

**INVERSE BOUNDARY-VALUE PROBLEM FOR
LINEARIZED EQUATION OF MOTION OF
A HOMOGENEOUS ELASTIC BEAM**

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Abstract: The present paper is devoted to the study of classical solution of an inverse boundary-value problem for the linearized equation of motion of a homogeneous elastic beam with an over-determination condition. The goal of the work is to determine both solution and the unknown coefficient together for the considered problem in the rectangular region. First, in order to investigate of solvability of the inverse problem, we reduce original problem to the auxiliary problem with trivial data. Applying the Fourier method and contraction mappings principle, the existence and uniqueness of the classical solution of the obtained equivalent problem is proved. Furthermore, using the equivalence, the unique solvability of the appropriate auxiliary inverse problem is shown.

AMS Subject Classification: 35L25, 35Q74, 35R30, 35A01, 35A02, 35A09

Key Words: inverse boundary-value problem; homogenous beam; classical solution; Fourier's method

Received: November 6, 2019

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1. Introduction and Preliminaries

In the rectangular region $D_T : 0 \leq x \leq 1, 0 \leq t \leq T$ we consider inverse boundary-value problem for the one-dimensional equation of motion of a homogeneous elastic beam [15]

$$u_{tt}(x, t) + u_{xxxx}(x, t) + \beta u_{xx}(x, t) = p(t)u(x, t) + f(x, t), \quad (x, t) \in D_T, \quad (1)$$

for the unknown functions $u(x, t)$, $p(t)$, with conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad 0 \leq t \leq T, \quad (3)$$

$$\int_0^1 g(x)u(x, t)dx = h(t), \quad 0 \leq t \leq T, \quad (4)$$

where T, β are positive integers, $f(x, t)$, $\varphi(x)$, $\psi(x)$, $g(x)$, and $h(t)$ are known functions of $x \in [0, 1]$ and $t \in [0, T]$.

In many cases the practical requirements lead to the problems of determining the coefficients or the right hand side of a differential equation (ordinary or in partial derivatives) from some known functional of its solution. Such problems are called inverse problems of mathematical physics. The practical importance of inverse problems is so great (they arise in various fields of human activity, such as seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, etc.) which puts them a series of the most actual problems of modern mathematics. The presence in the inverse problems of additional unknown functions requires that in the complement to the boundary conditions that are natural for a particular class of differential equations, impose some additional conditions - overdetermination conditions. The basics of the theory and practice of studying inverse problems of mathematical physics were established and developed in the fundamental works of the outstanding scientists A.N. Tikhonov [23], M.M. Lavrent'ev [16], V.K. Ivanov [11], and their followers.

Inverse problems associated with differential equations of various types, have been studied by many papers and monographs, in particular [1, 3, 4, 6, 7, 9, 10, 13, 14, 17, 19, 20]. But the problem statement and the proof techniques used in this paper are different from those presented in these works. The technique used in this paper is based on the passing from the original inverse

problem to the new equivalent one, the study of the solvability of the equivalent problem, and then in the reverse transition to the original problem.

Moreover, the vibrations and wave movements of an elastic beam were investigated by B.S. Bardin [5], T.P. Goy [8], D.V. Kostin [15], Y.T. Mehraliyev [2], Yu.A. Mitropolsky [18], J.M. Thompson [22], V.Z. Vlasov [24], and et al.

The simplest nonlinear model of the motion of a homogeneous beam is described by the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + k \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0,$$

where w is beam deflection (after the displacements of the points of the midline of the elastic beam located along the x -axis). Note that a similar equation also arises in the theory of crystals, in which w is parameter of order [12].

We introduce the set of functions

$$\tilde{C}^{2,4}(D_T) = \{u(x, t) : u(x, t) \in C^2(D_T), u_{xxxx}(x, t) \in C(D_T)\}.$$

Definition 1. The pair $\{u(x, t), p(t)\}$ defined in the domain D_T is said to be a classical solution of the problem (1)-(4), if the functions $u(x, t) \in \tilde{C}^{2,4}(D_T)$ and $p(t) \in C[0, T]$ satisfies Eq. (1), condition (2) on $[0, 1]$, and the statements (3)-(4) on the interval $[0, T]$.

Theorem 2. Suppose that $f(x, t) \in C(D_T)$, $\varphi(x), \psi(x) \in C[0, 1]$, $g(x) \in L_2(0, 1)$, $h(t) \in C^2[0, T]$, and the conditions

$$\int_0^1 g(x)\varphi(x)dx = h(0), \quad \int_0^1 g(x)\psi(x)dx = h'(0)$$

hold. Then the problem of finding a classical solution of (1)-(4) is equivalent to the problem of determining the functions $u(x, t) \in \tilde{C}^{2,4}(D_T)$ and $p(t) \in C[0, T]$ from the (1)-(3), and satisfying the condition

$$\begin{aligned} h''(t) + \int_0^1 g(x)u_{xxxx}(x, t)dx + \beta \int_0^1 g(x)u_{xx}(x, t)dx \\ = p(t)h(t) + \int_0^1 g(x)f(x, t)dx, \quad 0 \leq t \leq T. \end{aligned} \quad (5)$$

Proof. Let $\{u(x, t), p(t)\}$ be a classical solution of the problem (1)-(4). Differentiating both sides of (4) twice with respect to t , gives

$$\int_0^1 g(x) u_t(x, t) dx = h'(t),$$

$$\int_0^1 g(x) u_{tt}(x, t) dx = h''(t), \quad 0 \leq t \leq T. \quad (6)$$

Multiplying both sides of (1) by a special function $g(x)$ and integrating gives, in turn,

$$\begin{aligned} & \frac{d^2}{dt^2} \int_0^1 g(x) u(x, t) dx + \int_0^1 g(x) u_{xxxx}(x, t) dx + \beta \int_0^1 g(x) u_{xx}(x, t) dx \\ &= p(t) \int_0^1 g(x) u(x, t) dx + \int_0^1 g(x) f(x, t) dx, \quad 0 \leq t \leq T. \end{aligned} \quad (7)$$

Hence, taking into account (4) and (6), we conclude that condition (5) is satisfied.

Now, assume that $\{u(x, t), p(t)\}$ is the solution of problem (1)-(3), (5). Then from (5), taking into account (7), we have

$$\begin{aligned} & \frac{d^2}{dt^2} \left(\int_0^1 g(x) u(x, t) dx - h(t) \right) \\ &= p(t) \left(\int_0^1 g(x) u(x, t) dx - h(t) \right), \quad 0 \leq t \leq T. \end{aligned} \quad (8)$$

By virtue of (2) and

$$\int_0^1 g(x) \varphi(x) dx = h(0), \quad \int_0^1 g(x) \psi(x) dx = h'(0),$$

we find

$$\int_0^1 g(x) u(x, 0) dx - h(0) = \int_0^1 g(x) \varphi(x) dx - h(0) = 0,$$

$$\int_0^1 g(x)u_t(0,t)dx - h'(0) = \int_0^1 g(x)\psi(x)dx - h'(0) = 0. \quad (9)$$

From (8), taking into account (9), it is clear that the condition (4) is also satisfied. The proof is complete. \square

2. Classical Solvability of Inverse Boundary-value Problem

We seek the first component $u(x,t)$ of classical solution $\{u(x,t), p(t)\}$ of the problem (1)-(3), (5) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x, \quad \lambda_k = k\pi, \quad (10)$$

where

$$u_k(t) = 2 \int_0^1 u(x,t) \sin \lambda_k x dx, \quad k = 1, 2, \dots$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$u_k''(t) + (\lambda_k^4 - \beta \lambda_k^2) u_k(t) = F_k(t; u, p), \quad 0 \leq t \leq T; \quad k = 1, 2, \dots, \quad (11)$$

$$u_k(0) = \varphi_k, \quad u_k'(0) = \psi_k, \quad k = 1, 2, \dots, \quad (12)$$

where

$$F_k(t; u, p) = p(t)u_k(t) + f_k(t), \quad f_k(t) = \int_0^1 f(x,t) \sin \lambda_k x dx,$$

$$\varphi_k = 2 \int_0^1 \varphi(x) \sin \lambda_k x dx, \quad \psi_k = 2 \int_0^1 \psi(x) \sin \lambda_k x dx, \quad k = 1, 2, \dots$$

Suppose that $0 < \beta < \frac{\pi^2}{2}$. Solving the problem (11)-(12) gives

$$u_k(t) = \varphi_k \cos \beta_k t + \frac{1}{\beta_k} \psi_k \sin \beta_k t$$

$$+ \frac{1}{\beta_k} \int_0^t F_k(\tau; u, p) \sin \beta_k(t - \tau) d\tau, \quad k = 1, 2, \dots, \quad (13)$$

where

$$\beta_k = \sqrt{\lambda_k^4 - \beta \lambda_k^2}.$$

In order to determine the first component of the solution of the problem (1)-(3), (5) we substitute $u_k(t)$ ($k = 1, 2, \dots$) into (10) and obtain

$$\begin{aligned} u(x, t) = & \sum_{k=1}^{\infty} \left\{ \varphi_k \cos \beta_k t + \frac{1}{\beta_k} \psi_k \sin \beta_k t \right. \\ & \left. + \frac{1}{\beta_k} \int_0^t F_k(\tau; u, p) \sin \beta_k(t - \tau) d\tau \right\} \sin \lambda_k x. \end{aligned} \quad (14)$$

Now, from (5), taking into account (10), we have

$$\begin{aligned} p(t) = & [h(t)]^{-1} \left\{ h''(t) - \int_0^1 g(x) f(x, t) dx \right. \\ & \left. + \sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2) u_k(t) \int_0^1 g(x) \sin \lambda_k x dx \right\}. \end{aligned} \quad (15)$$

In this way to obtain the equation for the second component of the solution to the problem (1) - (3), (5) we substitute expression (13) into (17) and get

$$\begin{aligned} p(t) = & [h(t)]^{-1} \left\{ h''(t) - \int_0^1 g(x) f(x, t) dx \right. \\ & + \sum_{k=1}^{\infty} (\lambda_k^4 - \beta \lambda_k^2) \left[\varphi_k \cos \beta_k t + \frac{1}{\beta_k} \psi_k \sin \beta_k t \right. \\ & \left. \left. + \frac{1}{\beta_k} \int_0^t F_k(\tau; u, p) \sin \beta_k(t - \tau) d\tau \right] \int_0^1 g(x) \sin \lambda_k x dx \right\}. \end{aligned} \quad (16)$$

Thus, finding the solution of problem (1) - (3), (5) is reduced to the finding solution of system (14), (16) with respect to unknown functions $u(x, t)$ and $p(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (1)-(3), (5).

Lemma 3. *If $\{u(x, t), p(t)\}$ is a solution of (1)-(3), (5), then the functions*

$$u_k(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x dx, \quad k = 1, 2, \dots,$$

satisfy the system (13) on the interval $[0, T]$.

Proof. Let $\{u(x, t), p(t)\}$ be any solution to problem (1)-(3), (5). Multiplying both sides of the Eq. (1) by a function $2 \sin \lambda_k x$ ($k = 1, 2, \dots$), integrating by parts, and using the relations

$$\begin{aligned}
 & 2 \int_0^1 u_{tt}(x, t) \sin \lambda_k x dx \\
 &= \frac{d^2}{dt^2} \left(2 \int_0^1 u(x, t) \sin \lambda_k x dx \right) = u_k''(t), \quad k = 1, 2, \dots, \\
 & 2 \int_0^1 u_{xx}(x, t) \sin \lambda_k x dx \\
 &= -\lambda_k^2 \left(2 \int_0^1 u(x, t) \sin \lambda_k x dx \right) = -\lambda_k^2 u_k(t), \quad k = 1, 2, \dots, \\
 & 2 \int_0^1 u_{xxxx}(x, t) \sin \lambda_k x dx \\
 &= \lambda_k^4 \left(2 \int_0^1 u(x, t) \cos \lambda_k x dx \right) = \lambda_k^4 u_k(t), \quad k = 1, 2, \dots,
 \end{aligned}$$

we deduce that satisfies (11).

Similarly, from (2) we obtain that condition (12) is satisfied. Thus, the system of functions $u_k(t)$ ($k = 1, 2, \dots$) is a solution to problem (11), (12). From this it follows directly that the functions $u_k(t)$ ($k = 1, 2, \dots$) satisfy the system (13) on $[0, T]$. The lemma is proved. \square

Obviously, if $u_k(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x dx$, $k = 1, 2, \dots$ is a solution to system (13), then the pair $\{u(x, t), p(t)\}$ of functions $u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x$ and $p(t)$ is also a solution to system (14), (16).

It follows from Lemma 3 that

Remark 4. Suppose that system (14) and (16) has a unique solution. Then the problem (1)-(3), (5), can't have more than one solution, i.e. if problem (1)-(3), (7) has a solution, then it is unique.

Now, to study problem (1)-(3), (5), we consider the following spaces.

Denote by $B_{2,T}^5$ a set of all functions of the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x, \quad \lambda_k = \pi k,$$

considered in D_T where each of the function $u_k(t)$ ($k = 1, 2, \dots$) is continuous on $[0, T]$ and

$$J_T(u) \equiv \left(\sum_{k=1}^{\infty} (\lambda_k^2 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty.$$

The norm in this set is defined as follows

$$\|u(x, t)\|_{B_{2,T}^5} = J(u).$$

Next we denote by E_T^5 the space of vector-functions $z(x, t) = \{u(x, t), p(t)\}$, which $u(x, t) \in B_{2,T}^5$, $p(t) \in C[0, T]$.

The norm in the set E_T^5 will be

$$\|z(x, t)\|_{E_T^5} = \|u(x, t)\|_{B_{2,T}^5} + \|p(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^5$ and E_T^5 are the Banach spaces [21].

We now consider the operator

$$\Phi(u, p) = \{\Phi_1(u, p), \Phi_2(u, p)\},$$

in the space E_T^5 , where

$$\Phi_1(u, p) = \tilde{u}(x, t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t) \sin \lambda_k x, \quad \Phi_2(u, p) = \tilde{p}(t),$$

and the functions $\tilde{u}_k(t)$ ($k = 1, 2, \dots$) and $\tilde{p}(t)$ are equal to the right-hand sides of (13) and (16), respectively.

Then we find

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2 \left(\sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_k|)^2 \right)^{\frac{1}{2}} \\ & + \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} + \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \end{aligned}$$

$$+T \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \quad (17)$$

$$\begin{aligned} \|\tilde{p}(t)\|_{C[0,T]} &\leq \| [h(t)]^{-1} \|_{C[0,T]} \left\{ \left\| h''(t) - \int_0^1 g(x) f(x, t) dx \right\|_{C[0,T]} \right. \\ &\quad + \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (1 + \beta) \left[\left(\sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_k|)^2 \right)^{\frac{1}{2}} \right. \\ &\quad + \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\psi_{ik}|)^2 \right)^{\frac{1}{2}} + \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \\ &\quad \left. \left. + T \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \|g(x)\|_{L_2(0,1)} \right\}. \quad (18) \end{aligned}$$

Suppose that the data of problem (1)-(3), (5) satisfy the conditions

- (A₁) $\varphi(x) \in C^4[0, 1]$, $\varphi^{(5)}(x) \in L_2(0, 1)$,
 $\varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = \varphi^{(4)}(0) = \varphi^{(4)}(1) = 0$;
- (A₂) $\varphi(x) \in C^4[0, 1]$, $\psi'''(x) \in L_2(0, 1)$,
 $\psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0$;
- (A₃) $f(x, t), f_x(x, t), f_{xx}(x, t) \in C(D_T)$, $f_{xxx}(x, t) \in L_2(D_T)$, and $f(0, t) = f(1, t) = f_{xx}(0, t) = f_{xx}(1, t) = 0$, $0 \leq t \leq T$;
- (A₄) $0 < \beta < \frac{\pi^2}{2}$, $g(x) \in L_2(0, 1)$, $h(t) \in C^2[0, T]$,
 $h(t) \neq 0$, $0 \leq t \leq T$.

Then from relations (17) and (18), correspondingly we have

$$\begin{aligned} &\left(\sum_{k=1}^{\infty} (\lambda_k^5 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \\ &\leq 2 \left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \left\| \psi'''(x) \right\|_{L_2(0,1)} + \sqrt{T} \|f_{xxx}(x, t)\|_{L_2(D_T)} \\ &\quad + T \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \quad (19) \end{aligned}$$

$$\begin{aligned}
\|\tilde{p}(t)\|_{C[0,T]} &\leq \| [h(t)]^{-1} \|_{C[0,T]} \left\{ \left\| h''(t) - \int_0^1 g(x) f(x, t) dx \right\|_{C[0,T]} \right. \\
&\quad + \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (1 + \beta) \left[\left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \left\| \psi'''(x) \right\|_{L_2(0,1)} \right. \\
&\quad \left. + \sqrt{T} \|f_{xxx}(x, t)\|_{L_2(D_T)} + T \|p(t)\|_{C[0,T]} \right. \\
&\quad \left. \times \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \|g(x)\|_{L_2(0,1)} \left. \right\}. \tag{20}
\end{aligned}$$

We denote by

$$A_1(T) = 2 \left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \left\| \psi'''(x) \right\|_{L_2(0,1)} + \sqrt{T} \|f_{xxx}(x, t)\|_{L_2(D_T)},$$

$$B_1(T) = T,$$

$$\begin{aligned}
A_2(T) &= \| [h(t)]^{-1} \|_{C[0,T]} \left\{ \left\| h''(t) - \int_0^1 g(x) f(x, t) dx \right\|_{C[0,T]} \right. \\
&\quad + \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (1 + \beta) \left[\left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} \right. \\
&\quad \left. + \left\| \psi'''(x) \right\|_{L_2(0,1)} + \sqrt{T} \|f_{xxx}(x, t)\|_{L_2(D_T)} \right] \|g(x)\|_{L_2(0,1)} \left. \right\},
\end{aligned}$$

$$B_2(T) = \| [h(t)]^{-1} \|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (1 + \beta) T \|g(x)\|_{L_2(0,1)},$$

and rewrite (19), (20) as

$$\|\tilde{u}(x, t)\|_{B_{2,T}^5} \leq A_1(T) + B_1(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5}, \tag{21}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5}. \tag{22}$$

From the inequalities (21) and (22), we conclude

$$\|\tilde{u}(x, t)\|_{B_{2,T}^5} + \|\tilde{p}(t)\|_{C[0,T]}$$

$$\leq A(T) + B(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5}, \quad (23)$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

Thus, the following assertion is valid.

Theorem 5. *If conditions $(A_1) - (A_4)$ and*

$$B(T)(A(T) + 2)^2 < 1, \quad (24)$$

hold, then problem (1)-(3), (5) has a unique solution in the ball $K = K_R(\|z\|_{E_T^5} \leq R = A(T) + 2)$ of the space E_T^5 .

Proof. In the space E_T^5 , consider the equation

$$z = \Phi z, \quad (25)$$

where $z = \{u, p\}$, the components $\Phi_i(u, p)$, $i = 1, 2$, of operator $\Phi(u, p)$, defined by the right sides of equations (14) and (16), respectively.

Now, consider the operator $\Phi(u, p)$ in the ball $K = K_R$ of the space E_T^5 . Similarly to (23), we obtain that for any $z = \{u, p\}$, $z_1 = \{u_1, p_1\}$, and $z_2 = \{u_2, p_2\} \in K_R$ the following inequalities hold:

$$\|\Phi z\|_{E_T^5} \leq A(T) + B(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^5}, \quad (26)$$

$$\begin{aligned} & \|\Phi z_1 - \Phi z_2\|_{E_T^5} \leq B(T)R \\ & \times (\|p_1(t) - p_2(t)\|_{C[0,T]} + \|u_1(x, t) - u_2(x, t)\|_{B_{2,T}^5}). \end{aligned} \quad (27)$$

Then by (24), from (14) and (16) it follows that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{u, p\}$ in the ball $K = K_R$, which is a solution of equation (25), i.e. the pair $\{u, p\}$ is the unique solution of the systems (14), (16) in $K = K_R$.

Then the function $u(x, t)$ as an element of space $B_{2,T}^5$ is continuous and has continuous derivatives $u_x(x, t)$, $u_{xx}(x, t)$, $u_{xxx}(x, t)$, and $u_{xxxx}(x, t)$ in D_T .

From (9) it is easy to see that

$$\left(\sum_{k=1}^{\infty} (\lambda_k \|u''_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq (1 + \beta)$$

$$\times \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left\| \|f_x(x, t) + p(t)u_x(x, t)\|_{C[0,T]} \right\|_{L_2(0,1)}.$$

Hence, we conclude that the function $u_{tt}(x, t)$ is continuous in the domain D_T .

Further, it is easy to verify that equation (1), and conditions (2), (3), and (5) are satisfied in the usual sense. Consequently, $\{u(x, t), p(t)\}$ is a solution of (1)-(3), (5), and by Lemma 3 it is unique in the ball $K = K_R$. The proof is complete. \square

From Theorem 2 and Theorem 5, it implies the unique solvability of the original problem (1)-(4).

In summary, we conclude the following result.

Theorem 6. *Suppose that all assumptions of Theorem 5, and*

$$\int_0^1 g(x)\varphi(x)dx = h(0), \quad \int_0^1 g(x)\psi(x)dx = h'(0)$$

hold. Then problem (1)-(4) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^5} \leq R = A(T) + 2)$.

Acknowledgements. The authors are sincerely grateful to the editors and anonymous reviewers for carefully reading the manuscript, as well as for their valuable comments and suggestions that helped improve this article.

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