

**A STUDY OF EXTENDED BETA, GAUSS  
AND CONFLUENT HYPERGEOMETRIC FUNCTIONS**

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**Abstract:** In the present research note, we define a new extension of beta function by making use of the multi-index Mittag-Leffler function. Here, first we derive its fundamental properties and then we present a new type of beta distribution as an application of our proposed beta function. Moreover, we present a new extension of Gauss and confluent hypergeometric functions in terms of our newly introduced beta function. Some interesting properties of our extended hypergeometric functions (like integral representations, differential formulae, transformations and summation formulae and a generating relation) are also indicated in the last section.

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## 1. Introduction

Throughout in this paper, let  $N$ ,  $R$  and  $C$  be the sets of natural numbers, real numbers and complex numbers, respectively, and let

$$N := \{1, 2, 3, \dots\}, \quad N_0 := \{0, 1, 2, 3, \dots\} = N \cup \{0\}.$$

The classical beta function  $B(m, n)$  is defined by (see [10], see also [11])

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad (1)$$

$$(\Re(m) > 0, \Re(n) > 0).$$

Due to diverse applications of beta function in a wide range of engineering and sciences, a number of researchers have introduced and investigated several extensions of  $B(m, n)$  (see, for example, [1], [2], [7], [9] and [12]).

In 1997, Chaudhry et al. [2] introduced a useful generalization of (1) by

$$B_p(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \exp\left[-\frac{p}{x(1-x)}\right] dx \quad (2)$$

$$(\Re(m) > 0, \Re(n) > 0, \Re(p) > 0).$$

It is easily seen that for  $p = 0$ , (2) reduces to (1). By using  $B_p(m, n)$ , Chaudhry et al. [3] generalized the Gauss hypergeometric function and the confluent hypergeometric function, respectively, as follows:

$$F_p(c_1, c_2; c_3; z) = \sum_{n=0}^{\infty} \frac{(c_1)_n B_p(c_2 + n, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{z^n}{n!} \quad (3)$$

$$(p \geq 0, |z| < 1, \Re(c_3) > \Re(c_2) > 0)$$

and

$$\Phi_p(c_2; c_3; z) = \sum_{n=0}^{\infty} \frac{B_p(c_2 + n, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{z^n}{n!} \quad (4)$$

$$(p \geq 0, \Re(c_3) > \Re(c_2) > 0).$$

They also gave the following Euler's type integral representations:

$$F_p(c_1, c_2; c_3; z) = \frac{1}{B(c_2, c_3 - c_2)} \quad (5)$$

$$\times \int_0^1 x^{c_2-1} (1-x)^{c_3-c_2-1} (1-zx)^{-c_1} \exp\left[-\frac{p}{x(1-x)}\right] dx$$

$$(p \geq 0, |\arg(1-z)| < \pi, \Re(c_3) > \Re(c_2) > 0)$$

and

$$\begin{aligned} \Phi_p(c_2; c_3; z) &= \frac{1}{B(c_2, c_3 - c_2)} \\ &\times \int_0^1 x^{c_2-1} (1-x)^{c_3-c_2-1} \exp \left[ zx - \frac{p}{x(1-x)} \right] dx \\ &\quad (p \geq 0, \Re(c_3) > \Re(c_2) > 0). \end{aligned} \quad (6)$$

Recently, Shadab et al. [12] defined the following generalization of beta function:

$$\begin{aligned} B_\alpha^p(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} E_\alpha \left[ -\frac{p}{x(1-x)} \right] dx \\ &\quad (\alpha \in R_0^+, \Re(p) > 0), \end{aligned} \quad (7)$$

where  $E_\alpha(t)$  is the classical Mittag-Leffler function defined by (see [6])

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(1+\alpha k)} \quad (t \in C \text{ and } \alpha \geq 0). \quad (8)$$

They also investigated some interesting properties of  $B_\alpha^p(m, n)$  in [12]. The main motive of this paper is to introduce a further extension of beta function by making use of the multi-index Mittag-Leffler function and also to present a new generalization of Gauss and confluent hypergeometric functions.

The multi-index Mittag-Leffler function  $E_{(\frac{1}{a_i}, (b_i))}(t)$  is defined as follows (see [4], [5], see also [8]):

$$E_{(\frac{1}{a_i}, (b_i))}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(b_1 + \frac{k}{a_1}) \cdots \Gamma(b_s + \frac{k}{a_s})}, \quad (9)$$

where  $s > 1$  be an integer,  $a_1, \dots, a_s > 0$  and  $b_1, \dots, b_s$  be arbitrary real numbers. It is easily seen that, for  $s = 2$ , if we set  $\frac{1}{a_1} = \alpha$ ,  $\frac{1}{a_2} = 0$ , and  $b_1 = b_2 = 1$  in (9), then this multi-index Mittag-Leffler function reduces to the classical Mittag-Leffler function  $E_\alpha(t)$ .

## 2. A new type of extended beta function

In this section, we consider the following extension of beta function by making use of the multi-index Mittag-Leffler function  $E_{(\frac{1}{a_i}, (b_i))}(t)$ :

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} E_{(\frac{1}{a_i}, (b_i))} \left[ -\frac{p}{x(1-x)} \right] dx \quad (10)$$

$$(\Re(m) > 0, \Re(n) > 0, a_i > 0, b_i \in \mathbb{R}, p \geq 0).$$

For  $s = 2$ , if we set  $\frac{1}{a_1} = \alpha$ ,  $\frac{1}{a_2} = 0$ , and  $b_1 = b_2 = 1$  in (10) then we get the extended beta function defined by Shadab et al. [12].

Furthermore, we can consider the following variations in (10):

The multi-index Mittag-Leffler function  $E_{(\frac{1}{a_i}, (b_i))}(t)$  have the under mentioned connections with Wright hypergeometric function  ${}_1\Psi_s(t)$  and Fox H-function  $H_{p,q}^{m,n}(t)$  (see for details [4]):

$$E_{(\frac{1}{a_i}, (b_i))}(t) = {}_1\Psi_s \left[ \begin{matrix} (1, 1) \\ (b_i, \frac{1}{a_i})_1^s \end{matrix} \mid t \right] \quad (11)$$

and

$$E_{(\frac{1}{a_i}, (b_i))}(t) = H_{1,s+1}^{1,1} \left[ \begin{matrix} (0, 1) \\ -t \mid (0, 1), (1 - b_i, \frac{1}{a_i})_1^s \end{matrix} \right]. \quad (12)$$

Therefore, our newly extended beta function given in (10), is easily converted in terms of Wright hypergeometric function and Fox  $H$ -function as follows:

$$\begin{aligned} & B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) \\ &= \int_0^1 x^{m-1} (1-x)^{n-1} {}_1\Psi_s \left[ \begin{matrix} (1, 1) \\ (b_i, \frac{1}{a_i})_1^s \end{matrix} \mid -\frac{p}{x(1-x)} \right] dx \end{aligned} \quad (13)$$

and

$$\begin{aligned} & B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) \\ &= \int_0^1 x^{m-1} (1-x)^{n-1} H_{1,s+1}^{1,1} \left[ \begin{matrix} (0, 1) \\ \frac{p}{x(1-x)} \mid (0, 1), (1 - b_i, \frac{1}{a_i})_1^s \end{matrix} \right] dx. \end{aligned} \quad (14)$$

**Integral representation of  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$**

**Theorem 1.** For  $a_i > 0$ ,  $b_i \in \mathbb{R}$  and  $p \geq 0$ , we have the following integral representations of  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$ :

$$\begin{aligned} & B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = 2 \int_0^{\frac{\pi}{2}} \cos^{2m-1} t \sin^{2n-1} t \\ & \quad \times E_{(\frac{1}{a_i}, (b_i))}(-p \sec^2 t \csc^2 t) dt; \end{aligned} \quad (15)$$

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = \int_0^\infty \frac{w^{m-1}}{(1+w)^{m+n}} \times E_{(\frac{1}{a_i}), (b_i)} \left[ -p \left( 2 + w + \frac{1}{w} \right) \right] dw; \quad (16)$$

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = 2^{1-m-n} \int_{-1}^1 (1+w)^{m-1} (1-w)^{n-1} \times E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{4p}{(1-w^2)} \right] dw. \quad (17)$$

*Proof.* On setting  $x = \cos^2 t$ ,  $x = \frac{w}{1+w}$  and  $x = \frac{1+w}{2}$  in (10) yields, respectively, the integral representations (15)–(17).  $\square$

### 3. Properties of extended beta function $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$

This section deals with some basic properties of our newly introduced beta function  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$ .

**Theorem 2.** *The following result holds true for extended beta function  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$ :*

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = \sum_{s=0}^l \binom{l}{s} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+s, n+l-k), \quad (18)$$

where  $l \in N_0$ .

*Proof.* From (10), we have

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \{x + (1-x)\} \times E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx, \\ B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+1, n) + B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n+1). \quad (19)$$

Again, applying the same argument in the right side of (19), we get

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+2, n)$$

$$+2B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+1, n+1) + B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n+2),$$

continuing this process, by induction, we obtain the desired result.  $\square$

**Theorem 3.** *The following result holds true for extended beta function  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$ :*

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, 1-n) = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+k, 1). \quad (20)$$

*Proof.* On using (10) in the left side of (20), we get

$$\begin{aligned} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, 1-n) &= \int_0^1 x^{m-1} (1-x)^{-n} \\ &\quad \times E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx \\ &= \int_0^1 x^{m-1} \sum_{k=0}^{\infty} \frac{(n)_k x^k}{k!} E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx. \end{aligned}$$

Now interchanging the order of integration and summation in the above expression and then upon using (10), we arrive at our claimed result.  $\square$

**Theorem 4.** *The following result holds true for extended beta function  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n)$ :*

$$B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m, n) = \sum_{k=0}^{\infty} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(m+k, n+1). \quad (21)$$

*Proof.* This theorem can be established with the help of (10) by writing  $(1-x)^{n-1} = (1-x)^n \sum_{k=0}^{\infty} x^k$ . We omit the details.  $\square$

#### 4. The extended beta distribution

In this section, we define the following new extended beta distribution as an application of our extended beta function:

$$f(x) = \begin{cases} \frac{1}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} E_{(\frac{1}{a_i}, (b_i)} \left[ -\frac{p}{x(1-x)} \right] \\ \quad (0 < x < 1), \\ 0, \quad \text{otherwise} \end{cases} \quad (22)$$

$(\alpha, \beta \in R, a_i > 0, b_i \in R, p \geq 0).$

Next, we have presented here some fundamental properties of our extended beta distribution (22).

For  $n \in R$ , the  $n^{th}$  moment of  $X$  is given by

$$E(X^n) = \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + n, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} \quad (23)$$

$(\alpha, \beta \in R, a_i > 0, b_i \in R, p \geq 0).$

The particular case of (23) for  $n = 1$  yields the mean of our extended beta distribution

$$E(X) = \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + 1, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}. \quad (24)$$

The variance of our extended beta distribution is defined by

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2, \\ Var(X) &= \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + 2, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} \\ &\quad - \left[ \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + 1, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} \right]^2. \end{aligned} \quad (25)$$

The coefficient of variation of our introduced distribution (which is defined as the ratio of the standard deviation and mean) can be expressed as

$$C.V = \sqrt{\frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + 2, \beta) B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}{[B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + 1, \beta)]^2} - 1}. \quad (26)$$

The moment generating function (m.g.f) about origin of our extended beta distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r),$$

$$M_X(t) = \frac{1}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} \sum_{r=0}^{\infty} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + r, \beta) \frac{t^r}{r!}. \quad (27)$$

The characteristic function of the proposed distribution can be calculated as follows:

$$E(e^{itx}) = \sum_{r=0}^{\infty} \frac{i^r t^r}{r!} E(X^r),$$

$$E(e^{itx}) = \frac{1}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)} \sum_{r=0}^{\infty} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha + r, \beta) \frac{i^r t^r}{r!}. \quad (28)$$

The cumulative distribution function of our extended beta distribution (22) can be expressed as

$$F(x) = P[X < x] = \int_0^x f(x) dx,$$

$$F(x) = \frac{B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}, \quad (29)$$

where  $B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)$  denotes the incomplete extended beta function defined by

$$B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta) = \int_0^x x^{\alpha-1} (1-x)^{\beta-1} E_{(\frac{1}{a_i}, (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx.$$

The reliability function (which is simply the complement of the cumulative distribution function) of our newly introduced distribution is given by

$$R(x) = P[X \geq x] = 1 - F(x) = \int_x^{\infty} f(x) dx,$$

$$R(x) = \frac{B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)}, \quad (30)$$

where  $B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)$  is the incomplete extended beta function defined by

$$B_{p,x}^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta) = \int_x^{\infty} x^{\alpha-1} (1-x)^{\beta-1} E_{(\frac{1}{a_i}, (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx.$$

## 5. Extended hypergeometric functions and their associated properties

In this section, we present the following extensions of Gauss and confluent hypergeometric functions by making use of our extended beta function  $B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(\alpha, \beta)$ :

$$F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) = \sum_{l=0}^{\infty} \frac{(c_1)_l B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + l, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{t^l}{l!} \quad (31)$$

$(p \geq 0, |t| < 1, \Re(c_3) > \Re(c_2) > 0, a_i > 0, b_i \in R)$

and

$$\Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2; c_3; t) = \sum_{l=0}^{\infty} \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + l, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{t^l}{l!} \quad (32)$$

$$(p \geq 0, \Re(c_3) > \Re(c_2) > 0, a_i > 0, b_i \in R).$$

**Remark.** For  $s = 2$ , with  $\frac{1}{a_1} = 1, \frac{1}{a_2} = 0$ , and  $b_1 = b_2 = 1$ , (31) and (32) reduces to the known extensions of Gauss and confluent hypergeometric functions defined by Chaudhry et al. [3].

**Theorem 5.** *The following integral representations of our extended Gauss and confluent hypergeometric functions holds true:*

$$F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) = \frac{1}{B(c_2, c_3 - c_2)} \times \int_0^1 x^{c_2-1} (1-x)^{c_3-c_2-1} (1-tx)^{-c_1} E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx \quad (33)$$

$(p \geq 0, |\arg(1-t)| < \pi, \Re(c_3) > \Re(c_2) > 0, a_i > 0, b_i \in R)$

and

$$\Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2; c_3; t) = \frac{1}{B(c_2, c_3 - c_2)} \times \int_0^1 x^{c_2-1} (1-x)^{c_3-c_2-1} e^{tx} E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx \quad (34)$$

$(p \geq 0, \Re(c_3) > \Re(c_2) > 0, a_i > 0, b_i \in R).$

*Proof.* The above integral representations can be easily derived by using the integral representation of our extended beta function (10) in the right sides of (31) and (32), respectively.  $\square$

**Theorem 6.** *The following differential formulas for our extended Gauss and confluent hypergeometric functions holds true:*

$$\begin{aligned} \frac{d^k}{dt^k} \left\{ F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) \right\} &= \frac{(c_1)_k (c_2)_k}{(c_3)_k} \\ &\times F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + k, c_2 + k; c_3 + k; t) \\ (p \geq 0, a_i > 0, b_i \in R, k \in N_0) \end{aligned} \quad (35)$$

and

$$\begin{aligned} \frac{d^k}{dt^k} \left\{ \Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2; c_3; t) \right\} &= \frac{(c_2)_k}{(c_3)_k} \\ &\times \Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + k; c_3 + k; t) \\ (p \geq 0, a_i > 0, b_i \in R, k \in N_0). \end{aligned} \quad (36)$$

*Proof.* On differentiating (31) with respect to  $t$ , we get

$$\begin{aligned} \frac{d}{dt} \left\{ F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) \right\} \\ = \sum_{l=1}^{\infty} \frac{(c_1)_l B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + l, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{t^{l-1}}{(l-1)!}. \end{aligned}$$

On replacing  $l$  by  $l + 1$ , we have

$$\begin{aligned} \frac{d}{dt} \left\{ F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) \right\} \\ = \sum_{l=0}^{\infty} \frac{(c_1)_{l+1} B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + l + 1, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{t^l}{l!}. \end{aligned}$$

Now by using  $B(c_2, c_3 - c_2) = \frac{c_3}{c_2} B(c_2 + 1, c_3 - c_2)$  and  $(c_1)_{l+1} = c_1 (c_1 + 1)_l$ , we get

$$\begin{aligned} \frac{d}{dt} \left\{ F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) \right\} &= \frac{c_1 c_2}{c_3} \\ &\times \sum_{l=0}^{\infty} \frac{(c_1 + 1)_l B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + l + 1, c_3 - c_2)}{B(c_2 + 1, c_3 - c_2)} \frac{t^l}{l!}. \\ &= \frac{c_1 c_2}{c_3} F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + 1, c_2 + 1; c_3 + 1; t). \end{aligned} \quad (37)$$

Further differentiating (37) with respect to  $t$ , we get

$$\begin{aligned} \frac{d^2}{dt^2} \left\{ F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) \right\} &= \frac{c_1(c_1 + 1)c_2(c_2 + 1)}{c_3(c_3 + 1)} \\ &\times F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + 2, c_2 + 2; c_3 + 2; t). \end{aligned}$$

Continuing this process, by induction we obtain our claimed result (35).

Similarly we can establish the result (36).  $\square$

**Theorem 7.** *The following transformation formulas for our extended Gauss and confluent hypergeometric functions holds true:*

$$\begin{aligned} F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) &= (1-t)^{-c_1} \\ &\times F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)} \left( c_1, c_3 - c_2; c_2; -\frac{t}{(1-t)} \right) \\ &(p \geq 0, a_i > 0, b_i \in R) \end{aligned} \quad (38)$$

and

$$\begin{aligned} \Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2; c_3; t) &= e^t \Phi_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_3 - c_2; c_3; -t) \\ &(p \geq 0, a_i > 0, b_i \in R). \end{aligned} \quad (39)$$

*Proof.* On replacing  $x$  by  $1-x$  in (33) and then by using  $[1-t(1-x)]^{-c_1} = (1-t)^{-c_1} \left[ 1 + \frac{t}{1-t}x \right]^{-c_1}$ , we obtain

$$\begin{aligned} F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; t) &= \frac{(1-t)^{-c_1}}{B(c_2, c_3 - c_2)} \\ &\times \int_0^1 x^{c_3 - c_2 - 1} (1-x)^{c_2 - 1} \left( 1 + \frac{t}{1-t}x \right)^{-c_1} E_{(\frac{1}{a_i}), (b_i)} \left[ -\frac{p}{x(1-x)} \right] dx, \end{aligned}$$

which further on using (33), yields the needed result (38). In a similar way, we can establish (39).  $\square$

**Theorem 8.** *The following summation formula for our extended Gauss hypergeometric function holds true:*

$$\begin{aligned} F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1, c_2; c_3; 1) &= \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2, c_3 - c_1 - c_2)}{B(c_2, c_3 - c_2)} \\ &(p \geq 0, a_i > 0, b_i \in R, \Re(c_3 - c_1 - c_2) > 0). \end{aligned} \quad (40)$$

*Proof.* On putting  $t = 1$  in (33) and then by using (10), we get our required result (40).  $\square$

**Theorem 9.** *The following generating function for our extended Gauss hypergeometric function holds true:*

$$\begin{aligned} \sum_{l=0}^{\infty} (c)_l F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + l, c_2; c_3; t) \frac{x^l}{l!} &= (1 - x)^{-c_1} \\ &\times F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)} \left( c_1, c_2; c_3; \frac{t}{1 - x} \right) \\ (p \geq 0, \ a_i > 0, \ b_i \in R, \ |x| < 1). \end{aligned} \quad (41)$$

*Proof.* Using (31) on the left side of (41), we have

$$\begin{aligned} &\sum_{l=0}^{\infty} (c)_l F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + l, c_2; c_3; t) \frac{x^l}{l!} \\ &= \sum_{l=0}^{\infty} (c)_l \left[ \sum_{m=0}^{\infty} \frac{(c_1 + l)_m B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + m, c_3 - c_2)}{B(c_2, c_3 - c_2)} \frac{t^m}{m!} \right] \frac{x^l}{l!}. \end{aligned}$$

Now by using the identity  $(c)_m (c + m)_l = (c)_l (c + l)_m$ , in the above expression, we obtain

$$\begin{aligned} &\sum_{l=0}^{\infty} (c)_l F_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_1 + l, c_2; c_3; t) \frac{x^l}{l!} \\ &= \sum_{m=0}^{\infty} (c)_m \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + m, c_3 - c_2)}{B(c_2, c_3 - c_2)} \left[ \sum_{l=0}^{\infty} (c_1 + m)_l \frac{x^l}{l!} \right] \frac{t^m}{m!}. \\ &= \sum_{m=0}^{\infty} (c)_m \frac{B_p^{(a_1, \dots, a_s, b_1, \dots, b_s)}(c_2 + m, c_3 - c_2)}{B(c_2, c_3 - c_2)} (1 - x)^{-(c_1 + m)} \frac{t^m}{m!}, \end{aligned}$$

which, in view of (31), yields our claimed result (41).  $\square$

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