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NEW RESULTS ON SUPER EDGE MAGIC DEFICIENCY OF KITE GRAPHS

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Abstract: An *edge magic labeling* of a graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)|+|E(G)|\}$ such that $\lambda(u)+\lambda(uv)+\lambda(v)$ is constant, for every edge $uv \in E(G)$. The concept of edge magic deficiency was introduce by Kotzig and Rosas. Motivated by this concept Figueroa-Centeno, Ichishima and Muntaner-Batle defined a similar concept for super edge magic total labelings.

The *super edge magic deficiency* of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$, has a super edge magic total labeling or it is equal to $+\infty$ if there exists no such n. In this paper, we study the super edge magic deficiency of kite graphs.

AMS Subject Classification: 05C78

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1. Introduction

In this paper, we consider the graph G as a finite, simple and undirected graph

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and denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An *edge magic labeling* of a graph G is a bijection $\xi : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ such that $\xi(x) + \xi(xy) + \xi(y)$ constant, for every edge $xy \in E(G)$. A graph with an edge magic labeling is called *edge magic graph*. An edge magic labeling ξ is called *super edge magic* if $\xi(V(G)) = \{1, 2, \dots, p\}$. A graph with super edge magic labeling is called a *super edge magic graph*.

In [15], Kotzig and Rosa proved that for any graph G there exists an edge magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of edge magic deficiency of a graph G, which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge magic and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge magic}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge magic deficiency of a graph G with n vertices, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where F_n is the nth Fibonacci number. Motivated by Kotzig and Rosa's concept of edge magic deficiency, Figueroa-Centeno et al. [9] defined a similar concept for super edge magic labeling. The super edge magic deficiency of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge magic labeling or $+\infty$ if there exists no such n, formally defined as:

$$\text{Let } M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge magic graph}\}, \text{then}$$

$$\mu_s(G) = \begin{cases} \min\ M(G), & \text{if } M(G) \neq \phi; \\ +\infty, & \text{if } M(G) = \phi. \end{cases}$$

As a consequence of the above two definitions, we note that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [9, 10], Figueroa-Centeno et al. provided the exact values for the super edge magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. They proved that

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \equiv 0 \pmod 4 \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

For more details, the results on edge magic and super edge magic labeling of some graphs seen in [3, 5, 6, 7, 9, 12, 13, 16] and a complete survey [11].

In [18] Wallis posed the problem of investigating the edge magic properties of C_n with the path of length t attached to one vertex. Kim and Park [14] call such a graph an $(n,t)-$kite. The following proposition, proved by Ahmad and Muntaner-Batle [2], show that for an $(n,t)-$kite to be super edge-magic, n and t must have same parity.

Proposition 1. ([2]) *Let $G = (n,t)-$kite. If G is super edge-magic, then n and t have the same parity.*

In proving our results, we frequently use the following lemma:

Lemma 1. ([8]) *A graph G with p vertices and q edges is super edge magic total if and only if there exists a bijective function $\phi : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\phi(x) + \phi(y) : xy \in E(G)\}$ consists of q consecutive integers. In such a case, ϕ extends to super edge magic total labeling of G.*

Kim and Park [14] proved that an $(n,1)-$kite is super edge-magic if and only if n is odd and an $(n,3)-$kite is super edge magic if and only if n is odd and at least 5. Also, Park, Choi and Bae [17] proved that an $(n,2)-$kite is super edge magic if and only if n is even. From Proposition 1, $(n,t)-$kite is not super edge magic if n is odd and t is even.

In [2], Ahmad et al. determined the exact value of super edge magic deficiency of (n,t)-kite graph for all odd n; $t \equiv 0, 1 \pmod 4$ and also showed the upper bound for all odd n, $t \equiv 2, 3 \pmod 4$. In [4], Ahmad et al. determined the upper bound for all odd n and $t \equiv 3, 7 \pmod 8), t \neq 11$. In the next lemma, we determined the upper bound for $t = 11$.

Lemma 2. *For all odd $n \geq 3$, let $G = (n, 11)$ be a kite graph. Then $\mu_s(G) \leq 2$.*

Proof. Let $G^* = G \cup 2K_1$, the vertex set and edge set of G^* are defined as:

$$V(G^*) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq 11\} \cup \{z_1, z_2\},$$

$$E(G^*) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_j y_{j+1} : 1 \leq j \leq 10\} \cup \{x_n x_1, y_1 x_n\}.$$

We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 2\}$ of the graph G^* in the following way:

$$\psi(x_i) = \begin{cases} \frac{i}{2}, & \text{for even } i;\ \ 1 \leq i \leq n, \\ \frac{n+12+i}{2}, & \text{for odd } i;\ \ 1 \leq i \leq n, \end{cases}$$

$$\psi(y_j) = \begin{cases} \frac{n+j}{2}, & \text{for odd } j;\ 1 \le j \le 5, \\ \frac{2n+12+j}{2}, & \text{for even } j;\ 1 \le j \le 8, \end{cases}$$

$\psi(y_7) = \frac{n+11}{2}$, $\psi(y_9) = \frac{n+7}{2}$, $\psi(y_{11}) = \frac{n+9}{2}$, $\psi(y_{10}) = n + 13$.

The isolated vertices z_1, z_2 under the labeling ψ are labeled as

$$\psi(z_1) = n + 11, \quad \psi(z_1) = n + 12.$$

It is easy to check that the edge sums are $\frac{n+15}{2}, \frac{n+17}{2}, \frac{n+19}{2}, \ldots, \frac{3n+35}{2}$. Therefore by Lemma 1, ψ can be extended to a super edge magic total labeling. This shows that $\mu_s(G) \le 2$, which completes the proof. □

Ahmad et al. [1] found the exact value of super edge magic deficiency of $(n,t)-$kite graph for n even and $t = 1, 3$. In [4], Ahmad et al. found the upper bound and exact value of super edge magic deficiency of $(n,t)-$kite graph for $n \equiv 2 \pmod 4$, $t = 4$ and $t = 5$, respectively. In next lemma and theorems, we found the upper bound of super edge magic deficiency of $(n,t)-$kite graph for $n \equiv 2 \pmod 4$ and for all $t \ge 6$.

Lemma 3. *For all odd $t \ge 7$ and $n \ge 10$, $n \equiv 2 \pmod 4$ with $n > t$ and $n - t = 1, 3$. Then $\mu_s(G) \le 2$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be a vertex sequence of C_n and let $v_1, v_2, \ldots, v_t$ be the vertices of the path (the tail). Let $G^* = G \cup 2K_1$, the vertex set and edge set of G^* are defined as:

$$V(G^*) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le t\} \cup \{z_1, z_2\},$$

$$E(G^*) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{v_j v_{j+1} : 1 \le j \le t-1\} \cup \{u_n u_1, u_n v_1\}.$$

We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 2\}$ of the graph G^* in the following way:

$$\psi(u_i) = \begin{cases} \frac{n+t+3}{2}, & \text{for } \ i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 3 \le i \le n, \\ \frac{n+t+3+i}{2}, & \text{for even } i;\ 1 \le i \le n. \end{cases}$$

- When $n - t = 1$:

$$\psi(v_j) = \begin{cases} \frac{n}{2}, & \text{for } \ j=1, \\ \frac{n+1+j}{2}, & \text{for odd } j; \ 3 \le j \le t, \\ \frac{2n+t+3+j}{2}, & \text{for even } j; \ 1 \le j \le \frac{t-1}{2}, \\ \frac{2n+t+5+j}{2}, & \text{for even } j; \ \frac{t+1}{2} \le j \le t. \end{cases}$$

The isolated vertices are labeled as $\psi(z_1) = \frac{n+2}{2}$, $\psi(z_2) = \frac{4n+3t+5}{2}$.

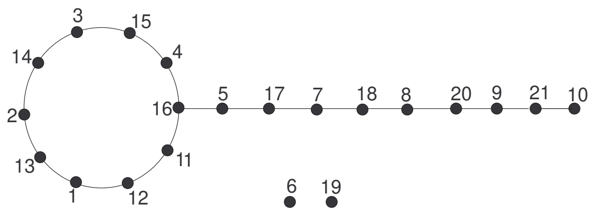

Figure 1: An illustration of the labeling for $n - t = 1$.

- When $n - t = 3$:

$$\psi(v_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } \ j; \ 1 \le j \le \frac{t-1}{2}, \\ \frac{n+1+j}{2}, & \text{for odd } j; \ \frac{t+1}{2} \le j \le t, \\ \frac{2n+t+5+j}{2}, & \text{for even } j; \ 1 \le j \le t. \end{cases}$$

The isolated vertices are labeled as $\psi(z_1) = \frac{2n+t+1}{4}, \psi(z_2) = \frac{2n+t+5}{2}$.

One can see that all edge sums form the set of q consecutive integers: $\{\frac{n+t+7}{2}, \frac{n+t+9}{2}, \ldots, \frac{3n+3t+5}{2}\}$. Applying Lemma 1, ψ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

Theorem 1. *For all odd $t \ge 7$ and $n \ge 14$, $n \equiv 2 \pmod 4$, the super edge magic deficiency of $G = (n,t)$-kite graph is*

$$\mu_s(G) \begin{cases} \le 2, & \textit{for } \ n > t \ \textit{ and } n - t = 4a+1, \ a = 1,2,3,\ldots \\ \le 3, & \textit{for } \ n > t \ \textit{ and } n - t = 4a+3, \ a = 1,2,3,\ldots \end{cases}$$

Proof. ***Case 1.*** When $n > t$ and $n - t = 4a+1$, $a = 1,2,3,\ldots$:

Let $G^* = G \cup 2K_1$, the vertex set and edge set of G^* are same as in Lemma 3. We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)|+2\}$ of the graph G^*

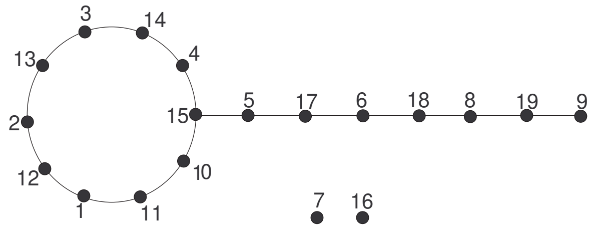

Figure 2: An illustration of the labeling for $n - t = 3$.

in the following way:

$$\psi(u_i) = \begin{cases} \frac{n+t+1}{2}, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 3 \le i \le n, \\ \frac{n+t+1+i}{2}, & \text{for even } i;\ 1 \le i \le n-2a, \\ \frac{n+t+3+i}{2}, & \text{for even } i;\ n-2a+1 \le i \le n, \end{cases}$$

$$\psi(v_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j;\ 1 \le j \le t, \\ \frac{2n+t+3+j}{2}, & \text{for even } j;\ 1 \le j \le \frac{t-1}{2}, \\ \frac{2n+t+5+j}{2}, & \text{for even } j;\ \frac{t+1}{2} \le j \le t. \end{cases}$$

The isolated vertices are labeled as $\psi(z_1) = \frac{2n+t-2a+3}{2}, \psi(z_2) = \frac{4n+3t+9}{4}$.

Case 2. When $n > t$ and $n - t = 4a + 3$, $a = 1, 2, 3, \ldots$:
Let $G^* = G \cup 3K_1$, the vertex set and edge set of G^* be defined as:

$$V(G^*) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le t\} \cup \{z_1, z_2, z_3\},$$

$$E(G^*) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{v_j v_{j+1} : 1 \le j \le t-1\} \cup \{u_n u_1, u_n v_1\}.$$

We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 3\}$ of the graph G^* in the following way:

$$\psi(u_i) = \begin{cases} \frac{n+t+3}{2}, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 3 \le i \le n, \\ \frac{n+t+3+i}{2}, & \text{for even } i;\ 1 \le i \le n-2a, \\ \frac{n+t+5+i}{2}, & \text{for even } i;\ n-2a+1 \le i \le n, \end{cases}$$

$$\psi(v_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j;\ 1 \le j \le t, \\ \frac{2n+t+5+j}{2}, & \text{for even } j;\ 1 \le j \le \frac{t+1}{2}, \\ \frac{2n+t+7+j}{2}, & \text{for even } j;\ \frac{t+3}{2} \le j \le t. \end{cases}$$

The isolated vertices are labeled as

$$\psi(z_1) = \frac{n+t+1}{2}, \psi(z_2) = \frac{2n+t-2a+5}{2}, \psi(z_3) = \frac{4n+3t+15}{4}.$$

The edge sums under the labeling ψ are q consecutive integers from the set

$$\{\frac{n+t+5}{2}, \frac{n+t+7}{2}, \dots, \frac{3n+3t+3}{2}\}$$

for *Case 1*, and

$$\{\frac{n+t+7}{2}, \frac{n+t+9}{2}, \dots, \frac{3n+3t+5}{2}\}$$

for *Case 2*. Then according to Lemma 1, ψ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

Theorem 2. *For all odd $t \ge 7$ and $n \ge 6$, $n \equiv 2 \pmod 4$ with $t > n$, the super edge magic deficiency of $G = (n,t)$-kite graph is $\mu_s(G) \le 2$.*

Proof. Let $G^* = G \cup 2K_1$, the vertex set and edge set of G^* are same as in Lemma 3. We define the labeling $\psi : V(G^*) \to \{1, 2, \dots, |V(G)| + 2\}$ of the graph G^* in the following way:

$$\psi(u_i) = \begin{cases} \frac{n+t+3}{2}, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 3 \le i \le n, \\ \frac{n+t+3+i}{2}, & \text{for even } i;\ 1 \le i \le n. \end{cases}$$

Case 1. When $t - n = 4s + 1$, $s = 0, 1, 2, \dots$:

$$\psi(v_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j;\ 1 \le j \le \frac{t-1}{2}, \\ \frac{n+1+j}{2}, & \text{for odd } j;\ \frac{t+1}{2} \le j \le t, \\ \frac{2n+t+3+j}{2}, & \text{for even } j;\ 1 \le j \le 2s+2, \\ \frac{2n+t+5+j}{2}, & \text{for even } j;\ 2s+4 \le j \le t. \end{cases}$$

The isolated vertices are labeled as $\psi(z_1) = \frac{2n+t+1}{2}$, $\psi(z_2) = \frac{2n+t+2s+7}{2}$.

Case 2. When $t - n = 4s - 1, \ s = 1, 2, 3, \ldots :$

$$\psi(v_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j; \ 1 \le j \le 2s+1, \\ \frac{n+1+j}{2}, & \text{for odd } j; \ 2s+3 \le j \le t, \\ \frac{2n+t+3+j}{2}, & \text{for even } j; \ 1 \le j \le \frac{t-1}{2}, \\ \frac{2n+t+5+j}{2}, & \text{for even } j; \ \frac{t+1}{2} \le j \le t. \end{cases}$$

The isolated vertices are labeled as $\psi(z_1) = \frac{n+2s+2}{2}, \psi(z_2) = \frac{4n+3t+9}{2}$.

It is easy to check that all edge sums in both cases form the same set of q consecutive integers as in Lemma 3. Therefore by Lemma 1, ψ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

Theorem 3. *For $n \ge 6$, $n \equiv 2 \pmod 4$ and $t \ge 8$, $t \equiv 0 \pmod 4$, the super edge magic deficiency of $G = (n,t)$-kite graph is*

$$\mu_s(G) \begin{cases} \le 2, & \text{for } \ n > t \ \text{ and } n - t = 2, \\ \le 3, & \text{for } \ n > t \ \text{ and } n - t = 4r + 2, \ \ r = 1, 2, 3, \ldots, \\ \le 2, & \text{for } \ t > n \ \text{ and } t - n = 4r + 2, \ \ r = 0, 1, 2, \ldots. \end{cases}$$

Proof. Let $a_1, a_2, \ldots, a_n$ be a vertex sequence of C_n and let $b_1, b_2, \ldots, b_t$ be the vertices of the path (the tail). Let $G = (n,t)$ kite graph, the vertex set and edge set of G are defined as:

$$V(G) = \{a_i : 1 \le i \le n\} \cup \{b_j : 1 \le j \le t\},$$

$$E(G) = \{a_i a_{i+1} : 1 \le i \le n-1\} \cup \{b_j b_{j+1} : 1 \le j \le t-1\} \cup \{a_n a_1, a_n b_1\}.$$

Case 1. When $G^* = G \cup 2K_1$:

We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 2\}$ of the graph G^* in the following way:

$$\psi(a_i) = \begin{cases} \frac{n+t+2}{2}, & \text{for } \ i = 1, \\ \frac{i-1}{2}, & \text{for odd } i; \ 3 \le i \le n, \\ \frac{n+t+2+i}{2}, & \text{for even } i; \ 1 \le i \le n, \end{cases}$$

$$\psi(b_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j;\ 1 \le j \le \frac{t-2}{2},, \\ \frac{n+1+j}{2}, & \text{for odd } j;\ \frac{t}{2} \le j \le t. \end{cases}$$

The isolated vertex c_1, is labeled as $\psi(c_1) = \frac{2n+t}{4}$.

- When $n > t$ *and* $n - t = 2$,

$$\psi(b_j) = \frac{2n+t+4+j}{2}, \quad \text{for even } j;\ 1 \le j \le t.$$

The isolated vertex c_2, is labeled as $\psi(c_2) = \frac{2n+t+4}{2}$.

- When $t > n$ *and* $t - n = 4r + 2$, $r = 0, 1, 2, \ldots,$

$$\psi(b_j) = \begin{cases} \frac{2n+t+2+j}{2}, & \text{for even } j;\ 1 \le j \le 2r+2,, \\ \frac{2n+t+4+j}{2}, & \text{for even } j;\ 2r+4 \le j \le t. \end{cases}$$

The isolated vertex c_2, is labeled as $\psi(c_2) = \frac{2n+2r+t+6}{2}$.

Case 2. When $G^* = G \cup 3K_1$ and $n > t$, $n - t = 4r + 2$, $r = 1, 2, 3, \ldots:$
We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 3\}$ of the graph G^* in the following way:

$$\psi(a_i) = \begin{cases} \frac{n+t+2}{2}, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 3 \le i \le n, \\ \frac{n+t+2+i}{2}, & \text{for even } i;\ 1 \le i \le n-2r, \\ \frac{n+t+4+i}{2}, & \text{for even } i;\ n-2r+2 \le i \le n, \end{cases}$$

$$\psi(b_j) = \begin{cases} \frac{n-1+j}{2}, & \text{for odd } j;\ 1 \le j \le t, \\ \frac{2n+t+4+j}{2}, & \text{for even } j;\ 1 \le j \le \frac{t}{2}, \\ \frac{2n+t+6+j}{2}, & \text{for even } j;\ \frac{t+2}{2} \le j \le t. \end{cases}$$

The isolated vertices are labeled as

$$\psi(c_1) = \frac{n+t}{2}, \psi(c_2) = \frac{2n-2r+t+4}{2}, \psi(c_3) = \frac{4n+3t+12}{2}.$$

The set of all edge sums generated by the above formula contains q consecutive integers $\frac{n+t+6}{2}, \frac{n+t+8}{2}, \ldots, \frac{3n+3t+4}{2}$. Therefore by Lemma 1, ψ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

Theorem 4. *For $n, t \geq 6$ and $n, t \equiv 2 \pmod 4$, the super edge magic deficiency of $G = (n,t)$-kite graph is $\mu_s(G) \leq 2$.*

Proof. Let $G^* = G \cup 2K_1$, the vertex set and edge set of G^* are defined as:

$$V(G^*) = \{a_i : 1 \leq i \leq n\} \cup \{b_j : 1 \leq j \leq t\} \cup \{c_1, c_2\},$$

$$E(G^*) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_j b_{j+1} : 1 \leq j \leq t-1\} \cup \{a_n a_1, a_1 b_t\}.$$

We define the labeling $\psi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 2\}$ of the graph G^* in the following way:

$$\psi(a_i) = \begin{cases} \frac{t+3+i}{2}, & \text{for odd } i;\ 1 \leq i \leq n-1, \\ \frac{t+n+4}{2}, & \text{for } \ i = n, \end{cases}$$

$$\psi(b_j) = \begin{cases} \frac{j+1}{2}, & \text{for odd } j;\ 1 \leq j \leq \frac{t}{2}, \\ \frac{j+3}{2}, & \text{for odd } j;\ \frac{t+2}{2} \leq j \leq t. \end{cases}$$

The isolated vertex c_1, is labeled as $\psi(c_1) = \frac{t+6}{4}$.

Case 1. When $t \geq n$ *and* $t - n = 4r$, $r = 0, 1, 2, \ldots$:

$$\psi(a_i) = \frac{2t+n+6+i}{2}, \qquad \text{for even } i;\ 1 \leq i \leq n-2,$$

$$\psi(b_j) = \begin{cases} \frac{t+n+4+j}{2}, & \text{for even } j;\ 1 \leq j \leq t-2r-2, \\ \frac{t+n+6+j}{2}, & \text{for even } j;\ t-2r \leq j \leq t. \end{cases}$$

The isolated vertex c_2, is labeled as $\psi(c_2) = \frac{2t+n-2r+4}{2}$.

Case 2. When $n > t$ *and* $n - t = 4r$, $r = 1, 2, 3, \ldots$:

$$\psi(b_j) = \frac{t+n+4+j}{2}, \qquad \text{for even } j;\ 1 \leq j \leq t,$$

$$\psi(a_i) = \begin{cases} \frac{2t+n+6+i}{2}, & \text{for even } i;\ 1 \leq i \leq n-2, r = 1, \\ \frac{2t+n+4+i}{2}, & \text{for even } i;\ 1 \leq i \leq 2r-1, r \neq 1, \\ \frac{2t+n+6+i}{2}, & \text{for even } i;\ 2r \leq i \leq n-2, r \neq 1. \end{cases}$$

The isolated vertex are labeled as $\psi(c_2) = \frac{2t+n+2r+4}{2}$.

As all the edge sums are q consecutive integers

$$\frac{n+t+8}{2}, \frac{n+t+10}{2}, \ldots, \frac{3n+3t+6}{2},$$

by Lemma 1, the vertex labeling ψ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

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References

[1] A. Ahmad, I. Javaid and M. F. Nadeem, Further results on super edge magic deficiency of unicyclic graphs, *Ars Combin.*, **99** (2011), 129–138.

[2] A. Ahmad and F. A. Muntaner-Batle, *On Super Edge-Magic Deficiency of Unicyclic Graphs*, Preprint.

[3] A. Ahmad, A. Q. Baig and M. Imran, On super edge-magicness of graphs, *Utilitas Math.*, **89** (2012), 373–380.

[4] A. Ahmad, M.K. Siddiqui, M. F. Nadeem and M. Imran, On super edge magic deficiency of kite graps, *Ars Combin.*, **107** (2012), 201–208.

[5] A. Q. Baig, *Super Edge-Mmagic Deficiency of Forests*, PhD Thesis (2011).

[6] A. Q. Baig, E. T. Baskoro and A. Semaničová–Feňovčíková, On the super edgemagic deficiency of a star forest, *Ars Combin.*, **106** (2012), 115–125.

[7] A. Q. Baig, A. Ahmad, E. T. Baskoro and R. Simanjuntak, On the super edgemagic deficiency of forests, *Utilitas Math.*, **89** (2011), 147–159.

[8] R. M. Figueroa, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labeling among other classes of labeling, *Discrete Math.*, **231** (2001), 153–168.

[9] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 98–409.

[10] R. M. Figueroa-Centeno, R. Ichishima, F. A. Muntaner-Batle, On the super Edge-Magic Deficiency of Graphs, *Ars Combin.*, **78** (2006), 33–46.

[11] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*; http://www.combinatorics.org/surveys/ds6.pdf (2010).

[12] M. Hussain, E. T. Baskoro, K. Ali, On super antimagic total labeling of Harary graph, *Ars Combin.*, **92** (2016), 120–130.

[13] M. Hussain, E. T. Baskoro, Slamin, On super edge-magic total labeling of banana trees, *Utilitas Math.*, **79** (2009), 243-251.

[14] S.-R. Kim and J. Y. Park, On super edge-magic graphs, *Ars Combin.*, **81** (2006), 113–127.

[15] A. Kotzig and A. Rosa, Magic valuaton of finite graphs, *Canad. Math. Bull.* **13(4)** (1970), 451–461.

[16] A.A.G. Ngurah, R. Simanjuntak, and E.T. Baskoro, On the super edge-magic deficiencies of graphs, *Australas. J. Combin.*, **40** (2008), 3–14.

[17] J. Y. Park, J. H. Choi and J-H. Bae, On super edge-magic labeling of some graphs, *Bull. Korean Math. Soc.*, **45** (2008), 11–21.

[18] W. D. Wallis, *Magic Graphs*, Birkhäuser, Boston, 2001.

[19] W. D. Wallis, E. T. Baskoro, M. Miller, and Slamin, Edge-magic total labelings, *Australas. J. Combin.*, **22** (2000), 177–190.