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FUNCTIONAL ORDER DERIVATIVES AND THE J^{α_m} OPERATOR

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Abstract: The bulk of theoretical physics involves the solutions of differential equations, which are traditionally derived from a set of theoretical axioms. The authors consider the possibility that a function and its derivative are known, possibly from experimental results, while the order of the derivative is not. In most such cases there is no constant solution. The functional calculus approach to fractional derivatives is used to develop a definition of the J^{α_m} operator in one dimension, which differentiates a function with a different order at each point in space.

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1. Motivation

The majority of theoretical physics involves solving equations of the form

$$\Omega f = g, \tag{1.1}$$

where Ω is a differential operator typically derived from a set of theoretical postulates, while f and g are functions of coordinates in phase space, which are themselves functions of time. An example of this is the Hamiltonian eigenvalue equation in non-relativistic quantum mechanics, where:

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$$\Omega = \frac{\hbar}{2\pi}\nabla^2 + V(\vec{r}) - E, \tag{1.2}$$

$$f = \psi(\vec{r}), \text{ and} \tag{1.3}$$

$$g = 0. \tag{1.4}$$

Most traditional methods in physics involve solving these equations for f and g. A typical experiment would start with a theoretical prediction for Ω and compare experimental results with those predictions. An additional approach would be the extraction an experimentally determined operator Ω from measured f and g. In this case the experimentally determined operator would be compared with the theoretical operator.

The simplest type of such equation is of the form:

$$\frac{d^\eta}{dx^\eta} f = g, \tag{1.5}$$

with the unknown to be evaluated being η.

1.1. Limitations on solvable equation

Unfortunately, in many cases there are no solutions to this equation. As an example, for any value of $\eta \geq 0$ and integer n:

$$\frac{d^\eta}{dx^\eta} \sin(x) = \sin[x + 2\pi\eta(n + \frac{1}{4})]. \tag{1.6}$$

Or, letting $n = 0$,

$$\frac{d^\eta}{dx^\eta} \sin(x) = \sin(x + \frac{\pi\eta}{2})]. \tag{1.7}$$

Thus any constant order derivative of the *sine* function results in a pure phase shift with no change in frequency, so there is no constant value of η that would solve the equation if, for example, the case where the frequency doubles: $f(x) = \sin(x)$ and $g(x) = \sin(2x)$.

2. J^{α_m} operator

However, this can be solved in terms of an operator that differentiates the function $f(x)$ with a coordinate dependent order, here referred to as the J^{α_m} operator where α_m represents the multiple values of the order of differentiation.

Definition 1. $J^{\alpha_m(x)} f(x)\Big|_{x=x_0} \equiv \frac{d^{\alpha_m(x_o)}}{dx^{\alpha_m(x_o)}} f(x)\Big|_{x=x_o}$.

In the case of $J^{\alpha_m}\sin(x) = \sin(2x)$ the solution is $\alpha(x) = \frac{2x}{\pi}$.

As the questions leading to this definition arose through questions in quantum mechanics the J^{α_m} operator will be described primarily through the functional calculus approach to non-integer differentiation (see Kempfle et al. [1] and references within) and the suitable functions $f(x)$ will be restricted to valid elements of the standard unbounded quantum mechanical physical Hilbert space, [2].

In this space K is the self-adjoint operator conjugate to the coordinate operator X. These operators satisfy the canonical commutation relation $[X, K] = iI$, where I is the identity operator. The standard, and simplest, form of the K operator is $K = -iD$, where the symbol i denotes the imaginary number. The eigenstates of the K and X operators are denoted by the kets $|k>$ and $|x>$ respectively, and for an arbitrary ket $|f>$ we can see that $f(x) =< f|x>$ and $\hat{f}(k) =< f|k>$. Following the results of Kempfle et al. [1], $\hat{f}(k) = \mathcal{F}(f(x))$ is the Fourier transform of $f(x)$ and the fractional differential operator may be defined as:

Definition 2. $D^q \equiv \mathcal{F}^{-1}(ik)^q \mathcal{F}$.

Simply replacing the constant q with a function of x, will, however, not reproduce the J^{α_m} operator.

Definition 3. $H(t, f(x)) \equiv D^t f(x) = \mathcal{F}^{-1}(ik)^t \mathcal{F}$, where t is an independent variable with the same units as x.

The H function is the two dimensional function containing all possible derivatives of the one dimensional function $f(x)$. The J^{α_m} operator defined in Definition 1 above may then be calculated by extracting the appropriate values form the H function:

$$J^{\alpha_m} f(x) = \int_{t=-\infty}^{t=\infty} dt[H(t, f(x))\delta(t - \alpha(x))], \tag{2.1}$$

where δ is the Dirac delta function.

3. Linearity

As D^t is a linear operator [1], from Definition 3:

$$H(t, af(x)) = D^t[af(x)] = aD^t f(x) = aH(t, f(x)), \text{ and} \tag{3.1}$$

$$H(t, f(x)) + H(t, g(x)) = D^t f(x) + D^t g(x) = D^t[f(x) + g(x)]. \tag{3.2}$$

This implies that the J^{α_m} operator is also linear. However, it does not follow the standard multiplication rule for differential operators. For example, one can quickly calculate the general J^{α_m} derivative of the sine function:

$$J^{\alpha_m} \sin(x) = \sin(x + \frac{\alpha\pi}{2}). \tag{3.3}$$

The first derivative of this function is also easily calculable:

$$D^1(\sin(x + \frac{\alpha\pi}{2})) = (1 + \frac{\pi}{2}D^1(\alpha))\sin(x + \frac{(\alpha+1)\pi}{2}). \tag{3.4}$$

However,

$$J^{\alpha_m+1}\sin(x) = \sin(x + \frac{(\alpha+1)\pi}{2}) = \frac{D^1 J^{\alpha_m}\sin(x)}{1 + \frac{\pi}{2}D^1(\alpha)}. \tag{3.5}$$

In general, $J^{\beta_m}(J^{\alpha_m} f(x)) \neq J^{\alpha_m+\beta_m} f(x)$ unless α_m is a constant.

4. Conclusion

In order to solve a differential equation with the unknown being the order of the derivative it was necessary to define an operator that differentiates functions with a non-constant order. The J^{α_m} operator acts on a function of x and, at point x_0 returns the derivative of order $\alpha(x_0)$.

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