

**GLOBAL ATTRACTIVITY OF SOLUTIONS OF
NONLINEAR FUNCTIONAL INTEGRAL
EQUATIONS IN TWO VARIABLES**

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Abstract: The purpose of this paper is to establish a generalization of Darbo's fixed point theorem and some new results on the existence and global attractivity of solution of functional integral equations in two variables by using this fixed point theorem and a measure of noncompactness. An example illustrating the results is also given.

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1. Introduction

In this work we consider the functional integral equation in two variables

$$x(t, s) = f \left(t, s, x(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) dv dw \right) \quad (1)$$

for $t, s \in \mathbb{R}_+ = [0, \infty)$. We will apply a Darbo type fixed point theorem and employ a measure of noncompactness to prove the existence of solutions and to show that solutions are uniformly locally attractive in the sense to be defined below. As a corollary, we also obtain sufficient conditions for the global attractivity of solutions. The form of equation (1) is very general and it can be seen to include a number of previously studied problems as special cases; we mention the papers [1, 3, 7, 15] as just a few examples. The approach of using measures of compactness to study various properties such as boundedness, monotonicity, stability, existence, uniqueness, attractivity, and asymptotic behavior of solutions of integral equations is well known in the literature; for example, see [1, 3, 4, 6, 9, 11, 12, 13, 16, 18, 19, 21, 22, 23, 24, 25] and the references contained therein.

Let E_1 be a real Banach space with the norm $\| \cdot \|$ and let $B(a, r)$ be a closed ball in E_1 centered at a with radius r . If X is a nonempty subset of E_1 , then by \bar{X} and $\text{Conv}X$ we denote the closure and convex closure of X . We let \mathcal{M}_{E_1} denote the family of all nonempty and bounded subsets of E_1 and let \mathcal{N}_{E_1} be its subfamily consisting of all relatively compact sets. The following definition of a measure of noncompactness can be found in [8].

Definition 1. A function $\mu : \mathcal{M}_{E_1} \rightarrow \mathbb{R}_+$ is called a measure of noncompactness in E_1 if it satisfies the following conditions:

- (i) For all $Y \in \mathcal{M}_{E_1}$, if $\mu(Y) = 0$, then Y is precompact;
- (ii) The family $\ker \mu = \{Y \in \mathcal{M}_{E_1} : \mu(Y) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_{E_1}$;
- (iii) $Y \subseteq Z$ implies $\mu(Y) \leq \mu(Z)$;
- (iv) $\mu(\bar{Y}) = \mu(Y)$;
- (v) $\mu(\text{Conv}Y) = \mu(Y)$;
- (vi) $\mu(\lambda Y + (1 - \lambda)Z) \leq \lambda\mu(Y) + (1 - \lambda)\mu(Z)$ for $\lambda \in [0, 1]$;
- (vii) If $Y_n \in \mathcal{M}_{E_1}$, $Y_n = \bar{Y}_n$, $Y_{n+1} \subset Y_n$ for $n = 1, 2, 3, \dots$, and $\lim_{n \rightarrow \infty} \mu(Y_n) = 0$, then $Y_\infty = \bigcap_{n=1}^{\infty} Y_n \neq \phi$.

The family $\ker \mu$ is called the *kernel of the measure* μ . Observe that the set Y_∞ in (vii) above is itself a member of the family $\ker \mu$. In fact, since $\mu(Y_\infty) \leq \mu(Y_n)$ for any n , we infer that $\mu(Y_\infty) = 0$. Thus, $Y_\infty \in \ker \mu$.

The Kuratowski measure of noncompactness for a bounded subset S of a metric space X is defined by (see [20])

$$\alpha(S) = \inf \left\{ \delta > 0 : S = \bigcup_{i=1}^n S_i, \text{diam}(S_i) \leq \delta, 1 \leq i \leq n, n \in \mathbb{N} \right\},$$

where $\text{diam}(S_i)$ denotes the diameter of the set S_i , that is,

$$\text{diam}(S_i) = \sup \{d(x, y) : x, y \in S_i\}.$$

The Hausdorff measure of noncompactness for a bounded set S is defined as

$$\chi(S) = \inf \{ \epsilon > 0 : S \text{ has a finite } \epsilon\text{-net in } X \}.$$

We recall the following important definitions and theorems.

Theorem 2. (Schauder [2]) *Let D be a nonempty, closed, and convex subset of a Banach space E . Then every compact, continuous map $T : D \rightarrow D$ has at least one fixed point.*

Theorem 3. (Darbo [10]) *Let Z be a nonempty, bounded, closed, and convex subset of a Banach space E and let $S : Z \rightarrow Z$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that*

$$\mu(SM) \leq k\mu(M), \quad M \subseteq Z.$$

Then S has a fixed point.

Definition 4. ([14]) Let \mathcal{F} be the class of all functions $H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (1) $\max\{x, y\} \leq H(x, y), \quad x, y \geq 0,$
- (2) H is continuous and nondecreasing,
- (3) $H(x + y, p, q) \leq H(x, p) + H(y, q).$

As an example of this definition, take $H(x, y) = x + y$.

Definition 5. ([17]) Let \mathcal{R} denote the class of functions $\beta : \mathbb{R}_+ \rightarrow [0, 1)$ satisfying $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 6. ([5]) *Let C be a nonempty, bounded, closed and convex subset of a Banach space E . Suppose $T : C \rightarrow C$ is continuous function such that for any $X \subseteq C$,*

$$\mu(TX) \leq \beta(\mu(X))\mu(X),$$

where μ is an arbitrary measure of noncompactness and $\beta \in \mathcal{K}$. Then T has at least one fixed point in C .

Remark 7. In [5], by taking $\beta(t) = k$ with $0 \leq k < 1$ for each $t \geq 0$ in Theorem 6, Darbo's fixed point theorem is obtained. But if we take $k = \frac{1}{2} \in [0, 1)$, then it is obvious that β does not satisfy the condition $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ as $n \rightarrow \infty$ in Definition 5.

2. Fixed point theorem

In this section we establish a fixed point theorem that is a generalization of Theorems 2 and 3 above.

Theorem 8. *Let E be a nonempty, closed, convex, and bounded subset of a Banach space X and let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $L : E \rightarrow E$ be continuous functions. Assume that for all $D \subseteq E$,*

$$H(\mu(LD), \psi(\mu(LD))) \leq \beta(\mu(D))H(\mu(D), \psi(\mu(D))),$$

where μ is an arbitrary measure of noncompactness, $\beta \in \mathcal{K}$, and $H \in \mathcal{F}$. Then L has at least one fixed point in D .

Proof. Let $E_0 = E$ and construct the sequence $\{E_n\}$ such that $E_{n+1} = \text{Conv}(LE_n)$ for $n \geq 0$. Now, $LE_0 = LE \subseteq E = E_0$ and $E_1 = \text{Conv}(LE_0) \subseteq E = E_0$. Continuing this process, we have $E_0 \supseteq E_1 \supseteq \dots \supseteq E_n \supseteq E_{n+1} \supseteq \dots$

If we can find $m \in \mathbb{N}$ such that $H(\mu(E_m), \psi(\mu(E_m))) = 0$, which in turn would give $\mu(E_m) = 0$, then we can conclude that E_m is relatively compact. Moreover, since $L(E_m) \subseteq \text{Conv}(LE_m) = E_{m+1} \subseteq E_m$, by Schauder's fixed point theorem L has a fixed point.

Let $0 < H(\mu(E_n), \psi(\mu(E_n)))$ for all $n \geq 1$. Then,

$$\begin{aligned} H(\mu(E_{n+1}), \psi(\mu(E_{n+1}))) &= H(\mu(\text{Conv}(LE_n)), \psi(\mu(\text{Conv}(LE_n)))) \\ &= H(\mu(LE_n), \psi(\mu(LE_n))) \end{aligned}$$

$$\begin{aligned} &\leq \beta(\mu(E_n))H(\mu(E_n), \psi(\mu(E_n))) \\ &< H(\mu(E_n), \psi(\mu(E_n))). \end{aligned}$$

Thus, for all $n \in \mathbb{N}$ we have

$$H(\mu(E_{n+1}), \psi(\mu(E_{n+1}))) < H(\mu(E_n), \psi(\mu(E_n))),$$

i.e., the sequence $\{H(\mu(E_n), \psi(\mu(E_n)))\}$ is non-negative and non-increasing. Therefore, there exists $h \geq 0$ such that

$$\lim_{n \rightarrow \infty} H(\mu(E_n), \psi(\mu(E_n))) = h.$$

If $h > 0$,

$$\frac{H(\mu(E_{n+1}), \psi(\mu(E_{n+1})))}{H(\mu(E_n), \psi(\mu(E_n)))} \leq \beta(\mu(E_n)).$$

As $n \rightarrow \infty$, we obtain $\beta(\mu(E_n)) \geq 1$, which is a contradiction to the fact that $\beta < 1$. Hence, $h = 0$ and $\lim_{n \rightarrow \infty} H(\mu(E_n), \psi(\mu(E_n))) = 0$, so $\lim_{n \rightarrow \infty} \mu(E_n) = 0$.

Since $E_n \supseteq E_{n+1}$ and $LE_n \subseteq E_n$ for all $n = 1, 2, \dots$, in view of Definition 1, we conclude that $E_\infty = \bigcap_{n=1}^\infty E_n$ is a nonempty, closed, and convex set and is invariant under L . Therefore, by Schauder's fixed point theorem, L has a fixed point in $E_\infty \subset E$. This completes the proof of the theorem. □

3. An application

In what follows, we will work in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ consisting of the set of bounded and continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+$. The space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ is equipped with the norm

$$\|x\| = \sup\{|x(t, s)| : t, s \geq 0\} \quad \text{for } x \in BC(\mathbb{R}_+ \times \mathbb{R}_+).$$

A measure of noncompactness μ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ is defined by Das et al. [13] (also see Arab et al. [6]) as follows.

Let X be a nonempty and bounded subset of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and for $x \in X$, $\epsilon > 0$, and $T > 0$, let

$$\begin{aligned} \omega^T(x, \epsilon) &= \sup\{|x(t, s) - x(u, v)| : t, s, u, v \in [0, T], \\ &\quad |t - u| \leq \epsilon, |s - v| \leq \epsilon\}; \\ \omega^T(X, \epsilon) &= \sup\{\omega^T(x, \epsilon) : x \in X\}; \end{aligned}$$

$$\begin{aligned}\omega_0^T(X) &= \lim_{\epsilon \rightarrow 0} \omega^T(X, \epsilon); \\ \omega_0(X) &= \lim_{T \rightarrow \infty} \omega_0^T(X); \\ X(t, s) &= \{x(t, s) : x \in X, t, s \in \mathbb{R}_+\}; \\ \mu(X) &= \omega_0(X) + \limsup_{t, s \rightarrow \infty} \text{diam} X(t, s);\end{aligned}$$

where

$$\text{diam} X(t, s) = \sup \{|x(t, s) - y(t, s)| : x, y \in X\}.$$

4. Solvability of equation (1)

Analogous to the definition of attractivity of solutions in $BC(\mathbb{R}_+)$ in [24], we need the following concepts in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$. Let G be an operator from $D_1 \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$ into $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and consider the solutions of the equation

$$(Gx)(t, s) = x(t, s). \quad (2)$$

Definition 9. Solutions of equation (2) are said to be locally attractive if there exists a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ such that, for arbitrary solutions $x = x(t, s)$ and $y = y(t, s)$ of equation (2) belonging to $B(x_0, r) \cap D_1$, we have

$$\lim_{t, s \rightarrow \infty} (x(t, s) - y(t, s)) = 0. \quad (3)$$

If the limit (3) is uniform with respect to $B(x_0, r) \cap D_1$, solutions of equation (2) are said to be uniformly locally attractive or that the solutions of (2) are asymptotically stable.

Definition 10. The solution $x = x(t, s)$ of equation (2) is said to be globally attractive if (3) holds for each solution $y = y(t, s)$ of (2). If condition (3) is uniformly satisfied with respect to the set D_1 , solutions of equation (2) are said to be globally asymptotically stable or uniformly globally attractive.

For the solvability of the functional integral equation (1), we will make use of the following conditions:

(H₁) The functions $a, b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous.

(H₂) The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist a nonnegative constant $K < 1$ and a continuous function $m : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|f(t, s, x, l) - f(t, s, y, p)| \leq K |x - y| + m(t, s) |l - p|$$

and $\sup \{f(t, s, 0, 0) : t, s \in \mathbb{R}_+\} < \infty$.

(H₃) The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists $l_0 \in \mathbb{R}$ and a positive constant D such that:

$$m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, l_0)| \, dv \, dw \leq D \text{ for all } t, s \in \mathbb{R}_+;$$

$$\lim_{t, s \rightarrow \infty} m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w)) - u(t, s, v, w, y(v, w))| \, dv \, dw = 0;$$

and

$$m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w)) - u(t, s, v, w, y(v, w))| \, dv \, dw \leq D$$

for all $t, s \in \mathbb{R}_+$ uniformly with respect to $x, y \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

Theorem 11. *Under conditions (H₁)–(H₃), equation (1) has at least one solution in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, the solutions of (1) are uniformly locally attractive.*

Proof. Consider the operator G defined by

$$(Gx)(t, s) = f \left(t, s, x(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) \, dv \, dw \right)$$

for any $x(t, s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$. By condition (H₂), Gx is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, and by (H₂)–(H₃), for arbitrary $t, s \in \mathbb{R}_+$, we have

$$\begin{aligned} & |(Gx)(t, s)| \\ & \leq \left| f \left(t, s, x(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) \, dv \, dw \right) \right| \end{aligned}$$

$$\begin{aligned}
 & - f(t, s, 0, 0) \Big| + |f(t, s, 0, 0)| \\
 & \leq K |x(t, s)| + m(t, s) \left| \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) dv dw \right| \\
 & \quad + |f(t, s, 0, 0)| \\
 & \leq K |x(t, s)| + 2D + |f(t, s, 0, 0)| \\
 & \leq K |x(t, s)| + \bar{M},
 \end{aligned}$$

where $\bar{M} = \sup \{ |f(t, s, 0, 0)| : t, s \in \mathbb{R}_+ \} + 2D$. Thus,

$$|(Gx)(t, s)| \leq K |x(t, s)| + \bar{M}.$$

Hence, $Gx \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

Let $B_r = B(0, r)$, where $r = \frac{\bar{M}}{1-K}$. Clearly G maps the ball B_r into itself. For a given $\epsilon > 0$, take $x, y \in B_r$ with $\|x - y\| < \epsilon$. Then,

$$\begin{aligned}
 & |(Gx)(t, s) - (Gy)(t, s)| \\
 & \leq \left| f \left(t, s, x(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) dv dw \right) \right. \\
 & \quad \left. - f \left(t, s, y(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, y(v, w)) dv dw \right) \right| \\
 & \leq K |x(t, s) - y(t, s)| + m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w)) \\
 & \quad - u(t, s, v, w, y(v, w))| dv dw.
 \end{aligned}$$

In view of (H₃), there exists $T > 0$ such that for $t, s \geq T$, we have

$$\begin{aligned}
 m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w)) \\
 - u(t, s, v, w, y(v, w))| dv dw \leq \epsilon.
 \end{aligned}$$

Thus,

$$|(Gx)(t, s) - (Gy)(t, s)| \leq (K + 1)\epsilon.$$

Now let $t, s \in [0, T]$. Then by the continuity of the function u on $[0, T] \times [0, T] \times [0, B] \times [0, A] \times [-r, r]$, where $A = \sup \{a(t) : t \in [0, T]\}$ and $B = \sup \{b(t) : t \in [0, T]\}$, we have

$$m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w))$$

$$- u(t, s, v, w, y(v, w))|dvdw \rightarrow 0$$

as $\epsilon \rightarrow 0$. Hence, G is continuous on B_r .

Now consider any nonempty set $X \subset B_r$ and fix $T > 0$ and $\epsilon > 0$. Choose $x \in X$ and $t_1, t_2, s_1, s_2 \in [0, T]$ with $|t_1 - t_2| \leq \epsilon$ and $|s_1 - s_2| \leq \epsilon$. Then, we have

$$\begin{aligned} & |(Gx)(t_2, s_2) - (Gx)(t_1, s_1)| \\ & \leq \left| f \left(t_2, s_2, x(t_2, s_2), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_2, s_2, v, w, x(v, w))dvdw, \right) \right. \\ & \quad \left. - f \left(t_2, s_2, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_2, s_2, v, w, x(v, w))dvdw, \right) \right| \\ & \quad + \left| f \left(t_2, s_2, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_2, s_2, v, w, x(v, w))dvdw, \right) \right. \\ & \quad \left. - f \left(t_1, s_1, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_2, s_2, v, w, x(v, w))dvdw, \right) \right| \\ & \quad + \left| f \left(t_1, s_1, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_2, s_2, v, w, x(v, w))dvdw, \right) \right. \\ & \quad \left. - f \left(t_1, s_1, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_1, s_1, v, w, x(v, w))dvdw, \right) \right| \\ & \quad + \left| f \left(t_1, s_1, x(t_1, s_1), \int_0^{a(s_2)} \int_0^{b(t_2)} u(t_1, s_1, v, w, x(v, w))dvdw, \right) \right. \\ & \quad \left. - f \left(t_1, s_1, x(t_1, s_1), \int_0^{a(s_1)} \int_0^{b(t_1)} u(t_1, s_1, v, w, x(v, w))dvdw, \right) \right| \\ & \leq K |x(t_2, s_2) - x(t_1, s_1)| + W_{r, \hat{D}}^T(f, \epsilon) \\ & \quad + m(t_1, s_1) \int_0^{a(s_2)} \int_0^{b(t_2)} |u(t_2, s_2, v, w, x(v, w)) \\ & \quad - u(t_1, s_1, v, w, y(v, w))|dvdw \\ & \quad + m(t_1, s_1) \int_{a(s_1)}^{a(s_2)} \int_{b(t_1)}^{b(t_2)} |u(t_1, s_1, v, w, y(v, w))| dvdw, \end{aligned}$$

where

$$\hat{D} = AB \sup\{|u(t, s, v, w, x)| : t, s \in [0, T], v \in [0, B], \\ w \in [0, A], x \in [-r, r]\},$$

$$W_{r, \hat{D}}^T(f, \epsilon) = \sup \left\{ \begin{array}{l} |f(t_2, s_2, x, y) - f(t_1, s_1, x, y)| : \\ t_1, t_2, s_1, s_2 \in [0, T], |t_2 - t_1| \leq \epsilon, \\ |s_2 - s_1| \leq \epsilon, x \in [-r, r], y \in [-\hat{D}, \hat{D}] \end{array} \right\},$$

$$\omega_r^T(u, \epsilon) = \sup \left\{ \begin{array}{l} m(t_1, s_1) |u(t_2, s_2, v, w, x) - u(t_1, s_1, v, w, y)| : \\ t_1, t_2, s_1, s_2 \in [0, T], |t_2 - t_1| \leq \epsilon, \\ |s_2 - s_1| \leq \epsilon, x \in [-r, r], v \in [0, B], w \in [0, A] \end{array} \right\},$$

$$U_r^T = \sup \left\{ \begin{array}{l} m(t_1, s_1) |u(t_1, s_1, v, w, y)| : t_1, s_1 \in [0, T], \\ x \in [-r, r], v \in [0, B], w \in [0, A] \end{array} \right\},$$

$$\omega^T(a, \epsilon) = \sup \{|a(s_2) - a(s_1)| : |s_2 - s_1| \leq \epsilon, s_2, s_1 \in [0, T]\},$$

$$\omega^T(b, \epsilon) = \sup \{|b(t_2) - b(t_1)| : |t_2 - t_1| \leq \epsilon, t_2, t_1 \in [0, T]\}.$$

Therefore,

$$\begin{aligned} & |(Gx)(t_2, s_2) - (Gx)(t_1, s_1)| \\ & \leq K\omega^T(x, \epsilon) + W_{r, \hat{D}}^T(f, \epsilon) + \omega_r^T(u, \epsilon) + U_r^T \omega^T(a, \epsilon) \omega^T(b, \epsilon). \end{aligned}$$

Using the above estimate, we have

$$\begin{aligned} W^T(GX, \epsilon) & \leq K\omega^T(X, \epsilon) + W_{r, \hat{D}}^T(f, \epsilon) \\ & \quad + \omega_r^T(u, \epsilon) + U_r^T \omega^T(a, \epsilon) \omega^T(b, \epsilon). \end{aligned}$$

From the continuity of f , u , a , and b , we see that

$$W_{r, \hat{D}}^T(f, \epsilon) \rightarrow 0, \omega_r^T(u, \epsilon) \rightarrow 0, \omega^T(a, \epsilon) \rightarrow 0, \text{ and } \omega^T(b, \epsilon) \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus, we obtain

$$\omega_0^T(GX) \leq K\omega_0^T(X).$$

As $T \rightarrow \infty$, we have

$$\omega_0(GX) \leq K\omega_0(X).$$

Again for $t, s \in \mathbb{R}_+$ and $x, y \in X$, we have

$$\begin{aligned} |(Gx)(t, s) - (Gy)(t, s)| &\leq K |x(t, s) - y(t, s)| \\ &\quad + m(t, s) \int_0^{a(s)} \int_0^{b(t)} |u(t, s, v, w, x(v, w)) \\ &\quad - u(t, s, v, w, y(v, w))| dv dw. \end{aligned}$$

Using condition (H₃) and letting $t, s \rightarrow \infty$ gives

$$\limsup_{t,s \rightarrow \infty} \text{diam}(GX)(t, s) \leq K \limsup_{t,s \rightarrow \infty} \text{diam}X(t, s).$$

Thus, we have

$$\mu(GX) \leq K\mu(X), \text{ where } K \in [0, 1).$$

Therefore, by taking $H(x, y) = x + y, \psi = 0$ and $\beta(t) = k, t \geq 0, k \in [0, 1)$ in Theorem 8, we can conclude that the operator G has at least one fixed point in B_r , so equation (1) has at least one solution in $B_r \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$. The solutions of equation (1) are uniformly locally attractive since all solutions of equation (1) in B_r belong to $\ker \mu$. □

Corollary 12. *If in addition to conditions (H₁)–(H₃), we have that $f(t, s, x, 0)$ is bounded, then the solutions of equation (1) are globally attractive.*

Proof. Let $Q = \sup \{|f(t, s, x, 0)| : t, s \in \mathbb{R}_+, x \in \mathbb{R}\}$. Then for any $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$, we have

$$\begin{aligned} |(Gx)(t, s)| &\leq \left| f \left(t, s, x(t, s), \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) dv dw \right) \right. \\ &\quad \left. - f(t, s, x(t, s), 0) \right| + |f(t, s, x(t, s), 0)| \\ &\leq m(t, s) \left| \int_0^{a(s)} \int_0^{b(t)} u(t, s, v, w, x(v, w)) dv dw \right| + Q \\ &\leq 2D + Q = r_1, \end{aligned}$$

so $G(BC(\mathbb{R}_+ \times \mathbb{R}_+)) \subset B_{r_1} = B(0, r_1)$. Thus, all solutions of equation (1) are in B_{r_1} . As in the proof of Theorem 11, we can show that $\mu(GX) \leq K\mu(X)$ for any subset of B_{r_1} , and we can find a subset C of B_{r_1} such that this set contains all solutions of equation (1) and $\mu(C) = 0$. That is, $\lim_{t,s \rightarrow \infty} |x(t, s) - y(t, s)| = 0$ for all $x, y \in C$. Thus, the solutions of equation (1) are globally attractive. □

We conclude this paper with an example.

Example 13. Consider the equation

$$x(t, s) = \frac{\sin(tsx(t, s))}{1 + t^2s^2} + \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} \frac{vwx^2(v, w) + st(1 + x^4(v, w))v^{11}w^{11}}{(1 + t^7s^7)(1 + x^4(v, w))} dvdw \quad (4)$$

for $t, s \in \mathbb{R}_+$. Here we have

$$a(t) = b(t) = \sqrt{t},$$

$$f(t, s, x, y) = \frac{\sin(tsx)}{1 + t^2s^2} + y,$$

and

$$u(t, s, v, w, x) = \frac{vwx^2 + st(1 + x^4)v^{11}w^{11}}{(1 + t^7s^7)(1 + x^4)}.$$

Now

$$\begin{aligned} \left| f(t, s, x, l) - f(t, s, y, p) \right| &= \left| \frac{\sin(tsx)}{1 + t^2s^2} + l - \frac{\sin(tsy)}{1 + t^2s^2} - p \right| \\ &\leq \frac{ts}{1 + t^2s^2} |x - y| + |l - p|. \end{aligned}$$

Here $K = \sup \left\{ \frac{ts}{1 + t^2s^2} : t, s, \in \mathbb{R}_+ \right\} < 1$ and $m(t, s) = 1$ for all $t, s \in \mathbb{R}_+$. Clearly, $f, u, m, a,$ and b are all continuous functions on their respective domains. We have

$$\sup \{ f(t, s, 0, 0) : t, s \in \mathbb{R}_+ \} = 0$$

and

$$\sup \{ f(t, s, x, 0) : t, s \in \mathbb{R}_+, x \in \mathbb{R} \}$$

is finite, i.e., $f(t, s, x, 0)$ bounded. Also,

$$\begin{aligned} \left| u(t, s, v, w, x) - u(t, s, v, w, y) \right| &= \frac{1}{1 + t^7s^7} \left| \frac{vwx^2}{1 + x^4} - \frac{vwy^2}{1 + y^4} \right| \\ &\leq \frac{vw}{1 + t^7s^7} \end{aligned}$$

and

$$\begin{aligned} \lim_{t,s \rightarrow \infty} \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} |u(t, s, v, w, x) - u(t, s, v, w, y)| \, dv \, dw \\ \leq \lim_{t,s \rightarrow \infty} \left\{ \frac{1}{1 + t^7 s^7} \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} v w \, dv \, dw \right\} = 0. \end{aligned}$$

Again,

$$\begin{aligned} \lim_{t,s \rightarrow \infty} \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} |u(t, s, v, w, x)| \, dv \, dw \\ \leq \lim_{t,s \rightarrow \infty} \left\{ \frac{1}{1 + t^7 s^7} \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} (v w + s t v^{11} w^{11}) \, dv \, dw \right\} \\ = \frac{1}{1 + t^7 s^7} \lim_{t,s \rightarrow \infty} \left\{ \frac{s t}{4} + \frac{s^7 t^7}{144} \right\}, \end{aligned}$$

i.e., $\lim_{t,s \rightarrow \infty} \int_0^{\sqrt{s}} \int_0^{\sqrt{t}} |u(t, s, v, w, x)| \, dv \, dw \leq \frac{1}{144}.$

Hence, the conditions of Corollary 12 are satisfied, and so equation (4) has at least one solution in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$, and solutions of this equation are globally attractive.

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