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CHARACTERIZATIONS OF γ-P_S-REGULAR AND γ-P_S-NORMAL SPACES

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Abstract: This paper defines some types of γ-P_S- separation axioms called γ-P_S-regular and γ-P_S-normal spaces using γ-P_S-open and γ-P_S-closed sets. Some relations, properties and characterizations of these spaces are discussed. Several examples are given to illustrate some of the results.

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0. Introduction

Kasahara [12] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. After this, Jankovic [10] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of function with α-closed graph. Later, Ogata [15] defined and studied the concept of γ-open sets, and applied it to investigate operation-functions and operation-separation axioms. Since that, γ operation on τ has attracted the

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attention of many researchers. Among them are γ-regular-open sets [6], α-γ-open sets [11], γ-preopen sets [13] and γ-semiopen sets [14]. Using some of these sets, several separation axioms such as γ-pre-regular and γ-pre-normal [8] were defined. In 2018, Haxha and Gjonbalaj [9] defined the concept of $\gamma^{\star}$-operation in the topological product of topological spaces and they also introduced some $\gamma^{\star}$-separation axioms. Ameen, Asaad and Muhammed [16] considered properties of δ-preopen, δ-semiopen, a-open and e^{*}-open sets in topological spaces.

Recently, the notion of γ-P_S-open sets had been defined by Asaad, Ahmad and Omar [4]. This set is stronger than γ-preopen set and weaker than γ-regular-open set. Besides that, they also introduced γ-locally indiscrete, γ-hyperconnected and γ-extremally disconnected spaces [6]. In addition, Asaad and Ahmad [1] studied further characterizations of γ-extremally disconnected spaces and investigated some relations of functions of γ-extremally disconnected spaces. Furthermore, Asaad, Ahmad and Omar ([3], [7]) introduced and studied the notion of γ-P_S-T_i spaces for $i = 0, \frac{1}{2}, 1, 2$.

The aim of this paper is to introduce some types of γ-P_S- separation axioms called γ-P_S-regular and γ-P_S-normal spaces using γ-P_S-open and γ-P_S-closed sets. Some relations and characterizations between these constructed γ-P_S-spaces with other existence γ-P_S- separation axioms as well as their properties have also been investigated.

1. Preliminaries and main definitions

In this paper, the pairs (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms assumed unless otherwise mentioned. An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [15]. A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-open [15] if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ-open set is called a γ-closed. The class of all γ-open sets of X is denoted by τ_γ. The τ_γ-closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ-closed sets containing A and it is denoted by τ_γ-$Cl(A)$ [15], and the τ_γ-interior of a subset A of X with an operation γ on τ is defined as the union of all γ-open sets containing A [14]. An operation γ on τ is said to be regular if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$, [15].

The following recalls necessary concepts and preliminaries required in the sequel of our work.

Definition 1. A subset A of a topological space (X, τ) with an operation γ on τ is said to be:

(i) γ-regular-open [6] if $A = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$.

(ii) γ-preopen [13] if $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$.

(iii) γ-semiopen [14] if $A \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A))$.

(iv) α-γ-open [11] if $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A)))$.

Definition 2. The complement of a γ-regular-open [6] (respectively, γ-preopen [13], γ-semiopen [14] and α-γ-open [11]) set is called γ-regular-closed (respectively, γ-preclosed, γ-semiclosed and α-γ-closed).

Definition 3. ([4]) A γ-preopen subset A of a topological space (X, τ) is called γ-P_S-open if for each $x \in A$, there exists a γ-semiclosed set F such that $x \in F \subseteq A$. The complement of a γ-P_S-open set is called γ-P_S-closed.

Definition 4. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the τ_γ-P_S-closure [4] (respectively, τ_γ-semiclosure [14] and $\tau_{\alpha-\gamma}$-closure [11]) of A is defined as the intersection of all γ-P_S-closed (respectively, γ-semiclosed and α-γ-closed) sets of X containing A and it is denoted by $\tau_\gamma\text{-}P_SCl(A)$ (respectively, $\tau_\gamma\text{-}sCl(A)$ and $\tau_{\alpha-\gamma}\text{-}Cl(A)$). The τ_γ-P_S-interior [4] of A is defined as the union of all γ-P_S-open sets of X contained in A and it is denoted by $\tau_\gamma\text{-}P_SInt(A)$.

Lemma 1.1. ([4]) Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then:

(i) A is γ-P_S-closed if and only if $\tau_\gamma\text{-}P_SCl(A) = A$.

(ii) $x \in \tau_\gamma\text{-}P_SCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-P_S-open set U of X containing x.

Definition 5. ([3]) A subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-P_S-generalized closed (shortly γ-P_S-g-closed) if $\tau_\gamma\text{-}P_SCl(A) \subseteq G$ whenever $A \subseteq G$ and G is γ-P_S-open set in X. The complement of a γ-P_S-g-closed set is called a γ-P_S-g-open.

For any topological space (X, τ) and any operation γ on τ, we denote the class of all γ-P_S-open (respectively, γ-P_S-closed, γ-P_S-g-open, γ-regular-open, α-γ-open and γ-semiopen) sets of X by τ_γ-$P_SO(X)$ (respectively, τ_γ-$P_SC(X)$, τ_γ-$P_SGO(X)$, τ_γ-$RO(X)$, $\tau_{\alpha-\gamma}$ and τ_γ-$SO(X)$).

Theorem 1.2. A set A is γ-P_S-g-open if $F \subseteq \tau_\gamma$-$P_SInt(A)$ whenever $F \subseteq A$ and F is a γ-P_S-closed set in X.

Definition 6. ([6]) A topological space (X, τ) is said to be γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed subset of X is γ-open.

Lemma 1.3. If (X, τ) is γ-locally indiscrete space, then γ is a regular operation. Therefore, if (X, τ) is γ-locally indiscrete, then τ_γ-$P_SO(X, \tau)$ is a topology on (X, τ).

Remark 1.4. ([6], [4]) If (X, τ) is γ-locally indiscrete space, then:

$$\tau_\gamma\text{-}RO(X) = \tau_\gamma = \tau_\gamma\text{-}P_SO(X) = \tau_{\alpha-\gamma} = \tau_\gamma\text{-}SO(X).$$

Definition 7. ([7]) A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-T_0 if for each pair of distinct points x, y in X, there exists a γ-P_S-open set G such that either $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

Definition 8. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-T_1, [7] (respectively, γ-semiT_1 [14]) if for each pair of distinct points x, y in X, there exist two γ-P_S-open (respectively, γ-semiopen) sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

Definition 9. ([7]) A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-T_2 if for each pair of distinct points x, y in X, there exist two γ-P_S-open sets G and H containing x and y respectively such that $G \cap H = \phi$.

Definition 10. ([3]) A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-$T_{\frac{1}{2}}$ if every γ-P_S-g-closed set in X is γ-P_S-closed.

Theorem 1.5. ([3]) For any topological space (X, τ) with an operation γ on τ. Then X is γ-P_S-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is

γ-P_S-closed or γ-P_S-open.

Theorem 1.6. ([4]) Let (X,τ) be a topological space and γ be an operation on τ. If (X,τ) is γ-semiT_1, then the concept of γ-P_S-open (respectively, γ-P_S-closed) set and γ-preopen (respectively, γ-preclosed) set are identical.

Lemma 1.7. ([7]) Let (X,τ) be a topological space and γ be an operation on τ. Then the following statements are hold:

(i) If X is γ-P_S-T_2, then it is γ-P_S-T_1.

(ii) If X is γ-P_S-T_1, then it is γ-P_S-$T_{\frac{1}{2}}$.

(iii) If X is γ-P_S-$T_{\frac{1}{2}}$, then it is γ-P_S-T_0.

2. Characterizations of γ-P_S-regular spaces

In this section, γ-P_S-regular space is a new type of γ-P_S- separation axioms will be defined. Some characterizations of γ-P_S-regular spaces are investigated.

Definition 11. A topological space (X,τ) with an operation γ on τ is said to be γ-P_S-regular if for each γ-P_S-closed set F of X not containing x, there exist disjoint γ-P_S-open sets G and H such that $x \in G$ and $F \subseteq H$.

The following are some significant characterizations of γ-P_S-regular spaces.

Theorem 2.1. Let (X,τ) be a topological space and γ be an operation on τ. Then the following conditions hold:

(i) X is γ-P_S-regular.

(ii) For each point x in X and each γ-P_S-open set U containing x, there exists a γ-P_S-open set V such that $x \in V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq U$.

(iii) For each γ-P_S-closed set F of X, $\cap \{\tau_\gamma$-$P_SCl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_SO(X)\} = F$.

(iv) For each subset S of X and each γ-P_S-open set G of X such that $S \cap G \neq \phi$, there exists a γ-P_S-open set H of X such that $S \cap H \neq \phi$ and τ_γ-$P_SCl(H) \subseteq G$.

(v) For each nonempty subset S of X and each γ-P_S-closed set F of X such that $S \cap F = \phi$, there exists γ-P_S-open sets M and N of X such that S $\cap$ $M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$.

(vi) For each $x \in X$ and each γ-P_S-closed set $F \subseteq X$ such that $x \notin F$, there exists γ-P_S-open set M and γ-P_S-g-open set N such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$.

(vii) For each $S \subseteq X$ and each γ-P_S-closed set $F \subseteq X$ such that $S \cap F = \phi$, there exists γ-P_S-open set M and γ-P_S-g-open set M such that $S \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$.

Proof. (i) $\Rightarrow$ (ii) Let x be any point in X and U be any γ-P_S-open set in X containing x. Then $X \backslash U$ is γ-P_S-closed set such that $x \notin X \backslash U$. Since X is γ-P_S-regular space, then there exists γ-P_S-open sets V and H such that $x \in V$ and $X \backslash U \subseteq H$ and $V \cap H = \phi$. This implies that $V \subseteq X \backslash H$ and hence $x \in V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq U$.

(ii) $\Rightarrow$ (iii) Let F be any γ-P_S-closed set of X. Suppose that $x \in X \backslash F$, where $X \backslash F$ is γ-P_S-open set and $x \in X$. Then by (ii), there exists a γ-P_S-open set V such that $x \in V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq X \backslash F$. This implies that $F \subseteq X \backslash \tau_\gamma$-$P_SCl(V)$. Let $H = X \backslash \tau_\gamma$-$P_SCl(V)$, then H is γ-P_S-open set such that $V \cap H = \phi$. Thus, by Lemma 1.1 (ii), $x \notin \tau_\gamma$-$P_SCl(H)$. Therefore, $\cap$ $\{\tau_\gamma$-$P_SCl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_SO(X)\} \subseteq F$.

(iii) $\Rightarrow$ (iv) Let S be any subset of X and G be any γ-P_S-open set of X such that $S \cap G \neq \phi$. Let $x \in S \cap G$. Then $x \notin X \backslash G$. Since $X \backslash G$ is γ-P_S-closed, then by (iii), there exists a γ-P_S-open set O such that $X \backslash G \subseteq O$ and $x \notin \tau_\gamma$-$P_SCl(O)$. If we take $H = X \backslash \tau_\gamma$-$P_SCl(O)$, then H is a γ-P_S-open set of X containing x and $S \cap H \neq \phi$. Now $H \subseteq X \backslash O$ and hence τ_γ-$P_SCl(H) \subseteq X \backslash O \subseteq G$.

(iv) $\Rightarrow$ (v) Let S be any nonempty subset of X and F be any γ-P_S-closed set of X such that $S \cap F = \phi$. Then $X \backslash F$ is γ-P_S-open set and $S \cap X \backslash F \neq \phi$. By (iv), there exists a γ-P_S-open set M of X such that $S \cap M \neq \phi$ and τ_γ-$P_SCl(M) \subseteq X \backslash F$. Let $N = X \backslash \tau_\gamma$-$P_SCl(M)$, then $F \subseteq N$ and $M \cap N = \phi$.

(v) $\Rightarrow$ (vi) Let x be any point X and a γ-P_S-closed set $F \subseteq X$ such that $x \notin F$. Then $\{x\}$ is a subset of X and $\{x\} \cap F = \phi$. Hence by using (v), there exists γ-P_S-open sets M and N of X such that $\{x\} \cap M \neq \phi$, $F \subseteq N$ and M $\cap$ $N = \phi$. Since every γ-P_S-open set is γ-P_S-g-open, then N is a γ-P_S-g-open set such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$.

(vi) $\Rightarrow$ (vii) For any subset S of X and any γ-P_S-closed set F in X such that $S \cap F = \phi$. Suppose $x \in S$, then $x \notin F$. By (vi), there exists γ-P_S-open set M and γ-P_S-g-open set N such that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$. Hence $S \cap M \neq \phi$.

(vii) $\Rightarrow$ (i) For each $x \in X$ and each γ-P_S-closed set F of X such that $x \notin F$. Then $\{x\}$ is a subset of X and $\{x\} \cap F = \phi$. So by using (vii), there exists γ-P_S-open set M and γ-P_S-g-open set N such that $\{x\} \cap M \neq \phi$, $F \subseteq N$ and $M \cap N = \phi$. This implies that $x \in M$, $F \subseteq N$ and $M \cap N = \phi$. Now, let $H = \tau_\gamma$-$P_S Int(N)$. Thus, by Theorem 1.2, we obtain $F \subseteq H$ and $M \cap H = \phi$. Therefore, a space X is γ-P_S-regular. □

Theorem 2.2. The following statements hold for any topological space (X, τ) and any operation γ on τ:

(i) X is γ-P_S-regular.

(ii) For each γ-P_S-closed set F of X, $\cap \{\tau_\gamma$-$P_S Cl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_S O(X)\} = F$.

(iii) For each γ-P_S-closed set F of X, $\cap \{\tau_\gamma$-$P_S Cl(H) : F \subseteq H$ and $H \in \tau_\gamma$-$P_S GO(X)\} = F$.

Proof. (i) $\Rightarrow$ (ii) It is clear from Theorem 2.1.

(ii) $\Rightarrow$ (iii) It is evident that every γ-P_S-open set is γ-P_S-g-open.

(iii) $\Rightarrow$ (i) Let F be a γ-P_S-closed set in X and let $x \in X$ such that $x \notin F$. Then by (iii), there exists a γ-P_S-g-open set H such that $F \subseteq H$ and $x \notin \tau_\gamma$-$P_S Cl(H)$. Since F is γ-P_S-closed and H is a γ-P_S-g-open, then by Theorem 1.2, we get $F \subseteq \tau_\gamma$-$P_S Int(H)$. Let $U = \tau_\gamma$-$P_S Int(H)$ which is γ-P_S-open and hence $F \subseteq U$. Also, let $V = X \backslash \tau_\gamma$-$P_S Cl(H)$ is a γ-P_S-open set containing x such that $V \cap U = \phi$. Hence X is γ-P_S-regular space. □

Theorem 2.3. Let (X, τ) be a γ-locally indiscrete space and γ be a regular operation on τ. Then X is γ-P_S-regular if and only if for each γ-P_S-closed set F of X not containing x in X, there exist γ-P_S-open sets G and H such that $x \in G$, $F \subseteq H$ and τ_γ-$P_S Cl(G) \cap \tau_\gamma$-$P_S Cl(H) = \phi$.

Proof. Let F be a γ-P_S-closed set in X and $x \in X$ does not belong to F. Since (X, τ) is γ-P_S-regular space, then there exist γ-P_S-open sets G_0 and H such that $x \in G_0$, $F \subseteq H$ and $G_0 \cap H = \phi$ which implies $G_0 \cap \tau_\gamma$-$P_S Cl(H) = \phi$. Since τ_γ-$P_S Cl(H)$ is γ-P_S-closed such that $x \notin \tau_\gamma$-$P_S Cl(H)$ and X is γ-P_S-regular, then there exist γ-P_S-open sets U and V such that $x \in U$, τ_γ-$P_S Cl(H) \subseteq V$ and $U \cap V = \phi$. This implies that τ_γ-$P_S Cl(U) \cap V = \phi$. If we take $G = G_0 \cap U$. Since (X, τ) is a γ-locally indiscrete space and γ is a regular operation on τ, then by Lemma 1.3, G is γ-P_S-open set. So G and H are γ-P_S-open sets such that $x \in G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

The converse is trivial. □

Notice that γ-regular-open, γ-open, α-γ-open and γ-semiopen sets can be substitute to γ-P_S-open set in Theorem 2.3 (this is because of Remark 1.4). Again, τ_γ-closure, $\tau_{\alpha-\gamma}$-closure and τ_γ-semi-closure of a set can be replaced by τ_γ-P_S-closure.

Theorem 2.4. The following statements are equivalent for any topological space (X, τ) with an operation γ on τ:

(i) X is γ-P_S-regular.

(ii) For each point x in X and each γ-P_S-open set H containing x, there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \subseteq H$.

(iii) For each point $x \in X$ and for each γ-P_S-closed set F of X not containing x, there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \cap F = \phi$.

Proof. (i) $\Rightarrow$ (ii) It is similar to Theorem 2.1.

(ii) $\Rightarrow$ (iii) Let $x \in X$ and let F be a γ-P_S-closed set of X such that $x \notin F$. Then $X \backslash F$ is γ-P_S-open set containing x. So by (ii), there exists a γ-P_S-open set $H \subseteq X$ containing x such that τ_γ-$P_SCl(G) \subseteq X \backslash F$. This implies that τ_γ-$P_SCl(G) \cap F = \phi$.

(iii) $\Rightarrow$ (i) Let F be any γ-P_S-closed set in X such that $x \notin F$. Then by (iii), there exists a γ-P_S-open set G containing x such that τ_γ-$P_SCl(G) \cap F = \phi$ which implies that $F \subseteq X \backslash \tau_\gamma$-$P_SCl(G)$. Since $X \backslash \tau_\gamma$-$P_SCl(G)$ is γ-P_S-open such that $X \backslash \tau_\gamma$-$P_SCl(G) \cap G = \phi$. Therefore, X is γ-P_S-regular space. □

The following example shows that the relation between the γ-P_S-regularity and γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$ are independent.

Example 2.5. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define an operation $\gamma \colon \tau \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$. Hence τ_γ-$P_SO(X) = \{\phi, X, \{a\}, \{b, c, d\}\}$. The space (X, τ) is γ-P_S-regular, but it is not γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$.

Example 2.6. Let a space $X = \{a, b, c\}$ with the topology τ=$\{\phi$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, c\}\}$. Define an operation $\gamma\colon \tau \to P(X)$ as follows:
For every $A \in \tau$,

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ Cl(A) & \text{if } a \notin A \end{cases}.$$

Clearly, $\tau_\gamma = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$. So τ_γ-$P_SO(X) = \{\phi,\ X,\ \{a\},\ \{a, c\}, \{b, c\}\}$. Therefore, the space (X, τ) is γ-P_S-T_i for $i = 0, \frac{1}{2}, 1, 2$, but it is not γ-P_S-regular since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

Definition 12. ([8]) A topological space (X, τ) with an operation γ on τ is said to be γ-pre-regular if for each γ-preclosed set F of X not containing x, there exist disjoint γ-preopen sets U and V such that $x \in U$ and $F \subseteq V$.

The relation between γ-P_S-regularity and γ-pre-regularity are independent while they are equivalent when a topological space (X, τ) is γ-semiT_1 as can be seen in the following two examples and theorem.

Example 2.7. In Example 2.6, the space (X, τ) is γ-pre-regular, but it is not γ-P_S-regular since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

Example 2.8. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\phi,\ \{a, b\}, \{c\},\ X\}$. Then $P_SO(X) = \tau$. Define an operation γ on τ by:
For every $A \in \tau$,

$$\gamma(A) = \begin{cases} A & \text{if } A = \{c\} \\ X & \text{if } A \neq \{c\} \end{cases}.$$

Obviously, $\tau_\gamma = \{\phi, X, \{c\}\}$ and τ_γ-$P_SO(X) = \{\phi, X\}$. Then the space (X, τ) is γ-P_S-regular, but it is not γ-pre-regular since the set $\{a, b\}$ is γ-preclosed not containing c, then there is no disjoint γ-preopen sets G and H such that $c \in G$ and $\{a, b\} \subseteq H$.

Theorem 2.9. Let (X, τ) be a γ-semiT_1 space. Then (X, τ) is γ-P_S-regular if and only if (X, τ) is γ-pre-regular.

Proof. The proof follows from Theorem 1.6. □

Definition 13. ([5]) Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-irresolute at a point $x \in X$ if for each β-P_S-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is (γ,β)-P_S-irresolute at every point x in X, then f is said to be (γ,β)-P_S-irresolute.

Theorem 2.10. ([5]) The following properties are equivalent for any function $f\colon (X,\tau) \to (Y,\sigma)$, where γ and β are operations on τ and σ, respectively:

(i) f is (γ,β)-P_S-irresolute.

(ii) The inverse image of every β-P_S-open set of Y is γ-P_S-open set in X.

(iii) The inverse image of every β-P_S-closed set of Y is γ-P_S-closed set in X.

Definition 14. ([5]) A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-open (respectively, (γ,β)-P_S-closed) if for every γ-P_S-open (respectively, γ-P_S-closed) set V of X, then $f(V)$ is β-P_S-open (respectively, β-P_S-closed) set in Y.

Lemma 2.11. ([5]) For a bijective function $f\colon (X,\tau) \to (Y,\sigma)$. The following properties of f are equivalent:

(i) f^{-1} is (γ,β)-P_S-irresolute.

(ii) f is (γ,β)-P_S-open.

(iii) f is (γ,β)-P_S-closed.

Lemma 2.12. ([5]) Let (Y,σ) be a topological space and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-closed if and only if for each subset S of Y and each γ-P_S-open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.

In the next results, we give some properties of γ-P_S-regular spaces which is related to γ-P_S-functions.

Theorem 2.13. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a bijective, (γ,β)-P_S-irresolute and (γ,β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X,τ) is γ-P_S-regular, then (Y,σ) is also β-P_S-regular.

Proof. Let E be a β-P_S-closed set of (Y, σ) such that $y \notin E$. Since f is (γ, β)-P_S-irresolute function, then by Theorem 2.10 (iii), $f^{-1}(E)$ is γ-P_S-closed in (X, τ). Since f is bijective, let $f(x) = y$, then $x \notin f^{-1}(E)$. Since (X, τ) is γ-P_S-regular space, then there exist disjoint γ-P_S-open sets G and H in X such that $x \in G$ and $f^{-1}(E) \subseteq H$. Since f is (γ, β)-P_S-closed, then by Lemma 2.12, we have there exists a β-P_S-open set U in Y containing E such that $f^{-1}(U) \subseteq H$. Since f is bijective (γ, β)-P_S-closed function. Then by Lemma 2.11, $y \in f(G)$ and $f(G)$ is β-P_S-open set in Y. Thus, $G \cap f^{-1}(U) = \phi$ and hence $f(G) \cap U = \phi$. Therefore, (Y, σ) is β-P_S-regular. □

Theorem 2.14. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a surjective (γ, β)-P_S-irresolute, (γ, β)-P_S-open and (γ, β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X, τ) is γ-P_S-regular, then (Y, σ) is also β-P_S-regular.

Proof. Similar to Theorem 2.13. □

Definition 15. ([8]) Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called:

(i) β-pre-anti-continuous if the inverse image of each β-preopen set in Y is open in X, or if the inverse image of each β-preclosed set in Y is closed in X.

(ii) γ-pre-anti-closed if the image of each γ-preclosed set in X is closed in Y.

Theorem 2.15. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a surjective β-pre-anti-continuous, β-P_S-open and β-P_S-closed function with the operation β on σ. If (X, τ) is regular, then (Y, σ) is β-P_S-regular.

Proof. Let E be a β-P_S-closed set of (Y, σ) containing y. Then E is β-preclosed. Since a function f is β-pre-anti-continuous, then $f^{-1}(E)$ is closed set in X containing $x \in X$, where $y = f(x)$. Since a space X is regular, there exists an open set V such that $x \in V \subseteq Cl(V) \subseteq f^{-1}(E)$ which implies that $y \in f(V) \subseteq f(Cl(V)) \subseteq E$. Since f is β-P_S-open and β-P_S-closed function, then $f(V)$ and $f(Cl(V))$ are β-P_S-open and β-P_S-closed sets in Y respectively. Then σ_β-$P_SCl(f(Cl(V))) \subseteq E$ and hence $y \in f(V) \subseteq \sigma_\beta$-$P_SCl(V) \subseteq \sigma_\beta$-$P_SCl(f(Cl(V))) \subseteq E$. Consequently, by Theorem 2.1, (Y, σ) is β-P_S-regular. □

Theorem 2.16. Let γ and β be operations on τ and σ respectively. Let

$f\colon (X,\tau) \to (Y,\sigma)$ be an injective, (γ,β)-P_S-irresolute and (γ,β)-P_S-closed. If (Y,σ) is β-P_S-regular, then (X,τ) is also γ-P_S-regular.

Proof. Suppose that a set F is any γ-P_S-closed set of (X,τ) and $x \notin F$. Since f is (γ,β)-P_S-closed function. Then $f(F)$ is β-P_S-closed set in Y and $f(x) \notin f(F)$. Since (Y,σ) is β-P_S-regular. Then there exist disjoint β-P_S-open sets U and V in Y such that $f(x) \in U$ and $f(F) \subseteq V$. Since f is injective, then $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Since f is (γ,β)-P_S-irresolute, then by Theorem 2.10 (ii), $f^{-1}(U)$ and $f^{-1}(V)$ are γ-P_S-open sets in X. So (X,τ) is γ-P_S-regular. □

Definition 16. ([2]) Let (X,τ) and (Y,σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called γ-P_S-continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is γ-P_S-continuous at every point x in X, then f is said to be γ-P_S-continuous.

Theorem 2.17. ([2]) For a function $f\colon (X,\tau) \to (Y,\sigma)$ and γ be an operation on τ, the following statements are equivalent:

(i) f is γ-P_S-continuous.

(ii) $f^{-1}(V)$ is γ-P_S-open set in X, for every open set V in Y.

(iii) $f^{-1}(F)$ is γ-P_S-closed set in X, for every closed set F in Y.

Theorem 2.18. Let γ be an operation on τ. Let $f\colon (X,\tau) \to (Y,\sigma)$ be an injective γ-P_S-continuous and γ-pre-anti-closed. If (Y,σ) is regular, then (X,τ) is γ-P_S-regular.

Proof. Let F be a γ-P_S-closed set of X containing x. Then F is γ-preclosed. Since a function f is γ-pre-anti-closed, then $f(F)$ is closed set in Y containing $y \in Y$, where $y = f(x)$. Since a space (Y,σ) is regular, then there exists an open set V such that $y \in V \subseteq Cl(V) \subseteq f(F)$ which implies that $x \in f^{-1}(V) \subseteq f^{-1}(Cl(V)) \subseteq F$. Since f is γ-P_S-continuous function, then by Theorem 2.17 (ii) and (iii), $f^{-1}(V)$ and $f^{-1}(Cl(V))$ are γ-P_S-open and γ-P_S-closed sets in X respectively. Then τ_γ-$P_SCl(f^{-1}(Cl(V))) \subseteq F$ and hence $x \in f^{-1}(V) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(V)) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(Cl(V))) \subseteq F$. Therefore, by Theorem 2.1, (X,τ) is γ-P_S-regular space. □

3. Characterizations of γ-P_S-normal spaces

Now, γ-P_S-normal space which is another type of γ-P_S- separation axioms is examined. Some characterizations of γ-P_S-normal spaces are investigated.

Definition 17. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-normal if for each pair of disjoint γ-P_S-closed sets E, F of X, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

Some important characterizations of γ-P_S-normal spaces are investigated as follows.

Theorem 3.1. Let (X, τ) be a γ-locally indiscrete space and γ be a regular operation on τ. Then X is γ-P_S-normal if and only if for each pair of disjoint γ-P_S-closed sets F and E of X, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

Proof. Let E and F be disjoint γ-P_S-closed sets in (X, τ). Since (X, τ) is γ-P_S-normal space, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq W$, $F \subseteq H$ and $W \cap H = \phi$. This implies that $W \cap \tau_\gamma$-$P_SCl(H) = \phi$. Since τ_γ-$P_SCl(H)$ is γ-P_S-closed set such that $E \cap \tau_\gamma$-$P_SCl(H) = \phi$ and X is γ-P_S-normal, then there exist γ-P_S-open sets U and V such that $E \subseteq U$, τ_γ-$P_SCl(H) \subseteq V$ and $U \cap V = \phi$ which implies that τ_γ-$P_SCl(U) \cap V = \phi$. If we put $G = W \cap U$. Since a space X is γ-locally indiscrete and γ is a regular operation on τ, then by Lemma 1.3, G is γ-P_S-open set in X. Hence G and H are γ-P_S-open sets such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$.

Conversely, suppose E and F are disjoint γ-P_S-closed sets in X. Then by hypothesis, we have there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$, $F \subseteq H$ and τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SCl(H) = \phi$. This implies that $G \cap H = \phi$ and hence (X, τ) is γ-P_S-normal space. □

Notice that γ-P_S-open set in Theorem 3.1 can be replaced by γ-regular-open, γ-open, α-γ-open and γ-semiopen sets according to Remark 1.4. Again, τ_γ-P_S-closure can be replaced by τ_γ-closure, $\tau_{\alpha-\gamma}$-closure and τ_γ-semi-closure of a set.

Theorem 3.2. The following conditions are equivalent for any topological space (X, τ) with an operation γ on τ:

(i) X is γ-P_S-normal.

(ii) For each γ-P_S-closed set F in X and each γ-P_S-open set G containing F, there exists a γ-P_S-open set H containing F such that τ_γ-$P_SCl(H) \subseteq G$.

(iii) For each pair of disjoint γ-P_S-closed sets F and E of X, there exists a γ-P_S-open set G containing E such that τ_γ-$P_SCl(G) \cap F = \phi$.

Proof. (i) $\Rightarrow$ (ii) Let F be any γ-P_S-closed set in X and G be any γ-P_S-open set in X such that $F \subseteq G$. Then $F \cap X\backslash G = \phi$ and $X\backslash G$ is γ-P_S-closed. By hypothesis, there exist γ-P_S-open sets H and W such that $F \subseteq H$ and $X\backslash G \subseteq W$ and $H \cap W = \phi$ which implies τ_γ-$P_SCl(H) \cap W = \phi$. Then τ_γ-$P_SCl(H) \subseteq X\backslash W \subseteq G$. So $F \subseteq H$ and τ_γ-$P_SCl(H) \subseteq G$.

(ii) $\Rightarrow$ (iii) Let E and F be γ-P_S-closed sets of X such that $E \cap F = \phi$. Then $E \subseteq X\backslash F$ and $X\backslash F$ is γ-P_S-open set containing E. By (ii), there exists a γ-P_S-open set $G \subseteq X$ containing E such that τ_γ-$P_SCl(G) \subseteq X\backslash F$. This implies that τ_γ-$P_SCl(G) \cap F = \phi$.

(iii) $\Rightarrow$ (i) Let E and F be disjoint γ-P_S-closed sets of X . Then by (iii), there exists a γ-P_S-open set G containing E such that τ_γ-$P_SCl(G) \cap F = \phi$ which implies that $F \subseteq X\backslash\tau_\gamma$-$P_SCl(G)$. Since $X\backslash\tau_\gamma$-$P_SCl(G)$ is γ-P_S-open such that $X\backslash\tau_\gamma$-$P_SCl(G) \cap G = \phi$. Therefore, X is γ-P_S-normal. □

Theorem 3.3. Let (X, τ) be a topological space and γ be an operation on τ. Then the following statements hold:

(i) X is γ-P_S-normal.

(ii) For each pair of disjoint γ-P_S-closed sets F and E of X, there exist disjoint γ-P_S-g-open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

(iii) For each γ-P_S-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq U$.

(iv) For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F, there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$.

(v) For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F, there exists a γ-P_S-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$.

(vi) For each γ-P_S-g-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq U$.

(vii) For each γ-P_S-g-closed set F in X and each γ-P_S-open set U containing F, there exists a γ-P_S-g-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq U$.

Proof. (i) $\Rightarrow$ (ii) It is obvious, since every γ-P_S-open set is γ-P_S-g-open.

(ii) $\Rightarrow$ (iii) Let F be any γ-P_S-closed set in X and let U be any γ-P_S-open set containing F. Then $X\backslash U$ is γ-P_S-closed set in X and hence by using (ii), there exist disjoint γ-P_S-g-open sets G and H such that $F \subseteq G$ and $X\backslash U \subseteq H$. Since $X\backslash U$ is γ-P_S-closed and H is γ-P_S-g-open. Then by Theorem 1.2, $X\backslash U \subseteq \tau_\gamma$-$P_SInt(H)$ and $G \cap \tau_\gamma$-$P_SInt(H) = \phi$. So, τ_γ-$P_SCl(G) \cap \tau_\gamma$-$P_SInt(H) = \phi$ which implies that τ_γ-$P_SCl(G) \subseteq X\backslash\tau_\gamma$-$P_SInt(H) \subseteq U$. Therefore, $F \subseteq G \subseteq \tau_\gamma$-$P_SCl(G) \subseteq U$.

(iii) $\Rightarrow$ (iv) Let $F \subseteq X$ be any γ-P_S-closed set and let U be any γ-P_S-open set $F \subseteq U$. Then τ_γ-$P_SInt(U)$ is γ-P_S-open. Then by (iii), we have there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$.

(iv) $\Rightarrow$ (v) For each γ-P_S-closed set F in X and each γ-P_S-g-open set U containing F. Then by (iv), there exists a γ-P_S-g-open set V such that $F \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U)$. Since $F \subseteq V$ where F is γ-P_S-closed set and U is γ-P_S-g-open set, then by Theorem 1.2, $F \subseteq \tau_\gamma$-$P_SInt(V)$. Since τ_γ-$P_SCl(\tau_\gamma$-$P_SInt(V)) \subseteq \tau_\gamma$-$P_SCl(V)$. Therefore, τ_γ-$P_SInt(V)$ is a γ-P_S-open set such that $F \subseteq \tau_\gamma$-$P_SInt(V) \subseteq \tau_\gamma$-$P_SCl(\tau_\gamma$-$P_SInt(V)) \subseteq \tau_\gamma$-$P_SInt(U)$.

(v) $\Rightarrow$ (vi) Suppose that $F \subseteq U$, where U is γ-P_S-open set and F is γ-P_S-g-closed set of X. So, τ_γ-$P_SCl(F) \subseteq U$. Since every γ-P_S-open set is γ-P_S-g-open (that is, U is γ-P_S-g-open) and τ_γ-$P_SCl(F)$ is γ-P_S-closed, then by (v), there exists a γ-P_S-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq \tau_\gamma$-$P_SInt(U) = U$.

(vi) $\Rightarrow$ (vii) The proof is clear.

(vii) $\Rightarrow$ (i) Suppose that E and F are two γ-P_S-closed sets of X such that $E \cap F = \phi$. This implies that $F \subseteq X\backslash E$, F is γ-P_S-g-closed and $X\backslash E$ is γ-P_S-open. Then by (vii), there exists a γ-P_S-g-open set V such that τ_γ-$P_SCl(F) \subseteq V \subseteq \tau_\gamma$-$P_SCl(V) \subseteq X\backslash E$. By Theorem 1.2, $F \subseteq \tau_\gamma$-$P_SInt(V)$. Now $E \subseteq X\backslash\tau_\gamma$-$P_SCl(V)$. Let $G = X\backslash\tau_\gamma$-$P_SCl(V)$ and $H = \tau_\gamma$-$P_SInt(V)$ are γ-P_S-open sets of X such that $E \subseteq G$, $F \subseteq H$ and $G \cap H = \phi$. Consequently, X is γ-P_S-normal space. □

Definition 18. ([8]) A topological space (X, τ) with an operation γ on τ is said to be γ-pre-normal if for each pair of disjoint γ-preclosed sets E, F of X, there exist disjoint γ-preopen sets U and V such that $E \subseteq U$ and $F \subseteq V$.

The relation between γ-P_S-normality and γ-pre-normality are independent. However, they are equivalent when a topological space (X, τ) is γ-semiT_1 as can be explained in the following two examples and theorem.

Example 3.4. Let (X, τ) be a topological space and γ be an operation on τ as in Example 2.6. Then the space (X, τ) is γ-pre-normal, but it is not γ-P_S-normal since the set $\{b\}$ is γ-P_S-closed not containing c, then there is no disjoint γ-P_S-open sets G and H such that $c \in G$ and $\{b\} \subseteq H$.

Example 3.5. In Example 2.8. Then the space (X, τ) is γ-P_S-normal, but it is not γ-pre-normal since $\{a\}$ and $\{b\}$ are disjoint γ-preclosed sets, then there is no disjoint γ-preopen sets G and H such that $\{a\} \in G$ and $\{b\} \subseteq H$.

Theorem 3.6. Let (X, τ) be a γ-semiT_1 space. Then (X, τ) is γ-P_S-normal if and only if (X, τ) is γ-pre-normal.

Proof. The proof follows from Theorem 1.6. □

Theorem 3.7. ([7]) Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-T_1 if and only if every singleton set in X is γ-P_S-closed.

Lemma 3.8. If (X, τ) is γ-P_S-T_1 space. Then every γ-P_S-normal space X is γ-P_S-regular.

Proof. Let F be a γ-P_S-closed set in X and $x \in X$ does not belong to F. Since X is γ-P_S-T_1. Then by Theorem 3.7, $\{x\}$ is γ-P_S-closed. So $\{x\}$ and F are two disjoint γ-P_S-closed sets of X. Since X is γ-P_S-normal, then there exist disjoint γ-P_S-open sets G and H such that $x \in \{x\} \subseteq G$ and $F \subseteq H$. Hence X is γ-P_S-regular. □

The following example shows that the converse of Lemma 3.8 is not true in general.

Example 3.9. In Example 2.5, the space (X, τ) is both γ-P_S-regular and

γ-P_S-normal, but (X,τ) is not γ-P_S-T_1 since every γ-P_S-open set containing b contains c also.

As in Lemma 1.7 (i) showed that every γ-P_S-T_2 space is γ-P_S-T_1 while the converse is true only when a space X is γ-P_S-regular as shown in the following Lemma 3.10.

Lemma 3.10. If (X,τ) is γ-P_S-regular and γ-P_S-T_1 space. Then it is γ-P_S-T_2.

Proof. Similar to Lemma 3.8. □

From Lemma 3.8 and 3.10, we have the following theorem.

Theorem 3.11. If (X,τ) is γ-P_S-normal and γ-P_S-T_1 space. Then it is γ-P_S-T_2.

Proof. Obvious. □

In the end of this section, we obtain some properties of γ-P_S-normal spaces which is related to γ-P_S-functions.

Theorem 3.12. For any operations γ and β on τ and σ respectively. If $f\colon (X,\tau)\to(Y,\sigma)$ is bijective (γ,β)-P_S-irresolute and (γ,β)-P_S-open and (X,τ) is γ-P_S-normal. Then (Y,σ) is also β-P_S-normal.

Proof. Suppose that E_1 and E_2 are disjoint β-P_S-closed sets of (Y,σ) and f is (γ,β)-P_S-irresolute function, then by Theorem 2.10 (iii), $f^{-1}(E_1)$ and $f^{-1}(E_2)$ are γ-P_S-closed sets in (X,τ). Since (X,τ) is γ-P_S-normal space, then there exist disjoint γ-P_S-open sets G_1 and G_2 in X such that $f^{-1}(E_1)\subseteq G_1$ and $f^{-1}(E_2)\subseteq G_2$. Since a function f is (γ,β)-P_S-open and bijective, then $f(G_1)$ and $f(G_2)$ are β-P_S-open sets in (Y,σ) such that $E_1\subseteq f(G_1)$, $E_2\subseteq f(G_2)$ and $f(G_2)\cap f(G_2)=\phi$. Then (Y,σ) is β-P_S-normal space. □

Corollary 3.13. Let $f\colon (X,\tau)\to(Y,\sigma)$ be a surjective (γ,β)-P_S-irresolute, (γ,β)-P_S-open and (γ,β)-P_S-closed, where γ and β are operations on τ and σ respectively. If (X,τ) is γ-P_S-normal, then (Y,σ) is also β-P_S-normal.

Proof. Directly follows from Theorem 3.12. □

Theorem 3.14. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a bijective β-pre-anti-continuous, β-P_S-open function with the operation β on σ. If (X,τ) is normal, then (Y,σ) is β-P_S-normal.

Proof. Let F_1 and F_2 are disjoint β-P_S-closed sets in (Y,σ). Then F_1 and F_2 are disjoint β-preclosed sets. Since f is β-pre-anti-continuous, then the sets $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed sets in (X,τ). Since (X,τ) is normal, then there exist disjoint open sets V_1 and V_2 such that $f^{-1}(F_1) \subseteq V_1$ and $f^{-1}(F_2) \subseteq V_2$. Since f is bijective β-P_S-open function, then $f(V_1)$ and $f(V_2)$ are β-P_S-open sets in (Y,σ) such that $F_1 \subseteq f(V_1)$, $F_2 \subseteq f(V_2)$ and $f(V_2) \cap f(V_2) = \phi$. Hence (Y,σ) is β-P_S-normal space. □

Theorem 3.15. Let γ and β be operations on τ and σ respectively. Let $f\colon (X,\tau) \to (Y,\sigma)$ be an injective, (γ,β)-P_S-irresolute and (γ,β)-P_S-closed. If (Y,σ) is β-P_S-normal, then (X,τ) is also γ-P_S-normal.

Proof. Assume that f is (γ,β)-P_S-closed and, F_1 and F_2 are disjoint γ-P_S-closed sets of (X,τ). Then $f(F_1)$ and $f(F_2)$ are β-P_S-closed sets in (Y,σ). Since (Y,σ) is β-P_S-normal space, then there exist disjoint β-P_S-open sets H_1 and H_2 in X such that $f(F_1) \subseteq H_1$ and $f(F_2) \subseteq H_2$. Since f is injective, then $F_1 \subseteq f^{-1}(H_1)$, $F_2 \subseteq f^{-1}(H_2)$ and $f^{-1}(H_1) \cap f^{-1}(H_2) = \phi$. Since f is (γ,β)-P_S-irresolute, then by Theorem 2.10 (ii), $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are γ-P_S-open sets in (X,τ). Therefore, (X,τ) is γ-P_S-normal. □

Theorem 3.16. For any operation γ on τ. If the function $f\colon (X,\tau) \to (Y,\sigma)$ is injective γ-P_S-continuous and γ-pre-anti-closed, and (Y,σ) is normal. Then (X,τ) is γ-P_S-normal.

Proof. Let F_1 and F_2 are disjoint γ-P_S-closed sets of (X,τ). Then F_1 and F_2 are disjoint γ-preclosed sets. Since f is γ-pre-anti-closed, then $f(F_1)$ and $f(F_2)$ are closed sets in (Y,σ). Since a space (Y,σ) is normal, then there exist open sets V_1 and V_2 in Y such that $f(F_1) \subseteq V_1$, $f(F_2) \subseteq V_2$ and $V_1 \cap V_2 = \phi$. Now since f is injective γ-P_S-continuous, then by Theorem 2.17 (ii), $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are γ-P_S-open sets in (X,τ) such that $F_1 \subseteq f^{-1}(V_1)$, $F_2 \subseteq f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. So (X,τ) is γ-P_S-normal space. □

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