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NEW STRUCTURES IN PSEUDO MAGIC SQUARES

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Abstract: A pseudo magic square (PMS) of order n is an $n \times n$ square matrix whose entries are integers such that the sum of the numbers of any row and any column is the same number, the magic constant. It is a generalization of the concept of magic squares. In this paper we investigate new algebraic structures of PMS's. We explore the group structure of PMS's to show that the quotient of the group of PMS's of order n by its subgroup with zero constant is isomorphic to the infinite additive group of integers, where theisomorphism is constructed by means of the magic constants of the corresponding PMS's. We investigate the ring structure of PMS's to characterize nilpotent and idempotent PMS's as well as we show that the set of PMS's of zero constant is a two-sided ideal in the ring of PMS's. Thus, we can define the quotient ring of PMS's. Moreover, we introduce an invariant and a weak invariant of PMS's and show some results derived from such definitions. In particular, we show that the set of weak invariants of PMS's forms a $\mathbb{Z}$-module under the pointwise addition and scalar multiplication.

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1. Introduction

The concept of magic square is well-known in the literature. The Loh-Shu magic square

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$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

is the oldest known magic square and its invention is attributed to Fuh-Hi (2858-2738 a.C.), [1]. There exist interesting papers available in the literature dealing with investigations on magic squares [3, 1, 10, 7, 2, 4]. Bremner [3] investigated magic squares of order three whose elements are all perfect squares, exhibiting a three-by-three magic square with seven distinct squares. Xin [10] constructed all magic squares of order three. Loly et al. [7] investigated the eigenvalues of low order singular and non-singular magic squares. Beck and Herick [2] enumerated the 4×4 magic squares. Lee et al. [6] investigated necessary and sufficient condition for a regular magic square to be nonsingular and the they construct some magic squares. Chan et al. [4] presented a construction of regular classical magic squares that are nonsingular for all odd orders. by utilizing circulant matrices.

In this paper, as it was mentioned in Abstract, we are interested in the investigation of new algebraic structures of pseudo magic squares, that is, new structures on $n \times n$ matrices with integer entries. In particular, we introduce an invariant and a weak invariant of PMS's and show some results derived from such definitions.

The paper is arranged as follows. Section 2 presents a brief review of basic concepts on pseudo magic squares. In Section 3 we present the contributions of the paper. More precisely, in Subsection 3.1, we characterize the set of PMS's based on its algebraic structures. For example, we show that the quotient group $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, +)$ is isomorphic to $(\mathbb{Z}, +)$, where $\mathcal{P}_n$ is the set of all PMS's of order n, and $\mathcal{P}_n^{(0)}$ is the set of PMS's of order n with zero constant (see Theorem 3.6). We study left, right and two-sided ideals on $\mathcal{P}_n$ (see Proposition 3.12). As a consequence, we show that the ordered triple $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, +, \cdot)$ is a ring with unit (see Theorem 3.13). We also investigate pseudo magic squares which are nilpotent (idempotent) (see Proposition 3.14). In Subsection 3.2, we introduce an invariant and a weak invariant of pseudo magic squares and we show some results derived from these definitions. In particular, we show that the set of invariants under composition of functions is a monoid (see Theorem 3.16). Moreover, we prove that the set of weak invariants of PMS's forms a $\mathbb{Z}$-module under the usual pointwise addition of functions and $\cdot_{sc}$ is the usual pointwise multiplication of a function by a (integer) scalar (see Theorem 3.19). Finally, in Section 4, we give a summary of the main results of this work.

2. Preliminaries on pseudo magic squares

Notation. In this paper, $\mathbb{Z}$ denotes the ring of integers endowed with the usual addition and multiplication. $\mathbb{R}$ means the field of real numbers under usual addition and multiplication. The set of square matrices of order n with entries in some ring $(R, +, \cdot)$ is denoted by $\mathbb{M}_n(R)$, and the set of $m \times n$ matrices with entries in R is denoted by $\mathbb{M}_{m\times n}(R)$.

Let us recall the concept of *pseudo magic square.*

Definition 2.1. ([5, Definition 2.1]) Let n be a positive integer. A pseudo magic square A_n of order n is an element of $\mathbb{M}_n(\mathbb{Z})$ such that the sum of the numbers of any row and any column is the same number a, the pseudo magic constant (constant for short).

The set of pseudo magic squares (PMS)'s of order n is denoted by $\mathcal{P}_n$.

Example 2.1. An interesting example is a PMS of order n whose rows are cyclic shifts of a given n-vector $(a_1, \dots, a_n)$ with integer entries:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}.$$

This square is, in fact, a Latin square of order n, *i.e.*, an $n \times n$ matrix such that every row and every column is a permutation of an n-set S (see [11] for more details about Latin squares).

3. The new results

In this section we present our contributions on pseudo magic squares. More precisely, we explore algebraic structures of PMS's, as it was said previously. We begin by recalling some basic concepts and by showing some new auxiliary results.

Let $(R, +, \cdot)$ be a ring and assume that $A_{m\times n} = [a_{ij}] \in \mathbb{M}_{m\times n}(R)$ and $B_{r\times s} = [b_{ij}] \in \mathbb{M}_{r\times s}(R)$. Recall that the Kronecker product $A_{m\times n} \otimes B_{r\times s}$ of

$A_{m\times n}$ and $B_{r\times s}$ is the $mr \times ns$ matrix given by

$$A_{m\times n} \otimes B_{r\times s} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Now, we define the *direct sum* of PMS's (see also [5, Theorem 3.3]).

Definition 3.1. Let A_n and B_n be two PMS's of order n. The direct sum $A_n \oplus B_n$ of A_n and B_n is a matrix in $\mathbb{M}_{2n}(\mathbb{Z})$ defined as

$$A_n \oplus B_n = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix}.$$

We next introduce the natural concept of *sub pseudo magic square*.

Definition 3.2. Let A_n be a PMS of order n. We say that B_k is a sub pseudo magic square (SPMS for short) of A_n of order k if $k \leq n$ and if B_k is itself a PMS of order k.

Keeping these concepts in mind, we are now able to show the first contribution of this paper.

Proposition 3.1. *Let $A_n = [a_{ij}]$ and $B_n = [b_{ij}]$ be two PMS's of order n and constants a and b, respectively. Then the following hold:*
(1) *The direct sum $A_n \oplus B_n$ is a PMS of order $2n$ with constant $a + b$.*
(2) *There exists a PMS of order $2^k n$ with constant $2^{k-1}(a + b)$, where $k \geq 1$ is a positive integer.*
(3) *The product $A_n B_n$ is a PMS of order n with constant ab. In particular, $(A_n)^m$ is a PMS of order n with constant a^m.*
(4) *The set of PMS's of order n is a $\mathbb{Z}$-algebra under the usual addition and multiplication of matrices and scalar multiplication of a matrix by a (integer) scalar.*
(5) *The Kronecker product $A_n \otimes B_n$ is a PMS of order n^2 with constant ab.*
(6) *There exists a PMS of order $n + 1$ derived from A_n with constant a.*
(7) *If A_n is a PMS of order n and constant a, then its transpose A_n^t is also a PMS of order n and constant a.*
(8) *Let $n \geq 1$ be an integer. Then there exists a PMS of order n and constant*

c, for every $c \in \mathbb{Z}$.
(9) Every PMS has at least one SPMS.

Proof. (1) This is part of the proof of Theorem 3.3 in [5].

(2) We utilize induction on k. From Item (1), it follows that $C_{2n} = A_n \oplus B_n$ is a PMS of order $2n$ with constant $a+b$; so the result holds for $k=1$. Assume that the result is true for $k=t$. Then there exists a PMS $P_{(2^t n)}$ of order $2^t n$ with constant $2^{t-1}(a+b)$, $t \geq 1$. Applying again Item (1) to the same PMS $P_{(2^t n)}$, one has a new PMS $Q_{(2^{t+1} n)}$ of order $2^{t+1} n$ with constant $2^t(a+b)$, as required.

(3) This result is well-known.

(4) This is a well-known result.

(5) We first compute the sum of the elements of the $n(i-1)+r$-th row, where $1 \leq r \leq n$:

$$\begin{aligned} a_{i1}[b_{r1} + b_{r2} + \ldots + b_{rn}] &+ \\ a_{i2}[b_{r1} + b_{r2} + \ldots + b_{rn}] &+ \\ + \ldots &+ \\ a_{in}[b_{r1} + b_{r2} + \ldots + b_{rn}] &+ \\ = [a_{i1} + a_{i2} + \ldots + a_{in}]b = ab. \end{aligned}$$

The sum of the elements of the $n(j-1)+t$-th column $(1 \leq t \leq n)$ is

$$\begin{aligned} a_{1j}[b_{1t} + b_{2t} + \ldots + b_{nt}] &+ \\ a_{2j}[b_{1t} + b_{2t} + \ldots + b_{nt}] &+ \\ + \ldots &+ \\ a_{nj}[b_{1t} + b_{2t} + \ldots + b_{nt}] &+ \\ = [a_{1j} + a_{2j} + \ldots + a_{nj}]b = ab. \end{aligned}$$

Therefore, it follows that $A_n \otimes B_n$ is a PMS with constant ab.

(6) It suffices to define the PMS A^*_{n+1} as follows:

$$A^*_{n+1} = \begin{bmatrix} 0_{1\times n} & a \\ A & 0_{n\times 1} \end{bmatrix}.$$

It is clear that A^*_{n+1} has order $n+1$ and constant a.

(7) Immediate.

(8) Let $n \geq 1$ be an integer. If $c=0$, then we take the zero matrix. If $c=1$, it suffices to consider the identity matrix I_n of order n. If $c=-1$ we consider

the diagonal matrix of order n with all entries equal to -1. Analogously, for each integer r, we take the diagonal matrix of order n with all entries equal to r, and we are done.

(9) Note that each PMS is a sub pseudo magic square over itself. □

Remark 3.2. Note that Items (1), (5), (6) of Proposition 3.1 are new methods to construct directly new PMS's from old ones. In fact, we can construct several new families of PMS's from these methods.

3.1. Algebraic structures in pseudo squares

In this section we present new results with respect to the algebraic structures in PMS's. We begin by recalling the following result of [5].

Theorem 3.3. ([5, Theorem 3.1] *The ordered pair $(\mathcal{P}_n, +)$ is an abelian group, where the operation $+$ is the usual addition of matrices.*

Although the group $(\mathcal{P}_n, +)$ is abelian, it is not cyclic, as states the following proposition.

Proposition 3.4. *The group $(\mathcal{P}_n, +)$ is not cyclic.*

Proof. Seeking a contradiction, assume that A_n is a generator of $\mathcal{P}_n$. We know that if A_n has constant c then their powers have constant mc, where $m \in \mathbb{Z}$ (in the case of $m = 0$ we assume that the zero power of A_n is the zero matrix). If A_n has constant zero, from Item 7 of Proposition 3.1 and from the previous discussion, A_n cannot be a generator. From the same argument, A_n cannot have constant $c > 1$ nor $c < -1$. Thus, let us consider that the generator A_n has constant $c = 1$. We know there exist at least two PMS's with constant $c = 1$, namely, the identity I_n of order n and the PMS J_n of order n given by

$$J_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Hence, A_n cannot be a generator. The same argument shows that none PMS with constant $c = -1$ can be a generator. Therefore, $(\mathcal{P}_n, +)$ is not cyclic, and the proof is complete. □

The set of PMS's with zero constant is a normal subgroup of $(\mathcal{P}_n,+)$, as states the next result.

Proposition 3.5. *The ordered pair* $(\mathcal{P}_n^{(0)},+)$ *is a normal subgroup of* $(\mathcal{P}_n,+)$*, where* $\mathcal{P}_n^{(0)}$ *is the set of all PMS's of order* n *with zero constant.*

Proof. We know that the identity PMS $[0]_{n\times n}$ belongs to $\mathcal{P}_n^{(0)}$. The sum of PMS's in $\mathcal{P}_n^{(0)}$ is also a PMS in $\mathcal{P}_n^{(0)}$. Furthermore, if $A_n \in \mathcal{P}_n^{(0)}$ then $-A_n \in \mathcal{P}_n^{(0)}$. Since $(\mathcal{P}_n,+)$ is an abelian group, then their subgroups are normal. In particular, $(\mathcal{P}_n^{(0)},+)$ is normal. The proof is complete. □

We next show that the quotient group is essentially the infinite cyclic group of integers.

Theorem 3.6. *Let* $n \geq 1$ *be an integer. Then the quotient group* $(\mathcal{P}_n/\mathcal{P}_n^{(0)},+)$ *is isomorphic to* $(\mathbb{Z},+)$.

Proof. First of all, notice that from Proposition 3.5, it follows that $\mathcal{P}_n^{(0)}$ is a normal subgroup of $(\mathcal{P}_n,+)$. Thus the quotient group makes sense.

Let $f : \mathcal{P}_n/\mathcal{P}_n^{(0)} \longrightarrow \mathbb{Z}$ be the function defined by

$$f(A_n + \mathcal{P}_n^{(0)}) = c_{A_n},$$

where c_{A_n} is the constant of A_n. We will show that f is an isomorphism.

(i) f is well-defined. Assume that $A_n+\mathcal{P}_n^{(0)} = B_n+\mathcal{P}_n^{(0)}$. Then $A_n - B_n \in \mathcal{P}_n^{(0)}$. This implies that A_n-B_n has constant equal to zero, that is, $c_{(A_n-B_n)} = 0$. From the proof of Theorem 3.3, we know that $c_{(A_n-B_n)} = c_{A_n} - c_{B_n}$; hence $c_{A_n} = c_{B_n}$. Therefore, $f(A_n + \mathcal{P}_n^{(0)}) = f(B_n + \mathcal{P}_n^{(0)})$.

(ii) f is homomorphism. Let $A_n + \mathcal{P}_n^{(0)}$ and $B_n + \mathcal{P}_n^{(0)}$ be two cosets in $\mathcal{P}_n/\mathcal{P}_n^{(0)}$. We have:

$$\begin{aligned} f([A_n + \mathcal{P}_n^{(0)}] + [B_n + \mathcal{P}_n^{(0)}]) &\overset{def}{=} f([A_n + B_n] + \mathcal{P}_n^{(0)}) \\ &\overset{def}{=} c_{(A_n+B_n)} \\ &= c_{A_n} + c_{B_n} \\ &= f(A_n + \mathcal{P}_n^{(0)}) + f(B_n + \mathcal{P}_n^{(0)}). \end{aligned}$$

(iii) f is injective. Assume that $f(A_n + \mathcal{P}_n^{(0)}) = f(B_n + \mathcal{P}_n^{(0)})$; so $c_{A_n} = c_{B_n}$, This implies that $c_{(A_n - B_n)} = 0$, *i.e.*, $A_n - B_n \in \mathcal{P}_n^{(0)}$. Thus it follows $A_n + \mathcal{P}_n^{(0)} = B_n + \mathcal{P}_n^{(0)}$, that is, f is injective.

(iv) f is surjective. Let z be an integer. From Proposition 3.1 Item 7, there exists a PMS A_n with constant z. Hence, it suffices to consider the coset $A_n + \mathcal{P}_n^{(0)}$ and we have $f(A_n + \mathcal{P}_n^{(0)}) = c_{A_n} = z$. Consequently, f is surjective.

These steps show that f is an isomorphism from $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, \oplus)$ to $(\mathbb{Z}, +)$, and theorem is proved.

An alternative proof (sketch): Proposition 3.1, Item 7 shows that f defined above is surjective. After this, it is easy to show that f is a homomorphism (as above) and $\ker f = \mathcal{P}_n^{(0)}$. We next apply the First Isomorphism Theorem showing that $\mathcal{P}_n/\mathcal{P}_n^{(0)} \cong \mathbb{Z}$ for each integer $n \geq 1$. □

If $(G, *)$ is a group and $x, y \in G$, then their *commutator* $[x, y]$ is the element $[x, y] = xyx^{-1}y^{-1}$. For subgroups X, Y of G, we define $[X, Y] = \langle [x, y] | x \in X, y \in Y \rangle$, where $\langle \cdot \rangle$ denotes the subgroup generated by the elements $[x, y]$. The *commutator subgroup* $G^{'}$ of G is defined as $G^{'} = [G, G]$. Recall that a *derived series* of G (see [8, 9]) is the sequence

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots G^{(i)} \geq G^{(i+1)} \geq \cdots,$$

where $G^{(0)} = G$, $G^{(1)} = G^{'}$, and for every $i \geq 0$, $G^{(i+1)} = (G^{(i)})^{'} = [G^i, G^i]$. The group $(G, *)$ is called *solvable* (see [9, p. 286] if its derived series reaches the trivial subgroup $\{e_G\}$ after a finite number of steps.

Adopting these definitions, we can state the following corollary of Theorem 3.6.

Corollary 3.7. *The following statements are true:*
(1) *For every integers $i, j \geq 1$, we have $\mathcal{P}_i/\mathcal{P}_i^{(0)} \cong \mathcal{P}_j/\mathcal{P}_j^{(0)}$.*
(2) *For every integer $n \geq 1$, the quotient group $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, \oplus)$ is cyclic.*
(3) *For every integer $n \geq 1$, the quotient group $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, \oplus)$ is solvable.*

Proof. (1) It follows directly from Theorem 3.6 and from the fact that isomorphisms are transitive.
(2) Follows from Theorem 3.6.
(3) Since from Item (2) the quotient group $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, \oplus)$ is abelian, the result follows. □

We next introduce the concept of *pseudo super-magic squares* (PSMS)'s

Definition 3.3. Let n be a positive integer. A pseudo super-magic square A_n of order n, is an element of $\mathbb{M}_n(\mathbb{Z})$ such that the sum of the numbers of each row, each column and also the sum of the numbers in both diagonals is the same number a, the pseudo super magic constant (constant for short).

The set of the PSMS's of order n is denoted by $\mathcal{P}_n^s$.

Proposition 3.8. *The ordered pair $(\mathcal{P}_n^s, +)$ is a subgroup of $(\mathcal{P}_n, +)$. Moreover, if the PSMS A_n has constant a, then $\mathrm{Tr}(A_n) = a$, where $\mathrm{Tr}(A_n)$ denotes the trace of the matrix A_n.*

Proof. The null matrix $[0]_{n\times n}$ of order n is in $(\mathcal{P}_n^s, +)$; it is the identity element of $(\mathcal{P}_n^s$. It is clear that the sum of two PSMS's A_n and B_n with constants a and b respectively, is a PSMS with constant $a+b$. If A_n is a PSMS with constant a, then its inverse $-A_n$ is also a PSMS with constant $-a$.

The second part follows immediately. □

The following result states that suitable subsets of PMS's are also a group under multiplication of matrices.

Theorem 3.9. *Let $\mathcal{P}_n$ be the set of PMS's of order n. Assume that $\mathcal{P}_n^* = \{A_n \in \mathcal{P}_n \mid (A_n)^{-1} \in \mathcal{P}_n\}$, where $(A_n)^{-1}$ denotes the inverse of A_n in $\mathbb{M}_n(\mathbb{Z})$ (if there exists). Then the ordered pair $(\mathcal{P}_n^*, \cdot)$ is a group, where $\cdot$ is the usual product of matrices.*

Proof. Assume that $A_n, B_n \in \mathcal{P}_n^*$. From Item 3 of Proposition 3.1, we know that $A_nB_n \in \mathcal{P}_n$. From definition of $\mathcal{P}_n^*$, we know that if $A_n \in \mathcal{P}_n^*$ then $(A_n)^{-1} \in \mathcal{P}_n^*$. Further, $(A_nB_n)^{-1} = (B_n)^{-1}(A_n)^{-1}$. Since both $(A_n)^{-1}$ and $(B_n)^{-1}$ belong to $\mathcal{P}_n$, applying again Item 3 of Proposition 3.1, we conclude that $(A_nB_n)^{-1} \in \mathcal{P}_n$. Hence $(A_nB_n)^{-1} \in \mathcal{P}_n^*$. It is clear that the identity matrix I_n of order n belongs to $\mathcal{P}_n^*$. Moreover, I_n is the identity element of $(\mathcal{P}_n^*, \cdot)$. Therefore, $\mathcal{P}_n^*$ is a group, as required. □

An interesting question that arises is as follows: is it true that if $A_n \in \mathcal{P}_n$ then $(A_n)^{-1} \in \mathcal{P}_n$? The answer for this question is no. A trivial counterexample is the Loh-Shu magic square

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix},$$

whose inverse is

$$\begin{bmatrix} 23/360 & -13/90 & 53/360 \\ 19/180 & 1/45 & -11/180 \\ -37/360 & 17/90 & -7/360 \end{bmatrix}.$$

However, although the latter matrix does not have integer entries, the sum of the elements of all rows and all columns remains the same. This induces us to prove the following result.

Proposition 3.10. *Let $A_n \in \mathcal{P}_n$ be a PMS of order n with constant a. Then the following hold:*
(1) If A_n has constant $a = 0$ then $\det A_n = 0$. In particular, if A_n has zero constant, then A_n is not invertible.
(2) If A_n is an invertible PMS of order n, then its inverse $(A_n)^{-1}$ is also a PMS of order n with entries in $\mathbb{R}$ and constant $1/a$.
(3) If $a \neq 0$, then the adjoint matrix $\widehat{A}_n$ is also a PMS of order n with entries in $\mathbb{R}$.

Proof. (1) Let

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be a PMS with constant $a = 0$. Since the determinant $\det A_n$ is n-linear and alternating, we have

$$\det A_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ a_{11}+a_{n1} & a_{12}+a_{n2} & \cdots & a_{1n}+a_{nn} \end{vmatrix}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ \sum_{i=1}^{n} a_{i1} & \sum_{i=1}^{n} a_{i2} & \cdots & \sum_{i=1}^{n} a_{in} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0.$$

Thus $\det A_n = 0$, as required.

(2) Let $A_n \in \mathcal{P}_n$ be an invertible PMS with inverse $(A_n)^{-1} = B_n$. From Item (1), it follows that $a \neq 0$. Hence,

$$A_n B_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

There are n systems of equations of the form:

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} &= 1 \\ a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} &= 0 \\ &\vdots \\ a_{n1}b_{11} + a_{n2}b_{21} + \ldots + a_{nn}b_{n1} &= 0 \\ &\Longrightarrow \\ b_{11}[a_{11} + a_{21} + \ldots + a_{n1}] &+ \end{aligned}$$

$$
\begin{aligned}
b_{21}[a_{12} + a_{22} + \ldots + a_{n2}] + \\
+ \ldots + \\
b_{n1}[a_{1n} + a_{2n} + \ldots + a_{nn}] = 1.
\end{aligned}
$$

Since A_n is a PMS, it follows that $[b_{11} + b_{21} + \ldots + b_{n1}] = 1/a$, where $a \neq 0$ is the constant of A_n.

The n-th system is given by

$$
\begin{aligned}
a_{11}b_{1n} + a_{12}b_{2n} + \ldots + a_{1n}b_{nn} = 0 \\
a_{21}b_{1n} + a_{22}b_{2n} + \ldots + a_{2n}b_{nn} = 0 \\
\vdots \\
a_{n1}b_{1n} + a_{n2}b_{2n} + \ldots + a_{nn}b_{nn} = 1 \\
\Longrightarrow \\
b_{1n}[a_{11} + a_{21} + \ldots + a_{n1}] + \\
b_{2n}[a_{12} + a_{22} + \ldots + a_{n2}] + \\
+ \ldots + \\
b_{nn}[a_{1n} + a_{2n} + \ldots + a_{nn}] = 1 \\
\Longrightarrow [b_{1n} + b_{2n} + \ldots + b_{nn}] = 1/a.
\end{aligned}
$$

Therefore, the sum of the elements of each column of B_n equals $1/a$.

We next compute the sum of the elements of each row of B_n. To do this, we consider the product B_nA_n:

$$
B_nA_n = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
$$

We have more n systems of equations where the first is:

$$
b_{11}a_{11} + b_{12}a_{21} + \ldots + b_{1n}a_{n1} = 1
$$

$$b_{11}a_{12} + b_{12}a_{22} + \ldots + b_{1n}a_{n2} = 0$$
$$\vdots$$
$$b_{11}a_{1n} + b_{12}a_{2n} + \ldots + b_{1n}a_{nn} = 0$$

which implies

$$b_{11}[a_{11} + a_{12} + \ldots + a_{1n}] + b_{12}[a_{21} + a_{22} + \ldots + a_{2n}] + \ldots$$
$$+b_{1n}[a_{n1} + a_{n2} + \ldots + a_{nn}] = 1 \Longrightarrow [b_{11} + b_{12} + \ldots + b_{1n}] = 1/a.$$

The n-th system is

$$b_{n1}a_{11} + b_{n2}a_{21} + \ldots + b_{nn}a_{n1} = 0$$
$$b_{n1}a_{12} + b_{n2}a_{22} + \ldots + b_{nn}a_{n2} = 0$$
$$\vdots$$
$$b_{n1}a_{1n} + b_{n2}a_{2n} + \ldots + b_{nn}a_{nn} = 1$$

which implies that

$$b_{n1}[a_{11} + a_{12} + \ldots + a_{1n}] + b_{n2}[a_{21} + a_{22} + \ldots + a_{2n}] + \ldots$$
$$+b_{nn}[a_{n1} + a_{n2} + \ldots + a_{nn}] = 1 \Longrightarrow [b_{n1} + b_{n2} + \ldots + b_{nn}] = 1/a.$$

Therefore, B_n is also a PMS of order n with entries in $\mathbb{R}$.

(3) Assume first that A_n is an invertible PMS of order n. We know that $\det A_n \neq 0$ and $\widehat{A}_n = \det A_n \cdot (A_n)^{-1}$. From Item (1), $a \neq 0$ and from Item (2), it follows that $(A_n)^{-1}$ is a PMS of order n and constant $1/a$. Therefore, $\widehat{A}_n$ is also a PMS of order n with constant $\widehat{a} = \det A_n \cdot 1/a$.

If A_n is not invertible, then $\det A_n = 0$; so $A_n \cdot \widehat{A}_n = \det A_n \cdot I_n = [0]_{n\times n}$. In order to avoid overload of the notation, we denote the entries of $\widehat{A}_n$ by b_{ij}. We first consider the case

$$A_n\widehat{A}_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We then have

$$\begin{aligned}
a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} &= 0\\
a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} &= 0\\
&\vdots\\
a_{n1}b_{11} + a_{n2}b_{21} + \ldots + a_{nn}b_{n1} &= 0
\end{aligned}$$

which implies that

$$\begin{aligned}
b_{11}[a_{11} + a_{21} + \ldots + a_{n1}] + b_{21}[a_{12} + a_{22} + \ldots + a_{n2}] + \ldots +\\
b_{n1}[a_{1n} + a_{2n} + \ldots + a_{nn}] = 0 \Longrightarrow a[b_{11} + b_{21} + \ldots + b_{n1}] = 0\\
\Longrightarrow b_{11} + b_{21} + \ldots + b_{n1} = 0.
\end{aligned}$$

Proceeding similarly to all columns of $\widehat{A}_n$, it follows that the sum of the elements of each column is equal to zero.

Analogously, let us consider the case

$$\widehat{A}_n A_n = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We have:

$$\begin{aligned}
b_{11}a_{11} + b_{12}a_{21} + \ldots + b_{1n}a_{n1} &= 0\\
b_{11}a_{12} + b_{12}a_{22} + \ldots + b_{1n}a_{n2} &= 0\\
&\vdots\\
b_{11}a_{1n} + b_{12}a_{2n} + \ldots + b_{1n}a_{nn} &= 0
\end{aligned}$$

which implies

$$b_{11}[a_{11} + a_{12} + \ldots + a_{1n}] + b_{12}[a_{21} + a_{22} + \ldots + a_{2n}] + \ldots +$$

$$\begin{aligned} b_{1n}[a_{n1} + a_{n2} + \ldots + a_{nn}] = 0 &\Longrightarrow a[b_{11} + b_{12} + \ldots + b_{1n}] = 0 \\ &\Longrightarrow b_{11} + b_{12} + \ldots + b_{1n} = 0. \end{aligned}$$

Proceeding similarly to all rows of $\widehat{A}_n$, it implies that the sum of the elements of each row is zero, and there is nothing more to prove. □

Remark 3.11. It is interesting to observe that Items (2) and (3) of Proposition 3.10 are direct methods to obtain new pseudo magic

Here, we consider *left ideals*, *right ideals* or *two-sided ideals* of PMS's, where the last means an ideal that is both left and right ideal at the same time.

Let us return our attention to the subgroup $\mathcal{P}_n^{(0)}$ of PMS's of order n with zero constant (cf. Proposition 3.5). Such subgroup is, in fact, an ideal in the ring $(\mathcal{P}_n, +, \cdot)$ of the PMS's of order n.

Proposition 3.12. *Let $(\mathcal{P}_n, +, \cdot)$ be the ring of PMS's of order n. Then the following hold:*
(1) *The set $\mathcal{P}_n^{(0)}$ of PMS's of order n with zero constant is a two-sided ideal of $\mathcal{P}_n$.*
(2) *The set $\mathcal{P}_n^{(0)}$ is closed under difference and under multiplication.*
(3) *$\mathcal{P}_n^{(0)}$ is not a subring of $(\mathcal{P}_n, +, \cdot)$.*

Proof. (1) From Proposition 3.5, we know that $(\mathcal{P}_n^{(0)}, +)$ is a subgroup of $(\mathcal{P}_n, +)$. Let R_n be an arbitrary PMS of order n and constant r, and A_n be a PMS of order n with constant $a = 0$. To complete the proof, we need to compute both products R_nA_n and A_nR_n. However, from Proposition 3.1 Item (3), it follows that both R_nA_n and A_nR_n are PMS's of order n with zero constant. Hence, R_nA_n and A_nR_n belongs to $\mathcal{P}_n^{(0)}$, which implies that $\mathcal{P}_n^{(0)}$ is a two-sided ideal, and we are done.

(2) If A_n and B_n have zero constant, it is clear that $A_n - B_n$, A_nB_n also have zero constant.

(3) Note that the identity matrix has constant equal to 1, i.e., I_n does not belong to $\mathcal{P}_n^{(0)}$. □

Based on the previous result, one can construct a *quotient ring* of PMS's:

Theorem 3.13. *Let $(\mathcal{P}_n, +, \cdot)$ be the ring of PMS's of order n. Then the ordered triple $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, +, \cdot)$ is a ring with unit. This ring is called a* *quotient ring of* $\mathcal{P}_n$

Proof. From Proposition 3.12, we know that $\mathcal{P}_n^{(0)}$ is a two-sided ideal. Thus, the multiplication on the quotient abelian group $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, +)$ is well-defined. From Theorem [5, Theorem 3.1], it follows that $(\mathcal{P}_n, +)$ is an abelian group. The identity is the left coset $I_n + \mathcal{P}_n^{(0)}$. The associativity of $\cdot$ and the distributivity of $\cdot$ with respect to $+$ follow from the properties of multiplication of matrices. Thus, $(\mathcal{P}_n/\mathcal{P}_n^{(0)}, +, \cdot)$ is a ring with unit, and the proof is complete. □

Proposition 3.14. *Let* $(\mathcal{P}_n, +, \cdot)$ *be the ring of PMS's of order* n *and assume that* $A_n \in \mathcal{P}_n$ *is a PMS with constant* a*. Then the following hold:*
(1) *If* A_n *is nilpotent then* $a = 0$*. The converse does not hold.*
(2) *If* A_n *is idempotent then* $a = 0$ *or* $a = 1$*. The converse does not hold.*

Proof. (1) Let $A_n \in \mathcal{P}_n$ be a nilpotent PMS with constant a. Then $A_n \neq [0]_{n\times n}$ and there exists some positive integer r such that $(A_n)^r = [0]_{n\times n}$. From Proposition 3.1 Item (3), it follows that $a^r = 0$. Since $\mathbb{Z}$ is an integral domain, one has $a = 0$.

To see that the converse is not true, it suffices to take the null PMS $[0]_{n\times n}$ of order n. This PMS has zero constant but it is not nilpotent.

(2) Let $A_n \in \mathcal{P}_n$ be an idempotent PMS with constant a; so $(A_n)^2 = A_n$. Again, from Proposition 3.1 Item (3), one has $a^2 = a$. Therefore, $a = 1$ or $a = 0$.

To verify that the converse is not true, let A_n be the PMS of order n given by

$$A_n = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix},$$

A_n has constant $a = 0$, and

$$(A_n)^2 = \begin{bmatrix} n(n-1) & -n & -n & \cdots & -n \\ -n & n(n-1) & -n & \cdots & -n \\ -n & -n & n(n-1) & \cdots & -n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -n & -n & -n & \cdots & n(n-1) \end{bmatrix}.$$

Since $(A_n)^2 \neq A_n$, it follows that A_n is not idempotent.

Let us now consider the PMS B_n of order n given by

$$B_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Thus B_n has constant $b = 1$ but $(B_n)^2 \neq B_n$, that is, B_n is not idempotent. □

Since the set of PMS's forms a $\mathbb{Z}$-module, we can define homomorphism and isomorphism among PMS's.

Definition 3.4. Let $(\mathcal{P}_n, +, \cdot)$ and $(\mathcal{P}_m, +, \cdot)$ be two $\mathbb{Z}$-modules of PMS's and let $h : \mathcal{P}_n \longrightarrow \mathcal{P}_m$ be a function. We say that h is a homomorphism of PMS's if:
(1) for all $A_n, B_n \in \mathcal{P}_n$, $h(A_n + B_n) = h(A_n) + h(B_n)$;
(2) for all $A_n \in \mathcal{P}_n$ and for all $k \in \mathbb{Z}$, $h(kA_n) = kh(A_n)$.

Definition 3.5. Let $h : \mathcal{P}_n \longrightarrow \mathcal{P}_m$ be a homomorphism of PMS's. We say that h is an isomorphism if h is bijective.

Remark 3.15. It is easy to see that the isomorphism between PMS's is an equivalence relation on the set of all PMS's.

3.2. Invariant and weak-invariant of PMS's

In this subsection, we introduce the concept of *invariance and weak invariance* of PMS's. Such definitions are natural, as can be seen in Definitions 3.6 and 3.7 given in the sequence.

Definition 3.6. Let $f : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ be a function. We say that f is a $\mathcal{P}_n$-invariant if f preserves constants, that is, if $A_n \in \mathcal{P}_n$ has constant a then $f(A_n) \in \mathcal{P}_n$ also has constant a.

The set of all $\mathcal{P}_n$-invariant functions is denoted by $\mathbb{I}(\mathcal{P}_n)$.

Example 3.1. As a trivial example, let $n \geq 1$ be a positive integer and consider the function $r_{ij} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ which changes row i and row j. It is clear that, if $A_n \in \mathcal{P}_n$ has constant a then the PMS $r_{ij}(A_n)$ has also constant a. Hence, r_{ij} is a $\mathcal{P}_n$-invariant.

The following result establishes that the set of invariant of order n under composition of functions is a monoid.

Theorem 3.16. *The ordered pair* $(\mathbb{I}(\mathcal{P}_n), \circ)$ *is a monoid, where* $\circ$ *is the composition of functions. Additionally,* $(\mathbb{I}(\mathcal{P}_n), \circ)$ *is a group if and only if* $n = 1$.

Proof. First of all, it is clear that $\mathbb{I}(\mathcal{P}_n)$ is nonempty, because the identity function $id_{\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ of $\mathcal{P}_n$ is $\mathcal{P}_n$-invariant. Further, $id_{\mathcal{P}_n}$ is the identity of $(\mathbb{I}(\mathcal{P}_n), \circ)$. Let us consider $f, g \in \mathbb{I}(\mathcal{P}_n)$ and let $A_n \in \mathcal{P}_n$ be a PMS with constant a. We first show that the composite $f \circ g$ is also a $\mathcal{P}_n$-invariant. In fact, $f \circ g(A_n) = f(g(A_n))$. Since $f \in \mathbb{I}(\mathcal{P}_n)$, it follows that $f(A_n)$ ia also a PMS of order n with constant a. Furthermore, because $g \in \mathbb{I}(\mathcal{P}_n)$, it implies that $f \circ g(A_n)$ is a PMS of order n whose constant is also a. Therefore, the composite $f \circ g$ is also a $\mathcal{P}_n$-invariant, that is, the composition is closed. The associativity follows directly from the associativity of composition of functions. Therefore, $(\mathbb{I}(\mathcal{P}_n), \circ)$ is a monoid.

To show the second part, consider that $n = 1$. In this case, we can identify the group $(\mathcal{P}_1, +)$ with the group of integers under the usual addition $(\mathbb{Z}, +)$. It is easy to see that the unique $\mathcal{P}_1$-invariant is the identity of $\mathcal{P}_1$. Hence $\mathbb{I}(\mathcal{P}_1) = id_{\mathcal{P}_1}$, that is, $(\mathbb{I}(\mathcal{P}_1), \circ)$ is a (trivial) group. To prove the converse, we seek a contradiction assuming that $n > 1$. We must show that $(\mathbb{I}(\mathcal{P}_n), \circ)$ fails to be a group. To see this, take the function $t_n : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ which associates to each PMS with constant $c \in \mathbb{Z}$ the fixed PMS cI_n. We know that for every integer constant c, there exist at least two PMS's of order n (> 1) with constant c. Thus, t_n is not injective, which implies that t_n does not have inverse. Therefore, $(\mathbb{I}(\mathcal{P}_n), \circ)$ is not a group. □

In the following, we introduce *weak invariance* for PMS's.

Definition 3.7. Let $h : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ be a function. We say that h is a weak $\mathcal{P}_n$-invariant if h preserves multiple of constants. More precisely, there exists a fixed integer t such that for all $A_n, B_n \in \mathcal{P}_n$ with constants a and b respectively, then $h(A_n), h(B_n) \in \mathcal{P}_n$ has constants ta and tb, respectively.

Remark 3.17. (1) Note that for each weak $\mathcal{P}_n$-invariant function h, the constant t is fixed.

(2) The set of all weak $\mathcal{P}_n$-invariant functions is denoted by $\mathbb{I}_w(\mathcal{P}_n)$.

Definition 3.7 motivates us to show the next results.

Proposition 3.18. (1) *For every integer t, there exists a weak $\mathcal{P}_n$-invariant function $h : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ whose constants are multiple of t.*
(2) *The identity $id_{\mathcal{P}_n}$ of $\mathcal{P}_n$ is a weak $\mathcal{P}_n$-invariant function with $t = 1$.*

Proof. (1) Define $h_t : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ as follows: if $A_n \in \mathcal{P}_n$ then $h_t(A_n) = tA_n$. It is obvious that is weak $\mathcal{P}_n$-invariant: if B_n has constant b then $h_t(B_n)$ has constant tb for all $B_n \in \mathcal{P}_n$.
(2) Immediate. □

Theorem 3.19. *The ordered triple $(\mathbb{I}_w(\mathcal{P}_n), +, \cdot_{sc})$ is a $\mathbb{Z}$-module, where $+$ is the usual pointwise addition of functions and $\cdot_{sc}$ is the usual pointwise multiplication of a function by a (integer) scalar.*

Proof. Let $f, g \in \mathbb{I}_w(\mathcal{P}_n)$. We first show that both $f + g$ and kf belong to $\mathbb{I}_w(\mathcal{P}_n)$, where $k \in \mathbb{Z}$. Let A_n be a PMS with constant a. Then $[f + g](A_n) = f(A_n) + g(A_n) = t_1 a + t_2 a = (t_1 + t_2)a$, *i.e.*, $f + g \in \mathbb{I}_w(\mathcal{P}_n)$.

Further, $[kf](A_n) = kf(A_n)$ is a PMS with constant kta, hence $kf \in \mathbb{I}_w(\mathcal{P}_n)$. The zero function which associates to every PMS of order n the zero PMS of order n (that is, the null square matrix of order n) is the identity of the group. Given $f \in \mathbb{I}_w(\mathcal{P}_n)$, then the function $-f$ is also weak $\mathcal{P}_n$-invariant and it is the additive inverse of f. The associativity and the commutativity of $+$ follow from the associativity and commutativity of the addition of matrices. The properties of scalar multiplication follow directly from the properties of addition of matrices and multiplication of a scalar by a matrix. □

4. Summary

In this paper, we have characterized the set of pseudo magic squares by showing new algebraic structures of them. We have shown that the quotient of the group of PMS's of order n by its subgroup with zero constant is isomorphic to the infinite additive group of integers, where the isomorphism is constructed by means of the magic constants of the corresponding PMS's. We have also explored the ring structure of PMS's to characterize nilpotent and idempotent PMS's. Additionally, we have introduced an invariant and a weak invariant of PMS's and we have shown that the set of weak invariants of PMS's forms a $\mathbb{Z}$-module under the pointwise addition and scalar multiplication. As future work, we intend to characterize pseudo magic squares in terms of Category Theory by investigating products, co-products, equalizers, pullbacks and pushouts. We will also investigate a possibility of correlation between PMS's and Matroid Theory.

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