

**EXTREMAL HYPER ZAGREB INDEX
FOR TRICYCLIC GRAPHS**

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Abstract: For a graph $G = (V(G), E(G))$, the first hyper Zagreb index is defined as $\sum_{uv \in E(G)} (d(u) + d(v))^2$, where $d(v)$ is the degree of the vertex v . The hyper Zagreb index is a kind of extensions of Zagreb index. In this paper, the monotonicity of the hyper Zagreb index under some graph transformations was studied. Using these mathematical properties, the extremal graph among tricyclic graphs are determined for hyper Zagreb index. Moreover, the sharp upper and lower bounds on the hyper Zagreb index of tricyclic graphs are provided.

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1. Introduction

Let G be a simple, finite and connected graph, with vertex set V and edge set E . In a graph G the number of element in V and E is called the *order* and *size*, respectively, of G . For a vertex $v \in V(G)$, the degree of v is the number of vertices attached to the vertex v and denoted as $d(v)$.

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A *topological index* of a graph G is a numerical quantity that associated to the graph G with the property that all the graphs isomorphic to G have same quantity. The *first* and *second Zagreb index* of a graph G is defined as:

$$M_1(G) = \sum_{v \in V} d(v)^2 \quad , \quad M_2(G) = \sum_{uv \in E} d(u)d(v).$$

In 1972, these topological indices appeared for the first time to find the total π -energy of molecular graphs [9]. Later, the Zagreb indices developed significant applications in QSPR/QSAR studies and a lot of research paper have been published on these, see [1, 2, 3, 4, 5, 6].

In 2013 Shirdel and co-authors [10] proposed a new version of Zagreb indices called *hyper Zagreb index* HM_1 . The hyper-Zagreb index is for a graph G defined as:

$$HM_1(G) = \sum_{e \in E} d(e)^2,$$

where $d(e) = d(u) + d(v)$ for the edge $e = uv$. Recently, in 2016 Jamil et al. introduced another version of hyper Zagreb index named as *second-hyper Zagreb index*, defined as

$$HM_2(G) = \sum_{e \sim f} d(e)d(f),$$

where $e \sim f$ represents that the edges e and f have a common vertex.

Recent results on the hyper Zagreb indices can be seen in [7, 8, 11, 12, 13, 14, 15].

We now define some notations which will use in later. Let Υ_n denotes the set of all connected tricyclic graphs with order n . For a tricyclic graph, the graph which is attained by removing its all pendent vertices is named as a *brace* of the graph. Denote by Υ_n^0 the set of all braces of tricyclic graphs as depicted in Fig. 6. Let Υ_n^1 and Υ_n^2 denote the sets of tricyclic graphs shown in Figs. 5 and 7, respectively.

In [8] authors studied the monotonicity of the hyper-Zagreb index with the help of some transformations, they also determined the upper and lower bounds of hyper-Zagreb index on acyclic, unicyclic and bicyclic graphs. In this research paper, we extend the results and find the lower and upper bounds on tricyclic graphs.

2. Methods and definitions

In this section, we introduced some graph transformations, with the help of these transformations we will investigate the extremal graphs with the first hyper-Zagreb index in the set of all connected tricyclic graphs. First we introduced the transformations which strictly increases the first hyper-Zagreb index of a graph.

Transformation 1. Let uv be an edge of connected graph G with $d_G(v) \geq 2$. Assume that $\{v, w_1, w_2, \dots, w_t\}$ are all the adjacent to the vertex u while w_1, w_2, \dots, w_t are pendant vertices. If

$$K = G - \{uw_1, uw_2, \dots, uw_t\} + \{vw_1, vw_2, \dots, vw_t\},$$

we say that K is attained from G by Transformation 1. As shown in Fig. 1.

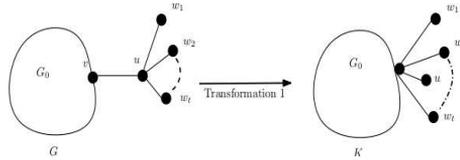


Figure 1: Transformation 1.

Transformation 1 strictly increases the HM_1 of a graph.

Lemma 1. ([8]) If K is attained from G by Transformation 1 as depicted in Fig. 1, then

$$HM_1(G) < HM_1(K).$$

Proof. Clearly, $d_G(v) < d_K(v)$ and $(d(u) + d(v))$ is not changed during Transformation 1. Hence,

$$\begin{aligned} HM_1(K) - HM_1(G) &> (t + 1)(d_G(v) + t + 1)^2 - (d_G(v) + d_G(u))^2 \\ &\quad - t(d_G(u) + 1)^2 \\ &= (t + 1)(d_G(v) + t + 1)^2 - (d_G(v) + t + 1)^2 \\ &\quad - t(t + 2)^2 \\ &= t(d_G(v) + t + 1)^2 - t(t + 2)^2 > 0. \end{aligned}$$

□

Transformation 2. Let G be nontrivial connected graph and $u, v \in V(G)$. Let $P_a = (u =)v_1v_2 \cdots v_a(= v)$ is a nontrivial a -length path of G connecting vertices u and v . If $K = G - \{v_1v_2, v_2v_3, \dots, v_{a-1}v_a\} + \{(u + v =)wv_1, wv_2, \dots, wv_a\}$, we say that K is attained from G by Transformation 2. As, shown in Fig. 2.

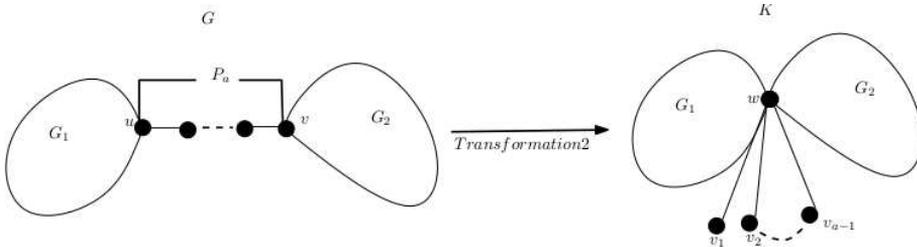


Figure 2: Transformation 2.

Lemma 2. ([8]) If K is obtained from G by Transformation 2 as depicted in Fig. 2, then

$$HM_1(K) > HM_1(G).$$

Proof. From Fig. 2, let $d_{G_1}(u) = x$ and $d_{G_2}(v) = y$ while $w = u + v$ (merge u and v to obtain w) with $d_K(w) = x + y + a - 1$, where $a \geq 2$. If $a = 2$,

$$HM_1(K) - HM_1(G) > (x + y + 2 - 1 + 1)^2 - (x + y + 2)^2 = 0.$$

If $a \geq 3$,

$$\begin{aligned} HM_1(K) - HM_1(G) &> (a - 1)(x + y + a - 1 + 1)^2 - (x + 3)^2 \\ &\quad - (y + 3)^2 + 16(a - 3) \\ &= (a - 1)(x + y + a)^2 - (x + 3)^2 - (y + 3)^2 \\ &\quad - 16(a - 3) \\ &> (x + y + a)^2 - (x + 3)^2 + (x + y + a)^2 \\ &\quad - (y + 3) > 0. \end{aligned}$$

□

Transformation 3. Let H be a nontrivial acyclic subgraph of G with $|H| = t$ which is attached at u_1 in graph G , u and v be two neighbors of u_1

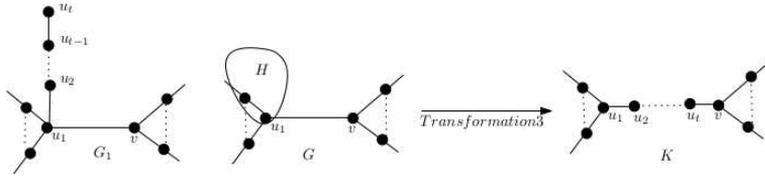


Figure 3: Transformation 3.

different from in H . If $K = G - (H - u_1) + u_1u_2 + u_2u_3 + \dots + u_tv$, we say that K is obtained from G by Transformation 3. As shown in Fig. 3.

Lemma 3. Let G and K be graphs, as depicted in Fig. 3, then

$$HM_1(G) > HM_1(K).$$

Proof. From Transformation 1 we know $HM_1(G) \geq HM_1(G_1)$. So, we only prove the following inequality:

$$HM_1(G_1) > HM_1(K).$$

By definition of HM_1 ,

$$\begin{aligned} HM_1(G_1) - HM_1(K) &= (d_{G_1}(u_{t-1}) + d_{G_1}(u_t))^2 \\ &\quad + (d_{G_1}(u_1) + d_{G_1}(u_2))^2 \\ &\quad + (d_{G_1}(u_1) + d_{G_1}(v))^2 \\ &\quad - (d_K(u_{t-1}) + d_K(u_t))^2 \\ &\quad - (d_K(u_t) + d_K(v))^2 \\ &\quad - (d_K(u_1) + d_K(u_2))^2 \\ &= (d_{G_1}(u_1) + 2)^2 + (d_{G_1}(u_1) + d_{G_1}(v))^2 \\ &\quad - (d_{G_1}(u_1) + 1)^2 - (d_{G_1}(v) + 2)^2 - 7 > 0. \end{aligned}$$

This completes the proof. □

Let G be a nontrivial connected graph. Two vertices u and v are said to be equivalent, if $G - u \cong G - v$. Clearly, $|N(u)| = |N(v)|$ and their neighbors have the same degree sequence.

Transformation 4. Let G_0 be a nontrivial connected graph and u and v are two vertices in G_0 with $d_{G_0}(u) = x, d_{G_0}(v) = y$ and $N_{G_0}(u) \subseteq N_{G_0}(v)$. Let

G be the graph obtained by attaching S_{k+1} and S_{l+1} at the vertices u and v of G_0 , respectively. If K is the graph attained by removing the l pendent vertices at v in G and connecting them to u of G , as shown in Fig. 4. We say that K is obtained from G by Transformation 4.

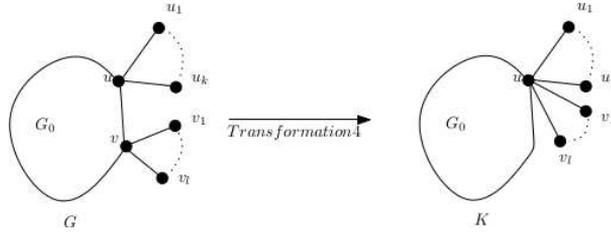


Figure 4: Transformation 4.

Lemma 4. If K is attained from G by Transformation 4 as depicted in Fig. 4, then

$$HM_1(G) < HM_1(K).$$

Proof. Since, $d_{G_0}(u) = x$, $d_{G_0}(v) = y$ and $N_{G_0}(v) \subseteq N_{G_0}(u)$. So by the definition of HM_1 ,

$$\begin{aligned} HM_1(K) - HM_1(G) &= \sum_{i=1}^k [d_K^2(uu_i) - d_G^2(uu_i)] \\ &+ \sum_{i=1}^l [d_K^2(uv_i) - d_G^2(vv_i)] \\ &+ \sum_{w \in N_{G_0}(v)} [d_K^2(uw) + d_K^2(vw)] \\ &- \sum_{w \in N_{G_0}(v)} [d_G^2(uw) + d_G^2(vw)] \\ &= (k+l)(k+l+x)^2 - k(k+x)^2 - l(l+y)^2 \\ &+ \sum_{w \in N_{G_0}(v)} [(k+l+x - d_{G_0}(w))^2 + (y + d_{G_0}(w))^2] \\ &\sum_{w \in N_{G_0}(v)} [(k+x + d_{G_0}(w))^2] \end{aligned}$$

$$+ (l + y + d_{G_0}(w))^2] > 2l(k + x - y) > 0.$$

This completes the proof. □

3. Results and discussion

In this section, we found the extremal graph for the first hyper Zagreb index HM_1 among all tricyclic graphs. To obtain the required result we will use the above discussed transformations.

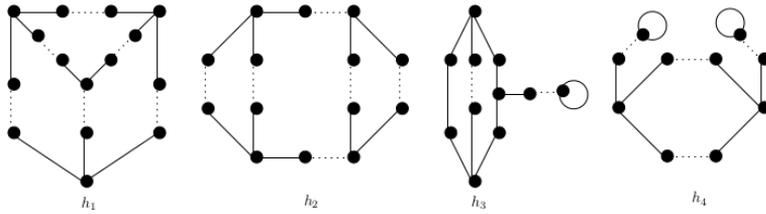


Figure 5: The h_i graphs in Υ_n^1 .

Theorem 5. Let G be any tricyclic graph with n vertices, then

$$16n + 140 \leq HM_1(G),$$

where the equality holds if and only if $G \in \Upsilon_n^1$.

Proof. Let G be a connected tricyclic graph. By Lemma 2, G can be converted to the one of the fifteen braces shown in Fig. 6. Meanwhile, for every graph G there exists a graph $g_i \in \Upsilon_n^0$, where $i \leq 15$, such that $HM_1(g_i) \leq HM_1(G)$ by 3. Clearly,

$$HM_1(h_i) = 16n + 140, \quad \text{for } i = 1, 2, 3, 4, 5.$$

So, the proof is completed. □

Theorem 6. Let G be a tricyclic graph of order n . Then

$$HM_1(G) \leq n^3 - n^2 + 12n + 120,$$

where the equality holds if and only if $G \cong S_n^{n+2}$ or $S_n^{K_4}$.

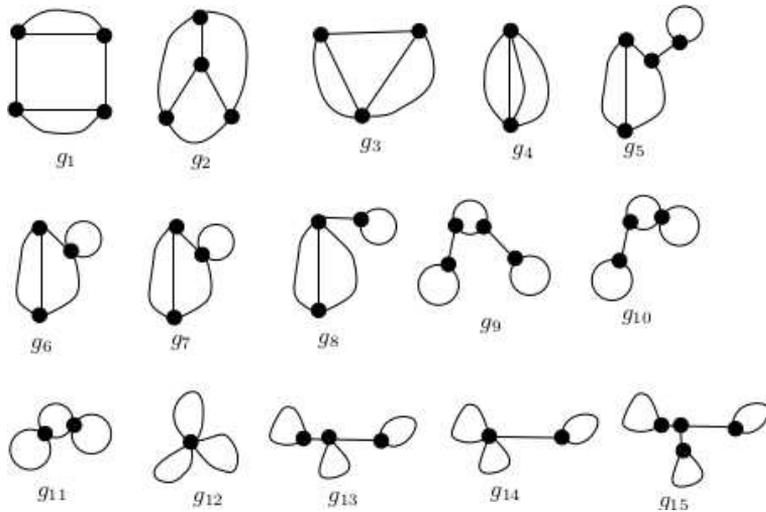


Figure 6: The g_i graphs in Υ_n^0

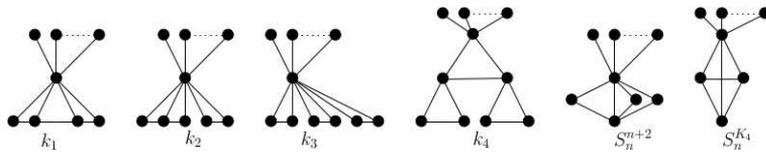


Figure 7: Some graphs.

Proof. For tricyclic graph G with given order n can be transformed to the one of the five graphs shown in Fig. 5 by repeated Transformation 2I and Transformation 4. In other words, for any tricyclic graph G with given order n there exists $k_i \in \Upsilon_n^2$, where $i \leq 6$ such that $HM_1(G) \leq HM_1(g_i)$ by Lemmas 2 and 4. Notice that,

$$\begin{aligned}
 HM_1(k_1) &= n^3 - n^2 + 12n + 96, \\
 HM_1(k_2) &= n^3 - n^2 + 12n + 74, \\
 HM_1(k_3) &= n^3 - n^2 + 12n + 54, \\
 HM_1(k_4) &= n^3 - 13n^2 + 68n + 130, \\
 HM_1(S_n^{n+2}) &= n^3 - n^2 + 12n + 120, \\
 HM_1(S_n^{K_4}) &= n^3 - n^2 + 12n + 120.
 \end{aligned}$$

Hence, the proof is completed. □

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