International Journal of Applied Mathematics

Volume 32 No. 5 2019, 847-856

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v32i5.11

MULTIPLICATION OPERATORS IN MEASURABLE SECTIONS SPACES

Marat Pliev

Southern Mathematical Institute of RAS
Vladikavkaz, 362027, RUSSIA

Abstract: In this article we consider linear operators in measurable section spaces. Let $\mathcal{X}$ be a liftable measurable bundle of Banach spaces and E be an order continuous Köthe function space over a finite measure space (A, Σ, μ). We prove that a linear continuous operator T in a measurable sections space $E(\mathcal{X})$ is a multiplication operator (by a function in $L_\infty(\mu)$) if and only if the equality $T(g\langle f, \phi^\star\rangle\phi) = g\langle T(f), \phi^\star\rangle\phi)$ holds for every $g \in L_\infty(\mu), f \in E(\mathcal{X})$, $\phi \in L_\infty(\mathcal{X})$ and $\phi^\star \in L_\infty(\mathcal{X}^\star)$.

AMS Subject Classification: 46B99, 47B99
Key Words: Köthe function space; measurable sections space' liftable measurable bundle of Banach spaces; multiplication operator

1. Introduction

Today the theory of linear and orthogonally additive operators in lattice-normed spaces is an active area of Functional Analysis (see for instance [1, 7, 8, 9, 10, 11, 12]). We remark that the spaces of continuous and measurable sections of Banach bundles are typical examples of lattice-normed spaces. The aim of this article is the investigation of operators between section spaces. We obtain necessary and sufficient conditions for a linear continuous operator $T : E(\mathcal{X}) \to E(\mathcal{X})$ to be a multiplication operator on a measurable function.

Received: October 19, 2019

2. Preliminaries

In this section we state some basic facts concerning Banach bundles and lattice-normed spaces. For notations and terminology not explained in the paper the reader can consult the book [2, 13]. All linear spaces we consider below are real.

Consider a vector space V and an Archimedean vector lattice E. A map $|\cdot| : V \to E_+$ is called a *vector norm*, if it satisfies the following conditions:

1) $|v| \geq 0$; $|v| = 0 \Leftrightarrow v = 0$; $v \in V$.

2) $|u + v| \leq |u| + |v|$; $u, v \in V$.

3) $|\lambda u| = |\lambda| \, |u|$; $\lambda \in R$, $u \in V$.

A vector norm is called *decomposable*, if

4) for all $e_1, e_2 \in E_+$ and $v \in V$ from $|v| = e_1 + e_2$ it follows that there exist $v_1, v_2 \in V$ such that $v = v_1 + v_2$ and $|v_k| = e_k$, $(k := 1, 2)$.

A triple $(V, |\cdot|, E)$ ((V, E) or V for brevity) is called a *lattice-normed space* if $|\cdot| : V \to E_+$ is a vector norm in the vector space V. If the norm $|\cdot|$ is decomposable then the space V is called *d*ecomposable. A net $(v_\alpha)_{\alpha\in\Delta}$ (bo)-*converges* to an element $v \in V$, if there exists a decreasing net $(e_\xi)_{\xi\in\Xi}$ in E_+ such that $\inf_{\xi\in\Xi}(e_\xi) = 0$ and for every $\xi \in \Xi$ there is an index $\alpha(\xi) \in \Delta$ such that $|v - v_{\alpha(\xi)}| \leq e_\xi$ for all $\alpha \geq \alpha(\xi)$. A net $(v_\alpha)_{\alpha\in\Delta}$ is called (bo)-*fundamental*, if the net $(v_\alpha - v_\beta)_{(\alpha,\beta)\in\Delta\times\Delta}$ (bo)-converges to zero. A lattice-normed space is called (bo)-*complete* if every (bo)-fundamental net (bo)-converges to an element of this space. Every decomposable (bo)-complete lattice-normed space is called a *Banach-Kantorovich space.*

Consider some examples of lattice-normed spaces. We start with two simple cases, namely vector lattices and normed spaces. If $V = E$ then the module of an element can be taken as its vector norm: $|v| := |v| = v \vee (-v)$; $v \in E$. The decomposability of this norm follows from the Riesz Decomposition Property holding in every vector lattice (see [2], Th. 1.13). If $E = R$, then V is a normed space.

Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach space E consisting of equivalence classes modulo equality almost everywhere integrable real-valued functions on Ω is called *Köthe function space* if E following conditions hold:

1. If $f \in L_0(\mu)$ and $|f| \leq |g|$ μ-a.e. for some $g \in E$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$;

2. for every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A belongs to E.

Definition 1. Let Ω be a nonempty set. A *bundle of Banach spaces over* Ω is a mapping $\mathcal{X}$ defined on Ω and associating a Banach space $\mathcal{X}_t := \mathcal{X}(t) := (\mathcal{X}(t), \|\cdot\|_{\mathcal{X}(t)})$ with every point $t \in \Omega$. The value $\mathcal{X}_t$ of bundle is called its *fiber* over t. A mapping s defined on a nonempty set $\mathrm{dom}(s) \subset \Omega$ is called a *section* over $\mathrm{dom}(s)$ if $s(t) \in \mathcal{X}_t$ for every $t \in \mathrm{dom}(s)$. A section over Ω is called *global*. Let $G(\Omega, \mathcal{X})$ standfor the set of all global sections of $\mathcal{X}$ endowed with the structure of vector space by letting $(\alpha u + \beta v)(t) = \alpha u(t) + \beta v(t)$, $(t \in \Omega)$, where $\alpha, \beta \in R$ and $u, v \in G(\Omega, \mathcal{X})$. For each section $s \in G(\Omega, \mathcal{X})$ we define its point-wise norm by $|||s||| : t \mapsto \|s(t)\|_{\mathcal{X}(t)}, (t \in \Omega)$. A set of sections $\mathcal{D}$ is called *fiberwise dense* in $\mathcal{X}$ if the set $\{s(t) : s \in \mathcal{D}\}$ is dense in $\mathcal{X}(t)$ for every $t \in \Omega$.

Definition 2. Now consider a nonzero σ-finite measure space (Ω, Σ, μ). Let $\mathcal{X}$ be a bundle of Banach spaces over Ω. A set of sections $\mathcal{I} \subset G(\Omega, \mathcal{X})$ is called a *measurability structure* on $\mathcal{X}$ if it satisfies the following conditions:

1. $\mathcal{I}$ is a vector space, i.e. $\lambda v + \mu u \in \mathcal{I}\,(\lambda, \mu \in R,\, u, v \in \mathcal{I})$;

2. $|||s||| : \Omega \to R$ is measurable for $s \in \mathcal{I}$;

3. the set $\mathcal{I}$ is fiberwise dense in $\mathcal{X}$. If $\mathcal{I}$ is a measurability structure in $\mathcal{X}$ then we call the pair $(\mathcal{X}, \mathcal{I})$ a *measurable bundle of Banach spaces* over (Ω, Σ, μ). We shall write simply $\mathcal{X}$ instead $(\mathcal{X}, \mathcal{I})$.

Definition 3. Let $(\mathcal{X}, \mathcal{I})$ be a measurable bundle of Banach spaces over Ω. Denote by $M(\Omega, \mathcal{X})$ the set of all section of $\mathcal{X}$ defined almost everywhere on Ω. We say that $s \in M(\Omega, \mathcal{X})$ is a *step-section*, if $s = \sum\limits_{i=1}^{n} \chi_{A_i} c_i$ for some $n \in N$, $A_1, \dots, A_n \in \Sigma$, $c_1, \dots, c_n \in \mathcal{I}$. The set all step-sections we denote by $S(\Omega, \mathcal{X})$. A section $u \in M(\Omega, \mathcal{X})$ is called *measurable* if for every $D \in \Sigma$, $\mu(D) < \infty$ there is a sequence $(s)_{n=1}^{\infty} \subset S(\Omega, \mathcal{X})$ such that $s(t) \to u(t)$ for almost all $t \in D$. The set of all measurable sections of $\mathcal{X}$ is denoted by $L_0(\Omega, \Sigma, \mu, \mathcal{X})$ or $L_0(\mu, \mathcal{X})$ for simplicity. For a Köthe function space E on (Ω, Σ, μ) we assign

$$E(\mathcal{X}) := \{f \in L_0(\mu, \mathcal{X}) : |\!|\!| f |\!|\!| \in E\}.$$

For measurable section $f \in L_0(\Omega, \Sigma, \mu, \mathcal{X})$ by $\mathrm{supp}(f)$ we denote the measurable set $\{t \in \Omega : f(t) \neq 0\}$. Let $\mathcal{X}$ be a measurable bundle of Banach spaces over Ω. The measurable bundle of Banach spaces $\mathcal{X}_0$ over Ω is called a *measurable subbundle* of $\mathcal{X}$, if an every fiber $\mathcal{X}_0(t)$ is a Banach subspace of the $\mathcal{X}(t)$ for every $t \in \Omega$ and $L_0(\mu, \mathcal{X}_0) = L_0(\mu, \mathcal{X}) \cap M(\Omega, \mathcal{X})$.

Let $\mathcal{X}$ be a measurable bundle of Banach spaces over Ω. Since the measure μ is σ-finite we can consider a fixed lifting $\rho : L_\infty(\mu) \to \mathcal{L}_\infty(\mu)$.

Definition 4. A mapping $\rho_{\mathcal{X}} : L_\infty(\mu, \mathcal{X}) \to \mathcal{L}_\infty(\mu, \mathcal{X})$ is called a lifting of $L_\infty(\mu, \mathcal{X})$ associated with ρ if, for all $u, v \in L_\infty(\mu, \mathcal{X})$ and $e \in L_\infty(\mu)$, the following hold:

(1) $\rho_{\mathcal{X}}(u) \in u$ and $\mathrm{dom}\rho_{\mathcal{X}}(u) = \Omega$;

(2) $|\rho_{\mathcal{X}}(u)| = \rho(|u|)$;

(3) $\rho_{\mathcal{X}}(u + v) = \rho_{\mathcal{X}}(u) + \rho_{\mathcal{X}}(v)$;

(4) $|\rho_{\mathcal{X}}(u)| = \rho_{\mathcal{X}}(|u|)\Omega$;

(5) $|\rho_{\mathcal{X}}(eu)| = \rho(e)\rho(|u|)$;

(6) $\{\rho_{\mathcal{X}}(u) : u \in L_\infty(\Omega, \mathcal{X})\}$ is fiberwise dense in $\mathcal{X}$.

We say that X is a *liftable* bundle of Banach spaces provided that there exists a lifting of $L_\infty(\mu)$ and a lifting of $L_\infty(\mu, \mathcal{X})$ associated with it. We refer a reader to [5, 13] for the detailed discussion of liftable bundles of Banach spaces and their connections with the theory of lattice-normed spaces.

Let X be a Banach space. Recall that a Markushevich basis (shortly M-basis) of X is a family $(x_i, x_i^\star)_{i\in I}$, where $x_i \in X$ and $x_i^\star \in X^\star$, such that:

1) $x_i(x_j^\star) = \delta_{ij}$ (the Kronecker symbol) for every $i, j \in I$;

2) $X = \mathrm{span}\{x_i : i \in I\}$;

3) $\{x_i^\star : i \in I\}$ separates the points of X (i.e. for each $x \in X\backslash\{0\}$ there is $i \in I$ such that $x_i^\star(x) \neq 0$.

It is well known that every separable Banach space has an M-basis, see [6]. More generally, every weakly compactly generated Banach space has an M-basis, see ([6], Cor. 5.2). For complete information on this topic, we refer the reader to [6].

Let E be a Köthe function space over finite measure space. It is well known that linear continuous operator in E is a multiplication operator if and only if it commutes with all multiplication operators in E ([11], Prop. 2.2). More precisely:

Corollary 5. *A linear continuous operator $T : E \to E$ is a multiplication operator if and only if $T(gf) = gT(f)$ for all $g \in L_\infty(\mu)$ and $f \in E$.*

3. Result

For the rest of the paper, (Ω, Σ, μ) is assumed to be a finite measure space, E is a order continuous Köthe function space over (Ω, Σ, μ). At first we introduce the kind of measurable bundles of Banach spaces, which we need. Liftable measurable bundle of Banach spaces $\mathcal{X}$ over (Ω, Σ, μ) is said to have a *Generalized M-basis* or *GM-basis* for short if there is a family $(\varphi_i, \varphi_i^\star)_{i\in I}$ measurable section, where $\varphi_i \in L_\infty(\mu, \mathcal{X})$ and $\varphi_i^\star \in L_\infty(\mu, \mathcal{X}^\star)$, such that

(i) $\langle \varphi_i^\star \varphi_j \rangle = \delta_{ij}\chi_\Omega$;

(ii) for every order ideal E of $L_0(\mu)$ there exists order dense subspace E_0 of E such that $\|f(\cdot) - \theta_k(\cdot)\|_{\mathcal{X}(\cdot)} \to 0$ a.e. as $n \to \infty$ for every step-section $f \in E(\mathcal{X})$; where $\theta_k = \sum\limits_{i=1}^{n(k)} h_i^k(\cdot)\varphi_i^k(\cdot), \varphi_i^k$ are elements of GM-basis, $h_i^k \in E_0$ for every $k \in N, 1 \le i \le n(k)$;

(iii) for every measurable section $f \in L_0(\mathcal{X})$, $\mu(\mathrm{supp}(f)) > 0$ there exist $\varphi_{i_0}^\star$ and measurable set $A \subset \mathrm{supp}(f)$, $\mu(A) > 0$ so that $g(\omega) := \langle \varphi_{i_0}^\star, f\rangle(\omega) > 0$ for every $\omega \in A$.

Corollary 6. *Suppose liftable measurable bundles of Banach spaces $\mathcal{X}$ has an GM-basis $(\varphi_i, \varphi_i^\star)_{i\in I}$. Let $T : E(\mathcal{X}) \to E(\mathcal{X})$ be operator satisfying*

$$T(g\langle f, \varphi_j^\star\rangle\varphi_i) = g(\langle T(f), \varphi_j^\star\rangle\varphi_i) \tag{1}$$

for every $g \in L_\infty(\mu)$, $f \in E(\mathcal{X})$ and $i, j \in I$. Then there is $g_0 \in L_\infty(\mu)$ such that $T(f) = g_0 f$ for every measurable section $f \in E(\mathcal{X})$.

Proof. We first notice that

$$\langle T(h\varphi_i), \varphi_j^\star\rangle = 0; \;\; i \ne j,\, h \in E. \tag{2}$$

Indeed, (3.1) applied to $f := h\varphi_i \in E(\mathcal{X})$ and $g := \chi_\Omega \in L_\infty(\mu)$ yields

$$\langle T(h\varphi_i), \varphi_j^\star\rangle\varphi_i = T(\langle h\varphi_i, \varphi_j^\star\rangle\varphi_i) = T(0) = 0,$$

and because $\operatorname{supp}(\varphi_i) = \Omega$ we have $\langle T(h\varphi_i), \varphi_j^\star\rangle) = 0$. For each $i \in I$ we define $S_i^T : E \to E$ by

$$S_i^T(h) := \langle T(h\varphi_i), \varphi_i^\star\rangle.$$

The mapping S_i^T is a linear continuous operator for every $i \in I$, because

$$\begin{aligned}\|S_i^T h\|_{E(\mathcal{X})} &= \|\langle T(h\varphi_i), \varphi_i^\star\rangle\|_E \leq \|\varphi_i^\star\|_{L_\infty(\mathcal{X})}\|T(h\varphi_i)\|_{E(\mathcal{X})} \\ &\leq \|\varphi_i^\star\|_{L_\infty(\mathcal{X})}\|T\|\|h\varphi_i\|_{E(\mathcal{X})} \leq K_i\|h\|_E\end{aligned}$$

for all $h \in E$, where $K_i := \|\varphi_i^\star\|_{L_\infty(\mathcal{X})}\|T\|\|\varphi_i^\star\|_{L_\infty(\mathcal{X}^\star)}$. We observe that each S_i^T satisfies

$$S_i^T(gh) = gS_i^T(h), \text{ for every } g \in L_\infty(\mu),\ h \in E. \tag{3}$$

Indeed, (3.1) with $i = j$ applied to $f := h\varphi_i \in E(\mathcal{X})$ yields

$$T(gh\varphi_i) = T(g\langle h\varphi_i, \varphi_i^\star\rangle\varphi_i) = g\langle T(h\varphi_i), \varphi_i^\star\rangle\varphi_i,$$

hence

$$\begin{aligned}S_i^T(gh) &= \langle T(gh\varphi_i), \varphi_i^\star\rangle = \langle(g\langle T(h\varphi_i), \varphi_i^\star\rangle\varphi_i), \varphi_i^\star\rangle \\ &= g\langle T(h\varphi_i), \varphi_i^\star\rangle = gS_i^T(h).\end{aligned}$$

For each $i \in I$ equality (3.3) allows us to apply Corollary 5 to find $g_i \in L_\infty(\mu)$ such that

$$\langle T(h\varphi_i, \varphi_i^\star)\rangle = S_i^T(h) = g_i h, \text{ for every } h \in E. \tag{4}$$

We claim that $g_i = g_j$ for every $i, j \in I$. Indeed, fix $i \neq j$ and consider the measurable section $f := \varphi_i + \varphi_j \in E(\mathcal{X})$. By (3.3), (3.2) and (3.1) (with $g := \chi_\Omega \in L_\infty(\mu)$ and $h := \chi_\Omega \in E$) we have

$$\begin{aligned}g_i\varphi_i &= \langle T(\varphi_i), \varphi_i^\star\rangle\varphi_i = \langle T(\varphi_i), \varphi_i^\star\rangle\varphi_i + \langle T(\varphi_j), \varphi_i^\star\rangle\varphi_i \\ &= \langle T(f), \varphi_i^\star\rangle)\varphi_j = T(\langle f, \varphi_i^\star\rangle\varphi_j) = T(\varphi_j),\end{aligned}$$

and similarly, we also have

$$\begin{aligned}g_j\varphi_i &= \langle T(\varphi_j), \varphi_j^\star\rangle\varphi_i = \langle T(\varphi_j), \varphi_j^\star\rangle\varphi_i + \langle T(\varphi_i), \varphi_j^\star\rangle\varphi_i \\ &= \langle T(f), \varphi_j^\star\rangle)\varphi_i = T(\langle f, \varphi_j^\star\rangle\varphi_i) = T(\varphi_i).\end{aligned}$$

Hence $g_i\varphi_i = g_j\varphi_i$ and because $\operatorname{supp}(\varphi_i) = \Omega$, we have $g_i = g_j$. Therefore, there is $g_0 \in L_\infty(\mu)$ such that

$$\langle T(h\varphi_i), \varphi_i^\star)\rangle = g_0 h, \text{ for every } h \in E,\ i \in I. \tag{5}$$

Let f be a step-section and $f \in E(\mathcal{X})$. To this end, fix $\varepsilon > 0$. We will prove that $T(f) = g_0 f$. Using conditions (ii) we have

$$\|f(\cdot) - \sum_{i=1}^{n(k)} h_i^k(\cdot)\varphi_i^k(\cdot)\|_{\mathcal{X}(\cdot)} \to 0, \ \ \mu - \text{a.e.}, \ \ k \to \infty.$$

Set

$$f_0 := \sum_{i=1}^{n(k)} h_i^k \varphi_i^k, \ \text{ and } \ f_1 := g_0 f_0 = \sum_{i=1}^{n(k)} g_0 h_i^k \varphi_i^k \in E(\mathcal{X}).$$

For each $i \in I$ equalities (3.2) and (3.5) yield

$$\langle T(f_0), \varphi_j^\star\rangle = \sum_{i=1}^{n(k)} \langle T(h_i^k \varphi_i^k), \varphi_j^\star\rangle = \sum_{i=1}^{n(k)} \delta_{i,j} g_0 h = \langle f_1, \varphi_j^\star\rangle, \text{ for every } j \in I,$$

and using (iii) we have $T(f_0) = f_1 = g_0 f_0$. Therefore,

$$\begin{aligned}
\|T(f) - g_0 f\|_{E(\mathcal{X})} &= \|T(f) - Tf_0 + g_0 f_0 - g_0 f\|_{E(\mathcal{X})} \\
&\leq \|T(f) - T(f_0)\|_{E(\mathcal{X})} + \|g_0 f_0 - g_0 f\|_{E(\mathcal{X})} \\
&\leq \|T\|\|f - f_0\|_{E(\mathcal{X})} + \|g_0\|_E \|f_0 - f\|_{E(\mathcal{X})} \\
&= \|T\|(\|\|f(\cdot) - f_0(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) + \|g_0\|_E(\|\|f(\cdot) - f_0(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) \\
&= \|T\|(\|\|f(\cdot) - \sum_{i=1}^{n(k)} h_i^k(\cdot)\varphi_i^k(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) \\
&\quad + \|g_0\|_E(\|\|f(\cdot) - \sum_{i=1}^{n(k)} h_i^k(\cdot)\varphi_i^k(\cdot))\|_{\mathcal{X}(\cdot)}\|_E).
\end{aligned}$$

Using fact that E is a order continuous we can find $k \in N$ such that

$$\|\|f(\cdot) - \sum_{i=1}^{n(k)} h_i^k(\cdot)\varphi_i^k(\cdot))\|_{\mathcal{X}(\cdot)}\|_E < \varepsilon,$$

and finally we have

$$\|T(f) - g_0 f\|_{E(\mathcal{X})} \leq C\varepsilon$$

where $C := (\|T\| + \|g_0\|_E)$. It follows at once that $T(f) = g_0 f$ for every $f \in S(\mathcal{X})$. In addition $S(\mathcal{X})$ is dense in $E(\mathcal{X})$ with respect the norm $\|\cdot\|_{E(\mathcal{X})}$ and so the equality $T(f) = g_0 f$ holds for every $f \in E(\mathcal{X})$, because both T and $M_{g_0} : E(\mathcal{X}) \to E(\mathcal{X})$ are continuous operators. □

Now we need the following corollary.

Corollary 7. *Let $\mathcal{X}$ be liftable measurable bundles of Banach spaces, $T: E(\mathcal{X}) \to E(\mathcal{X})$ a linear continuous operator satisfying*

$$T(g, \langle f, \phi^\star \rangle \phi) = g(\langle T(f), \phi^\star \rangle \phi) \tag{6}$$

for every $g \in L_\infty(\mu)$, $f \in E(\mathcal{X})$, $\phi \in L_\infty(\mu, \mathcal{X})$ and $\phi^\star \in L_\infty(\mu, \mathcal{X}^\star)$. Let $\mathcal{Y}$ be an measurable subbundle of $\mathcal{X}$. Then T maps $E(\mathcal{Y})$ into itself.

Proof. Fix $f \in E(\mathcal{Y})$ of the form $f = h\psi$ for some $h \in E$ and $\psi \in L_\infty(\mu, \mathcal{Y})$, $\operatorname{supp}(\psi) = \Omega$. Then there exists $\psi^\star \in L_\infty(\mu, \mathcal{X}^\star)$ such that $\langle \psi(\cdot), \psi^\star(\cdot) \rangle = \chi_\Omega$. By applying (3.6) with $g := \chi_\Omega$ and $f = \psi$ we get

$$T(f) = T(\langle h\psi, \psi^\star \rangle \psi) = T(\langle f, \psi^\star \rangle \psi) = (\langle T(f), \psi^\star \rangle \psi) \in E(\mathcal{Y}).$$

□

Let (Ω, Σ, μ) be a finite measure space and $\mathcal{X}$ a liftable measurable bundle of Banach spaces over Ω. $\mathcal{X}$ is said to have *R-property* if for every measurable sections $f, g \in L_0(\mu, \mathcal{X})$ there exists a measurable subbundle $\mathcal{Y}_{f,g}$ of $\mathcal{X}$ such that $f(t), g(t) \in \mathcal{Y}_{f,g}(t)$ for almost every all $t \in \Omega$ and $\mathcal{Y}_{f,g}$ has a *GM*-basis.

Now we can prove the main result.

Theorem 8. *Let $\mathcal{X}$ be liftable measurable bundle of Banach spaces and $\mathcal{X}$ has a R-property. Let $T: E(\mathcal{X}) \to E(\mathcal{X})$ be a linear continuous operator. The following statements are equivalent:*

1) *T is a multiplication operator, that is, there is $g_0 \in L_\infty(\mu)$ such that $T(f) = g_0 f$ for all $f \in E(\mathcal{X})$;*
2) *The equality $T(g\langle f, \varphi^\star \rangle \varphi) = g(\langle T(f), \varphi^\star \rangle \varphi)$ holds for every $g \in L_\infty(\mu)$, $f \in E(\mathcal{X})$, $\varphi \in L_\infty(\mu, \mathcal{X})$ and $\varphi^\star \in L_\infty(\mu, \mathcal{X}^\star)$.*

Proof. 1) $\Rightarrow$ 2) is straightforward. 2) $\Rightarrow$ 1). Fix $\psi \in L_\infty(\mu, \mathcal{X})$, $\operatorname{supp}(\psi) = \Omega$ and $\psi^\star \in L_\infty(\mu, \mathcal{X}^\star)$ with $\langle \psi, \psi^\star \rangle = \chi_\Omega$. As in the proof of Corollary 6, we can define an operator

$$S^T: E \to E; \;\; S^T(h) := \langle T(h\psi), \psi^\star \rangle$$

which satisfies

$$S^T(gh) = gS^T(h); \text{ for every } g \in L_\infty(\mu),\ h \in E.$$

By Corollary 5 there the exists $g_0 \in L_\infty(\mu)$ such that

$$\langle T(h\psi), \psi^\star \rangle = S^T(h) = g_0 h. \tag{7}$$

We claim that $T(f) = g_0 f$ for all $f \in E(\mathcal{X})$. Indeed, take any $f \in E(\mathcal{X})$. Since $\mathcal{X}$ has the R-property, there is a measurable subbundle $\mathcal{Y}_{f,\psi}$ of X such that $f(t), \psi(t) \in \mathcal{Y}_{f,\psi}(t)$ for almost every all $t \in \Omega$ and $\mathcal{Y}_{f,\psi}$ has a GM-basis. By Corollary 7 we have $T(E(\mathcal{Y}_{f,\psi})) \subset E(\mathcal{Y}_{f,\psi})$. Clearly,

$$T|_{E(\mathcal{Y}_{f,\psi})}(g\langle f_1, \varphi^\star \rangle \varphi) = g\langle T|_{E(\mathcal{Y}_{f,\psi})}(f_1), \varphi^\star \rangle \varphi$$

for every $g \in L_\infty(\mu)$, $f_1 \in E(\mathcal{Y}_{f,\psi})$, $\varphi^\star \in L_\infty(\mu, \mathcal{X}^\star)$, $\varphi \in L_\infty(\mu, \mathcal{Y}_{f,\psi})$. Corollary 6 applied to $T|_{E(\mathcal{Y}_{f,\psi})}$ ensures the existence of $g \in L_\infty(\mu)$ so that

$$T(f_1) = gf_1; \ f_1 \in E(\mathcal{Y}_{f,\psi}).$$

Since both ψ and f belong to $E(\mathcal{Y}_{f,\psi})$, we have $T(\psi) = g\psi$ and $T(f) = gf$. On the other hand, (3.7) applied to $h := \chi_\Omega \in E$ yields

$$g_0 = \langle T(\psi), \psi^\star \rangle = \langle g\psi, \psi^\star \rangle = g,$$

and so $T(f) = g_0 f$, as claimed. □

Notice that the particular case of Theorem 8 for a constant Banach bundle X was proved in ([3], Th. 1.4).

Acknowledgments. The research was supported by the Russian Foundation for Basic Research (Grant Number 17-51-12064).

References

[1] N. Abasov, A.M. Megahed, M. Pliev, Dominated operators from lattice-normed spaces to sequence Banach lattices, *Annals of Functional Analysis*, **7**, No 4 (2016), 646-655.

[2] C.D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht (2006).

[3] J.M. Calabuig, J. Rogriges, E.A. Sanchez-Perez, Multiplication operators in Köthe-Bochner spaces, *J. Math. Anal. Appl.*, **373** (2011), 316-321.

[4] B. De Pagter, W.J. Ricker, Algebras of multiplication operators in Banach function spaces, *J. Operator Theory*, **42**, No 2 (1999), 245-267.

[5] A.E. Gutman, Banach bundles in the theory of lattice-normed spaces, In: *Linear Operators Compatible with Order*, Sobolev Institute Press, Novosibirsk (1995), 63-211.

[6] P. Hajek, V. Montesinos Santalucia, J. Vanderwerff, V. Zizler, *Biorthogonal Systems in Banach Spaces*, Springer, New York (2008).

[7] V. Orlov, M. Pliev, D. Rode, Domination problem for AM-compact abstract Uryson operators, *Archiv der Math.*, **107**, No 5 (2016), 543-552.

[8] M. Pliev, Narrow operators on lattice-normed spaces, *Open Math.*, **9**, No 6 (2011), 1276-1287.

[9] M. Pliev, M. Popov, On extension of abstract Urysohn operators, *Siberian Math. J.*, **57**, No 3 (2016), 552-557.

[10] M.A. Pliev, M.R. Weber, Disjointness and order projections in the vector lattices of abstract Uryson operators, *Positivity*, **20**, No 3 (2016), 695-707.

[11] M. Pliev, K. Ramdane, Order unbounded orthogonally additive operators in vector lattices, *Mediterranean J. of Math.*, **15**, No 2 (2018), 1-19.

[12] M. Pliev, X. Fang, Narrow orthgonally additive operators in lattice-normed spaces, *Siberian Math. J.*, **58**, No 1 (2017), 134-141.

[13] A.G. Kusraev, *Dominated Operators*, Springer, Dordrecht (2000).