+i) P^- \left(\frac{1}{x+i} \ln \left[\left(\frac{x-i}{x+i} \right)^x K(x) \right] \right) \right\},$$

$$(P^\pm V)(x) = \frac{1}{2}V(x) \pm \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{V(s)}{s-x} ds, \quad V(x) \in L_2(R).$$

If the equation (11') is solvable, the extremum problem (11), (12) becomes trivial and equivalent to this equation. If the equation is unsolvable, it follows from Theorem 1 for

$$H = L_2(R_+), \quad H_+ = \{v(x) \in L_2(R_+) : x \in \mu \Rightarrow v(x) = 0\},$$

$$K = W, \quad Av(x) = \sqrt{\rho(x)/2}v(x), \quad A = B, g = f.$$

Theorem 2. *Let the characteristic function of the set μ be $\chi_\mu(x)$. If the system of functions*

$$\{\chi_\mu(x) \psi_j(x)\}, \quad j = 1, \dots, |\chi| \quad (14)$$

is linearly independent, the extremum problem (11), (12) is solvable. The only solution to the problem can be found from the Wiener-Hopf equation

$$(Wu)(x) = \chi_{R_+ \setminus \mu}(x) g(x) + w_-(x), \quad x \in R_+,$$

where

$$w_-(x) = \left(g(x) - \frac{1}{\rho(x)} \sum_{j=1}^{|\chi|} \lambda_j \psi_j(x) \right) \chi_\mu(x),$$

the coefficients λ_j , $j = 1, \dots, |\chi|$ are uniquely determined from the system of equations

$$\sum_{j=1}^{|\chi|} \lambda_j \int_\mu \frac{1}{\rho(x)} \overline{\psi_l(x)} \psi_j(x) dx = \int_0^\infty \overline{\psi_l(x)} g(x) dx, \quad l = 1, \dots, |\chi|.$$

Let us give an example showing that the condition of Theorem 2 is essential, although the system of functions (13) is linearly independent. Let the following be

$$\overline{X^-(x)} = \frac{(x+i)(1+ix-e^{ix})}{x^2}.$$

Hence,

$$\psi_1(x) = F^{-1} \frac{1+ix-e^{ix}}{x^2} = \begin{cases} 1-x, & x \in [0, 1], \\ 0, & x \notin [0, 1], \end{cases}$$

and, therefore, system (14) is linearly dependent for any $\mu \subset [1, \infty)$.

In conclusion, let us indicate two particular cases, where the condition of Theorem 2 is certainly satisfied.

Lemma 1. *If $K(x)$ is a rational function, the system of functions is linearly independent on any set $\mu \subset R_+$ ($\text{mes } \mu > 0$).*

Proof. Suppose that for some $A_j \in C$, $j = 1, \dots, |\chi|$, the following equality is the case:

$$0 = \chi_\mu(x) \sum_{j=1}^{|\chi|} c_j \psi_j(x) = \chi_\mu(x) F^{-1} \sum_{j=1}^{|\chi|} \overline{X^-(x)} \frac{c_j}{(x+i)^j}. \quad (15)$$

Let us take advantage of the fact that $\overline{X^-(x)}$ is a rational function analytically continuing to the upper half-plane, the degrees of the polynomials of its numerator and denominator are equal. Representing the right rational function

$$\sum_{j=1}^{|\chi|} \overline{X^-(x)} \frac{c_j}{(x+i)^j}$$

in the form of a sum of simple fractions and using the well-known Fourier transform formulas

$$F^{-1} \frac{1}{(x+\alpha i)} = \frac{x^{n-1} e^{-\alpha x}}{i^n (n-1)!} \chi_{R_+}(x),$$

from (15) we obtain

$$0 = \chi_\mu(x) F^{-1} \sum_{j=1}^{|\chi|} \overline{X^-(x)} \frac{c_j}{(x+i)^j} = \chi_\mu(x) \sum_{k=1}^m e^{-\alpha_k x} P_k(x),$$

where α_k – pairwise unequal positive numbers, $P_k(x)$ – some polynomials $k = 1, \dots, m$. Since μ is a set of positive measure,

$$\sum_{k=1}^m e^{-\alpha_k x} P_k(x) \equiv 0 \quad \Rightarrow \quad \sum_{j=1}^{|\chi|} \overline{X^-(x)} \frac{c_j}{(x+i)^j} \equiv 0.$$

It follows that

$$c_1 = \dots = c_{|\chi|} = 0. \quad (16)$$

We proved that equality (15) is satisfied only under condition (16), which was to be proved. \square

Lemma 2. *If $\mu = [0, \varepsilon]$, the system of functions (14) is linearly independent for any $\varepsilon > 0$.*

Proof. Let us assume the contrary that there are not all zero complex numbers $c_1, \dots, c_{|\chi|}$ such that

$$\chi_\mu(x) \sum_{j=1}^{|\chi|} c_j \psi_j(x) = 0.$$

Then

$$\sum_{j=1}^{|\chi|} c_j \psi_j(x) = \varphi(x), \quad -\infty < x < \infty, \quad (17)$$

where $\varphi(x) \in L_2(R)$, $\text{supp } \varphi(x) \subset [\varepsilon, \infty)$. Let us divide the Fourier transform of equation (17) divided by $\overline{X^-(x)}$. Returning to the original, we obtain

$$\sum_{j=1}^{|\chi|} \frac{c_j x^{j-1} e^{-x}}{i^j (j-1)!} \chi_{R_+}(x) = \int_{\varepsilon}^x r(x-s) \varphi(s) ds \quad (18)$$

where $r(x) = F^{-1} \left(\overline{X^-(x)} \right)^{-1}$ is a generalized function, $\text{supp } r(x) \subset R_+$. It follows from (18) that

$$\sum_{j=1}^{|\chi|} \frac{c_j x^{j-1} e^{-x}}{i^j (j-1)!} = 0$$

for $x \in [0, \varepsilon]$, that is $c_1 = \dots = c_{|\chi|} = 0$, and this contradicts our assumption. The lemma is proved. \square

We note as a consequence that if $\mu = [0, \varepsilon]$, the extremum problem (11), (12) has a unique solution and is solvable in quadratures for any $\varepsilon > 0$. The statement is also true for $\varepsilon = \infty$. An analysis of the problem in this case is given in [10].

4. Conclusions

In this work, a partially overspecified problem with a Noetherian operator in a complex Hilbert space is posed and solved. Necessary and sufficient conditions for its solvability are obtained by the factorization method and formulated in terms of the basis of the kernel of the adjoint operator. As a result, it can be shown as the example that the Wiener-Hopf equation with the sought u and given g functions of the space $L_2(0, \infty)$, $k \in L(-\infty, \infty)$ is solvable for arbitrary $\varepsilon > 0$. In case of a negative index of the operator K , the problem of determination of u is solvable in quadratures and has a unique solution.

As a result of the formulation and solution of the extremum problem in convolutions with an additional condition on the axis, within the scope of complex Hilbert spaces, its solution is obtained in an unlimited range.

Acknowledgements

Special thanks for the statement of the extremum problem with convolution operators, as well as the discussion of the results to the supervisor, Dr. of Phys. and Math., Professor Chersky Yuri Iosifovich, who would have turned 90 on December 8, 2019.

References

- [1] V.M. Alekseev, V.M. Tikhomirov, S.V. Fomin, *Optimal Control*, Nauka, Moscow (1979).
- [2] A. Berline, C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer Academic Publishers, Boston/Dordrecht/London (2004).
- [3] R. Bielawski, Hyperkähler manifolds of curves in Twistor spaces, *SIGMA*, **10** (2014), Art. 033.
- [4] R. Bielawski, C. Peternell, Differential geometry of Hilbert schemes of curves in a projective space, *Complex Manifolds*, **6**, No 1 (2019), 335-347; doi: 10.1515/coma-2019-0018.
- [5] Yu.I. Chersky, Extremum problems for a Noetherian operator, In: *Research Methods for Differential and Integral Operators*, Naukova Dumka, Kiev (1989), 196-200.
- [6] G. Costakis, On a conjecture of D. Herrero concerning hypercyclic operators, *Comptes Rendus de l'Academie des Sciences. Ser. I. Mathematique*, **330**, No 3 (2000), 179-182.
- [7] S. Engelberg, Extrema and the method of Lagrange multipliers, In: *Random Signals and Noise*, CRC Press, Boca Raton (2007), 73-87; doi: 10.1201/b15871-6.
- [8] J.D. Fay, *Theta Functions on Riemann Surfaces*, Lecture Notes in Math., Vol. 352, Springer-Verlag, Berlin/Heidelberg/New York (1973).
- [9] F.D. Gakhov, Yu.I. Chersky, *Convolution Type Equations*, Nauka, Moscow (1978).

- [10] Yu.A. Grigoryev, An extremum problem with the Wiener-Hopf operator, *Izvestiya Vuzov. Maths.*, **10** (1989), 68-70.
- [11] Yu.A. Grigoryev, Some signs of the solvability of convolution type equations, *Differ. Equations*, **26** No 1 (1990), 178-179.
- [12] S.G. Kreyn, S.Ya. Lvin, Overdetermined and underdetermined equations in Hilbert spaces, *Izvestiya Vuzov. Maths.*, **9** (1987), 59-66.
- [13] M. Masujima, Wiener-Hopf method and Wiener-Hopf integral equation, In: *Applied Mathematical Methods in Theoretical Physics*, Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim (2005), 177-247.
- [14] A. Peris, Multi-hypercyclic operators are hypercyclic. *Mathematische Zeitschrift*, **236**, No 4 (2001), 779-786.
- [15] L. Piela, Lagrange multipliers method, In: *Ideas of Quantum Chemistry*, Elsevier, Oxford, UK (2014), e121-e125; doi:10.1016/b978-0-444-59436-5.00034-9.
- [16] S. Saitoh, Applications of the reproducing kernel theory to inverse problems, *Comm. Korean Math. Soc.*, **16** (2001), 371-383.
- [17] S. Saitoh, Nonlinear transforms and analyticity of functions, In: *Nonlinear Mathematical Analysis and Applications*, Hadronic Press, Palm Harbor (1998), 223-234.
- [18] S. Saitoh, One approach to some general integral transforms and its applications, *Integral Transforms and Special Functions*, **3** (1995), 49-84.
- [19] S. Saitoh, M. Yamada, Inversion formulas for a linear system determined by input and response relations, by using suitable function spaces, *Hokkaido University Technical Report Series in Mathematics*, **118** (2007), 18-21.
- [20] S. Saitoh, M. Yamamoto, Integral transforms involving smooth functions, In: *Reproducing Kernels and Their Applications*, Springer, Boston, MA (1999), 149-164.
- [21] P.A. Santos, B. Silbermann, Galerkin method for Wiener-Hopf operators with piecewise continuous symbol, *Integral Equations and Operator Theory*, **38** No 1 (2000), 66-80; doi: 10.1007/bf01192302.

- [22] E. Trlat, C. Zhang, E. Zuazua, Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces, *SIAM Journal on Control and Optimization*, **56**, No 2 (2018), 1222-1252.
- [23] G. Wahba, *Spline Models for Observational Data*, SIAM, Philadelphia (1990).
- [24] Q.-F. Wang, Optimal control for Klein-Gordon-Schrödinger quantum system in complex Hilbert space, In: *2018 37th Chinese Control Conference (CCC)*, IEEE (2018); doi:10.23919/chicc.2018.8483916.
- [25] N. Wilberth, M. Mpimbo, S. Kumar, Somewhere dense orbit that is not dense on a complex Hilbert space, *Concrete Operators*, **6**, No 1 (2019), 58-63; doi: 10.1515/conop-2019-0005.
- [26] A. Yamada, Fay's trisecant formula and Hardy H^2 reproducing kernels, In: *Reproducing Kernels and their Applications*, Springer, Boston, MA (1999), 165-188.
- [27] M. Yamada, S. Saitoh, Identification of non-linear systems, *J. Computational, Math. Optimization*, **4** (2008), 47-60.

