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MATHEMATICAL STRUCTURES DEFINED BY IDENTITIES III

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Abstract: We extend the theory (**formal part only**) of algebras with one binary operation (our paper, Petridi [8]) to algebras with several operations of any arity.

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1. Introduction

We refer to our papers Petridi [8] and Petridi [9] for concepts, notations, definitions, notes and remarks.

In Subsection 4.1 of Petridi [8], we briefly outlined our ideas of generalizing the method of tables to algebras with several operations

$$V_1(x_1, x_2, \cdots, x_{\alpha_1}), V_2(x_1, x_2, \cdots, x_{\alpha_2}), \cdots, V_k(x_1, x_2, \cdots, x_{\alpha_k})$$

satisfying axiomatically defined identities and indicated the way of how to proceed. The project is now carried out. The technique applied is the same as in Formal Part of Petridi [8]. The crucial fact that the number $I_n^{V_1 V_2 \cdots V_k}$ of **formally** reducible identities can be calculated by exactly the same method used for $I_n^{V_1}(= I_n)$ seems to hold true. Algebras with only binary operations are discussed. For algebras with two binary operations $V(x, y)$ and $W(x, y)$, the proof is given in detail.

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Algebras with operations of any arity can be treated by reduction to a well defined set of algebras with binary operations.

Research and exposition of the general theory are impeded by problems of construction and inspection of the tables T_n whenever n is greater than 3. This is due to the fast growth of the Catalan numbers ($S_n \sim \frac{4^n}{\pi^{\frac{1}{2}} n^{\frac{3}{2}}}$) and their generalizations, let alone problems of printing and publication. Programs designed to seek the structures resulting from a given identity failed after a few steps (blow-ups). Exposition therefore is limited to illustrate the theory on the worked example of table T_3.

Still, the concrete new findings reached in this case corroborate further our fundamental thesis that there is a scarcity of existing mathematical structures in the sense that the frequency of **irreducible** identities goes to zero with increasing n. Seen historically, this also explains why mathematics, in the course of time, has developed the way it did with associativity $V(V(x, y), z) = V(x, V(y, z))$, the simplest structure, reigning supreme over the mathematical landscape. All other essential mathematical structures, found or created by research such as *e.g.* Groups, Fields, Vector Spaces, Lie Algebras, *etc*, ... include in their axiom system (signature) at least one binary operation obeying the law of associativity. We conclude with a note on the connection with Formal Languages.

Given k operations $V_1^{\alpha_1}(x_1, x_2, \cdots, x_{\alpha_1})$ of arity α_1, $V_2^{\alpha_2}(x_1, x_2, \cdots, x_{\alpha_2})$ of arity $\alpha_2, \cdots, V_k^{\alpha_k}(x_1, x_2, \cdots, x_{\alpha_k})$ of arity α_k, their n-iterates containing the operation $V_1^{\alpha_1}$ p_1- times, the operation $V_2^{\alpha_2}$ p_2- times, $\cdots$, the operation $V_k^{\alpha_k}$ p_k- times are symbolized by

$$J_i^n \begin{pmatrix} V_1^{\alpha_1} & V_2^{\alpha_2} & \cdots & V_k^{\alpha_k} \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}, \ \alpha_i \geq 0, \, p_i \geq 0.$$

The order of the iterate is

$$n = p_1 + p_2 + \cdots + p_k$$

and the number of its variable places is

$$(\alpha_1 - 1)p_1 + (\alpha_2 - 1)p_2 + \cdots + \alpha_k(p_k - 1) + 1.$$

The index i runs from 1 to $S_n^{V_1 \cdots V_k}$. We call $S_n^{V_1 \cdots V_k}$ the **n-th Catalan number** of the structure.

The numbers $S_n^{V_1 \cdots V_k}$ are the Taylor coefficients, at $t = 0$, of the formal generating function

$$\phi_{V_1\cdots V_k}(t) = \sum_{n=0}^{\infty} S_n^{V_1\cdots V_k} t^n.$$

The sequence $S_n^{V_1\cdots V_k}$ can be calculated recursively from

$$\begin{aligned} S_{n+1}^{V_1\cdots V_k} = & \sum_{\substack{x_1+x_2+\cdots+x_{\alpha_1}=n \\ x_i\geq 0}} S_{x_1}^{V_1\cdots V_k} S_{x_2}^{V_1\cdots V_k} \cdots S_{x_{\alpha_1}}^{V_1\cdots V_k} + \\ & \sum_{\substack{x_1+x_2+\cdots+x_{\alpha_2}=n \\ x_i\geq 0}} S_{x_1}^{V_1\cdots V_k} S_{x_2}^{V_1\cdots V_k} \cdots S_{x_{\alpha_2}}^{V_1\cdots V_k} + \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \sum_{\substack{x_1+x_2+\cdots+x_{\alpha_k}=n \\ x_i\geq 0}} S_{x_1}^{V_1\cdots V_k} S_{x_2}^{V_1\cdots V_k} \cdots S_{x_{\alpha_k}}^{V_1\cdots V_k}. \end{aligned}$$

According to E. Catalan, see Dickson [6] for more details, the number of solutions of the Diophantine equation $x_1 + x_2 + \cdots + x_{\alpha_i} = n$ is $\binom{a_i + n - 1}{n}$.

The function $\phi_{V_1\cdots V_k}(t)$ is a solution of the functional equation

$$\frac{\phi_{V_1\cdots V_k}(t) - 1}{t} = \sum_{i=1}^{k} (\phi_{V_1\cdots V_k}(t))^{\alpha_i}$$

with the initial condition $\phi(0) = 1$.

For $k = 1, \alpha_1 = 2$ we get the classical Catalan numbers $S_n = \frac{1}{n+1}\binom{2n}{n}$, which count the n-iterates (parenthesizing) of $V_1(x, y)$. Their recursion formula is

$$S_{n+1}^{V_1} = \sum_{\substack{x_1+x_2=n \\ x_i\geq 0}} S_{x_1}^{V_1} S_{x_2}^{V_1}$$

and the functional equation becomes

$$\frac{\phi_{V_1}(t) - 1}{t} = (\phi_{V_1}(t))^2,$$

giving

$$\phi_{V_1}(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n=0}^{\infty} S_n t^n.$$

For $k = 1$, $\alpha_1 = \alpha$, we obtain the higher Catalan numbers $\dfrac{1}{(\alpha-1)n+1}\left(\begin{array}{c}\alpha n\\ n\end{array}\right)$, whose generating function $\phi_\alpha(t)$ satisfies

$$\frac{\phi_\alpha(t)-1}{t} = (\phi_\alpha(t))^\alpha.$$

2. Binary operations

We will now examine the case of two binary operations $V(x, y)$ and $W(x, y)$. Since $k = 2$, $\alpha_1 = \alpha_2 = 2$ the corresponding generating function which gives the number of iterates of order n is

$$\frac{\phi_{VW}(t)-1}{t} = 2(\phi_{VW}(t))^2, \ \phi(0) = 1.$$

Solving the quadratic equation we obtain

$$\phi_{VW}(t) = \frac{1-\sqrt{1-8t}}{4t} = \sum_{n=0}^{\infty} 2^n\, S_n t^n,$$

where $S_n = \dfrac{1}{n+1}\left(\begin{array}{c}2n\\ n\end{array}\right)$ are the ordinary Catalan numbers.

Hence the number of iterates of order n is $S_n^{VW} = 2^n S_n$, $n \geq 1$, the first of which are

n	S_n^{VW}
1	2
2	8
3	40
4	224
5	1344
$\vdots$	$\vdots$

.

Following the same rules of formation as done in Petridi [8], the first three A–tables are:

T_1	T_2	
Vxx	$VVxxx$	$WVxxx$
Wxx	$VWxx$	$WWxxx$
	$VxVxx$	$WxVxx$
	$VxWxx$	$WxWxx$

$\underline{T_3}$

$VVVxxxx$	$WVVxxxx$	$\cdots$	$VVxxWxx$	$WVxxWxx$
$VVWxxxx$	$WVWxxxx$	$\cdots$	$VWxxWxx$	$WWxxWxx$
$VVxVxxx$	$WVxVxxx$	$\cdots$	$VxWVxxx$	$WxWVxxx$
$VVxWxxx$	$WVxWxxx$	$\cdots$	$VxWWxxx$	$WxWWxxx$
$VVxxVxx$	$WVxxVxx$	$\cdots$	$VxWxVxx$	$WxWxVxx$
$VVxxWxx$	$WVxxWxx$	$\cdots$	$VxWxWxx$	$WxWxWxx$

Because of lack of space, in T_3 figure only the first two and the last two columns, the four columns in the middle having been omitted. After labeling these word expressions from 1 to 40, the tables can be perused as seen below. The importance of perusal and inspection of tables was aptly pointed out by D.H. Lehmer in his article in *MAA Studies in Mathematics*, Vol. **6**, 1969, see [7] and the references therein.

$\underline{T_1}$

1
2

$\underline{T_2}$

1	2
3	4
5	6
7	8

$\underline{T_3}$

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
5	6	13	14	33	34	35	36
7	8	15	16	37	38	39	40

The general table of order n has $2n$ lines and $2^{n-1}\,S_{n-1}$ columns, that is a total of $2^n\,n\,S_{n-1}$ entries. For $n \geq 3$ it is easy to see that some n–iterates appear in table T_n with multiplicities higher than 1, as can be verified in table T_3. To prove it, we have to show that $2^n\,n\,S_{n-1} > 2^n\,S_n$ for $n \geq 3$. The easy proof is as follows. Using the recursion $S_n = \dfrac{2(2n-1)}{n+1}S_{n-1}$ for the Catalan numbers we have

$$\begin{aligned} 2^n\,n\,S_{n-1} &> 2^n\,S_n \\ n\,S_{n-1} &> \frac{2(2n-1)}{n+1}S_{n-1} \\ n(n+1) &> 2(2n-1) \end{aligned}$$

The last inequality holding true for $n \geq 3$, application of the pigeonhole principle does the rest.

All concepts and definitions of Petridi [8] relating to one binary operation $V(x, y)$ can be carried over literally to the present case. Regrettably, because of the reasons explained in Section 1, we were unable to go further than table T_3. We were lucky, however, to discover that already for the incidence matrix of this table, the fundamental theorem of Subsection 2.4 of Petridi [8], which is the key enabling to calculate the number I_n of formally reducible identities, remains true. Because of the highly peculiar nature of this property we surmise that it is equally true for all higher tables T_n. For easy reference we repeat the theorem hereunder.

Theorem 1. *Let*

1. $\delta(J_i^n, J_j^n) = \begin{cases} 1 \textit{ if } J_i^n = J_j^n \textit{ reducible} \\ 0 \textit{ if } J_i^n = J_j^n \textit{ irreducible} \end{cases}$
2. $M(J_i^n) =$ *the multiplicity of* J_i^n *in table* A_n
3. $I_n = \sum_{i,j}^{S_n} \delta(J_i^n, J_j^n) =$ *the number of reducible* n*–identities*
4. $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) =$ *the number of reducible* n*–identities on the* i*–th line of the incidence matrix of table* A_n,

then

$$\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) = \sum_{\nu=1}^{M(J_i^n)} (-1)^{\nu-1} \binom{M(J_i^n)}{\nu} S_{n-\nu}.$$

Expressed in words, the theorem says that the number of reducible n–identities on the i–line of the incidence matrix of table A_n does not depend on J_i^n but only on its multiplicity $M(J_i^n)$. An immediate consequence is that

$$I_n = \sum_{k=1}^{[\frac{n+1}{2}]} T_{nk} \, (\sum_{\nu=1}^{k} (-1)^{\nu-1} \binom{k}{\nu} S_{n-\nu}),$$

where T_{nk} is the number of iterates in table T_n having multiplicity k. As proved in Petridi [8], I_n is

$$I_n = o(1 - e^{-\frac{n}{16}})$$

and the scarcity of the reducible identities is evinced by

$$S_n^2 - I_n = o(e^{-\frac{n}{26}}).$$

We now will prove the validity of this theorem for table T_3. To this end we have calculated the incidence matrix relative to table T_3 as shown in the appendix attached hereto.

Proof. The proof leaps to the eye. Indeed, the four iterates $J_5^3 = 5$, $J_6^3 = 6$, $J_7^3 = 7$, $J_8^3 = 8$ have all multiplicity 2, giving a sum $\sum 1 = 14$. Similarly for the iterates J_{13}^3, J_{14}^3, J_{15}^3, J_{16}^3. All other 32 iterates have multiplicity 1 with a sum $\sum 1 = 8$. Hence the number I_3^{VW} of reducible 3–identities is

$$I_3^{VW} = 32 \cdot 8 + 8 \cdot 14 = 368$$

and the relative frequency is

$$\frac{I_3^{VW}}{(2^3 \ S_3)^2} = \frac{368}{1600} = 0.28.$$

The main objective is of course to prove that

$$\lim_{n \to \infty} \frac{I_n^{VW}}{(2^n \ S_n)^2} = 1,$$

which would imply that **irreducible** identities are getting scarce with increasing n. Expressed otherwise, this would mean that there are no algebras defined by **lengthy** identities involving two binary operations.

The case of algebras with more than two binary operations $V_1, V_2, \cdots, V_\lambda$, $\lambda > 2$, can be dealt with in the same way we did for $\lambda = 2$. The functional equation for the generating function $\phi_{V_1 \, V_2 \cdots V_\lambda}(t)$ turns out to be

$$\frac{\phi_{V_1 \, V_2 \cdots V_\lambda}(t) - 1}{t} = \lambda(\phi_{V_1 \, V_2 \cdots V_\lambda}(t))^2$$

which after solving gives the corresponding "Catalan" numbers of the structure

$$S_n^{V_1 \, V_2 \cdots V_\lambda} = \lambda^n \, S_n \;\; (S_n = n - \text{th Catalan number}).$$

□

3. Operations of any arity

A direct approach to form the tables for several operations of arities higher than two is well nigh impossible without the use of powerful computers. If at all even then. We may circumvent, however, the obstacle by reducing the problem to the binary case as follows.
Given the operations

$$\begin{array}{r} V^{\alpha_1}(x_1,\, x_2,\, \cdots,\, x_{\alpha_1}) \\ V^{\alpha_2}(x_1,\, x_2,\, \cdots,\, x_{\alpha_2}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ V^{\alpha_m}(x_1,\, x_2,\, \cdots,\, x_{\alpha_m}) \end{array}$$

we form their respective $\binom{\alpha_i}{2}$, projections on the subspaces of the variable places

$$V^{\alpha_1}_{jkj}(x,\, y) \;=\; V^{\alpha_1}(\underbrace{c,\, \cdots,\, c}_{i},\, x,\, \underbrace{c,\, \cdots,\, c}_{j}\, x\, \underbrace{c,\, \cdots,\, c}_{k}),$$

taken over all solutions of $i + j + k = \alpha_1 - 2$,

$$V^{\alpha_2}_{jkj}(x,\, y) \;=\; V^{\alpha_2}(\underbrace{c,\, \cdots,\, c}_{i},\, x,\, \underbrace{c,\, \cdots,\, c}_{j}\, x\, \underbrace{c,\, \cdots,\, c}_{k}),$$

taken over all solutions of $i + j + k = \alpha_2 - 2$,

$$\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots$$

$$V^{\alpha_m}_{jkj}(x,\, y) \;=\; V^{\alpha_m}(\underbrace{c,\, \cdots,\, c}_{i},\, x,\, \underbrace{c,\, \cdots,\, c}_{j}\, x\, \underbrace{c,\, \cdots,\, c}_{k}),$$

taken over all solutions of $i + j + k = \alpha_m - 2$. Since all these projections are binary operations, we can apply to them the results of the previous section 3 and conclude that the fundamental theorem holds true.

4. Connection with formal languages

Seen from the angle of Formal Languages, a set of operations and their iterates is just the Language L generated by the grammar $G(V_1^{\alpha_1},\, V_2^{\alpha_2},\, \cdots,\, V_k^{\alpha_k},\, x)$ with

$$x \in L \;:\; \text{the starting word}$$

and the derivation rules of words

$$\begin{array}{ll} \text{If} & W_1 \in L \\ \text{If} & W_2 \in L \\ & \cdots\ \cdots\ \cdots \\ \text{If} & W_k \in L \end{array}$$

$$\begin{array}{ll} \text{Then} & V_1^{\alpha_1} W_{x_1} W x_2 \cdots W_{x_{\alpha_1}} \in L \\ \text{Then} & V_2^{\alpha_2} W_{y_1} W y_2 \cdots W_{y_{\alpha_2}} \in L \\ & \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \text{Then} & V_k^{\alpha_k} W_{z_1} W z_2 \cdots W_{z_{\alpha_k}} \in L \end{array}$$

where the indexes $x_1, \cdots x_{\alpha_1}$, $y_1, \cdots y_{\alpha_2}, \cdots z_1, \cdots z_{\alpha_k}$, run over all permutations with repetitions of $\{1,\ 2,\ \cdots,\ k\}$.

The reverse is also true. If in the alphabet of the grammar all non-terminal symbols are replaced by x and the terminal symbols are replaced respectively by $V_1^{\alpha_1}$, $V_2^{\alpha_2}$, $\cdots$ $V_k^{\alpha_k}$, we obtain the structure with operations $V_1^{\alpha_1}$, $V_2^{\alpha_2}$, $\cdots$ $V_k^{\alpha_k}$.

For further reading on the subject see Sub-section 4.2 of Petridi [8]. Whether precise analytical results, analogous to those of Petridi [8] and the present paper, hold for all Formal Language as well as their uses in Information Theory is, to our knowledge, an open field to be explored.

A. Appendix

Incidence matrix $\|\delta(J_i^3,\, J_j^3)\|$ relative to T_3
(J_i^3 is denoted by i. Blank spaces mean $0's$)

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	20	1	2	3	4	5	6	7	8	9	30	1	2	3	4	5	6	7	8	9	40	$\sum_i 1$	$M(J_j^3)$
1	1	1	1	1	1	1	1	1																																			8	1
2	1	1	1	1	1	1	1	1																																			8	1
3	1	1	1	1	1	1	1	1																																			8	1
4	1	1	1	1	1	1	1	1																																			8	1
5	1	1	1	1	1	1	1	1					1	1																					1	1	1	1					14	2
6	1	1	1	1	1	1	1	1					1	1																					1	1	1	1					14	2
7	1	1	1	1	1	1	1	1							1	1																							1	1	1	1	14	2
8	1	1	1	1	1	1	1	1							1	1																							1	1	1	1	14	2
9									1	1	1	1	1	1	1	1																											8	1
10									1	1	1	1	1	1	1	1																											8	1
11									1	1	1	1	1	1	1	1																											8	1
12									1	1	1	1	1	1	1	1																											8	1
13					1	1			1	1	1	1	1	1	1	1																	1	1	1	1					14	2
14					1	1			1	1	1	1	1	1	1	1																	1	1	1	1					14	2
15							1	1	1	1	1	1	1	1	1	1																					1	1	1	1	14	2
16							1	1	1	1	1	1	1	1	1	1																					1	1	1	1	14	2
17																	1	1	1	1	1	1	1	1																			8	1
18																	1	1	1	1	1	1	1	1																			8	1
19																	1	1	1	1	1	1	1	1																			8	1
20																	1	1	1	1	1	1	1	1																			8	1
21																	1	1	1	1	1	1	1	1																			8	1
22																	1	1	1	1	1	1	1	1																			8	1
23																	1	1	1	1	1	1	1	1																			8	1
24																	1	1	1	1	1	1	1	1																			8	1
25																									1	1	1	1	1	1	1	1											8	1
26																									1	1	1	1	1	1	1	1											8	1
27																									1	1	1	1	1	1	1	1											8	1
28																									1	1	1	1	1	1	1	1											8	1
29																									1	1	1	1	1	1	1	1											8	1
30																									1	1	1	1	1	1	1	1											8	1
31																									1	1	1	1	1	1	1	1											8	1
32																									1	1	1	1	1	1	1	1											8	1
33		1	1							1	1																					1	1	1	1					8	1
34		1	1							1	1																					1	1	1	1					8	1
35		1	1							1	1																					1	1	1	1					8	1
36		1	1							1	1																					1	1	1	1					8	1
37				1	1							1	1																									1	1	1	1	8	1
38				1	1							1	1																									1	1	1	1	8	1
39				1	1							1	1																									1	1	1	1	8	1
40				1	1							1	1																									1	1	1	1	8	1

From the above table we get

$$I_3^{VW} \;=\; \sum_{i=1}^{40}\sum_{j=1}^{40} 1 \;=\; 368.$$

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