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$\gamma-$OPEN FUNCTION AND $\gamma-$CLOSED FUNCTIONS

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Abstract: In this paper we define two types of functions of topological spaces: $\gamma-$open functions and $\gamma-$closed functions. In addition, we examine the relation of these functions among themselves and their relation with $\gamma-$continuous functions. In the following, we study some properties of $\gamma-$open and $\gamma-$closed functions.

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1. Introduction

The notion of $\gamma-$operation in a topological space and the notion of the $\gamma-$open set were introduced by the Japanese mathematician H. Ogata in 1991. Further, through the notion of the $\gamma-$open set, Ogata defined the $\gamma - T_i$ $(i = 0, \frac{1}{2}, 1, 2)$ spaces.

In 1992 F.U. Rehman and B. Ahmad defined the $\gamma-$interior, $\gamma-$exterior, $\gamma-$closure and $\gamma-$boundary of a subset of a topological space (see [9]).

In 2003 B. Ahmad and S. Hussain studied many properties of a $\gamma-$operation in a topological space, they defined the meaning of $\gamma-$neighborhood as well as $\gamma-$neighborhood of the base (see [1]).

In 2009 C.K. Basu, B.M. Uzzal Afsan and M.K. Chash defined $\gamma-$continuity of function (see [3]).

In our previous work [4]: "γ^*-operation in the product of topological

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spaces", we have defined the concept of $\gamma 6*-$operation in the topological product of topological spaces $(X_i, \mathcal{T}_i), i \in I$, through γ_i-operations of spaces $(X_i, \mathcal{T}_i)$, $i \in I$.

As noted before, in this paper we define two types of functions of topological spaces: $\gamma-$open functions and $\gamma-$closed functions and also we study some properties of these functions.

2. Preliminaries

Definition 1. ([11]) Let $(X, \mathcal{T})$ be the topological space. The function $\gamma : \mathcal{T} \to \mathcal{P}(X), (\mathcal{P}(X)$, is the partition of $X)$, such that for each $G \in \mathcal{T}, G \subseteq \gamma(G)$, is called $\gamma-$operation in the topological space $(X, \mathcal{T})$.

Examples of $\gamma-$operation in a topological space $(X, \mathcal{T})$, are functions: $\gamma : \mathcal{T} \to \mathcal{P}(X)$, given by: $\gamma(G) = G, \gamma(G) = clG, \gamma(G) = int(clG)$, etc.

Definition 2. ([11]) Let $(X, \mathcal{T})$ be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a $\gamma-$operation in this space. The set $G \subseteq X$ is called $\gamma-$open, if for every $x \in G$, there exists the set $U \in \mathcal{T}$, such that $x \in U$ and $\gamma(U) \subseteq G$.

The family of all $\gamma-$open sets of topological space $(X, \mathcal{T})$, is denoted by: $\mathcal{T}_\gamma$. This means:

$$\mathcal{T}_\gamma = \{G \subseteq X : \forall x \in G, \exists U \in \mathcal{T} : x \in U \subseteq \gamma(U) \subseteq G\}. \tag{1}$$

Definition 3. ([2]) The set F, is called $\gamma-$closed, if its complement $F^C = X \setminus F$ is the $\gamma-$open set in X.

By Definition 3, it turns out that if the set F is $\gamma-$closed, it is closed set. True, if F is $\gamma-$closed, then $F^C = X \setminus F$ is a $\gamma-$open set and as such is open set. It means F is closed set. By the definition of $\gamma-$operation, $\gamma-$open set and $\gamma-$closed set some simple statements are derived, which are given by the following theorem.

Theorem 4. ([2]) *Let $(X, \mathcal{T})$ be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a $\gamma-$operation in this space, also let be $\mathcal{T}_\gamma$ as given in equation (1), then:*

1. $\mathcal{T}_\gamma \subseteq \mathcal{T}$

2. *$\emptyset, X \in \mathcal{T}_\gamma$ ($\emptyset$ and X are $\gamma-$closed sets),*

3. *If $\gamma(G) = G, \forall G \in \mathcal{T}$, then $\mathcal{T}_\gamma = \mathcal{T}$,*

4. *If $\mathcal{T} = \{\emptyset, X\}$ (in discrete topology), then for each $\gamma-$operation, the only $\gamma-$open and $\gamma-$closed sets are $\emptyset$ and X.*

Proof. 1. From $G \in \mathcal{T}_\gamma$ it follows that for every $x \in G$, exists $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq \gamma(U_x) \subseteq G$, so $G = \cup\{U_x : x \in G\} \in \mathcal{T}$ (as a union of sets $U_x \in \mathcal{T}$). This means $\mathcal{T}_\gamma \subseteq \mathcal{T}$.

2. Since $\emptyset$ has no element, we can consider that each "element" satisfies the condition of Definition 1. It means $\emptyset \in \mathcal{T}_\gamma$. On the other hand, $\forall x \in X$, there exists an open set U (at least set X), so $x \in U$ and $\gamma(U) \subseteq X$. It means $X \in \mathcal{T}_\gamma$.

3. Let $\gamma(G) = G, \forall G \in \mathcal{T}$. From $G \in \mathcal{T}$ and $x \in G$ we have: $x \in G$ and $\gamma(G) = G \subseteq G$, it means $\mathcal{T} \subseteq \mathcal{T}_\gamma$. From statement 1. it follows that $\mathcal{T}_\gamma = \mathcal{T}$.

4. It follows directly from the statements 1. and 2.

□

Theorem 5. ([1]) *Let $(X, \mathcal{T})$ be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a $\gamma-$operation in this space, then:*

1. *$\mathcal{T}_\gamma \subseteq \mathcal{T}$,*

2. *$\emptyset, X \in \mathcal{T}_\gamma$ ($\emptyset$ and X are $\gamma-$closed sets).*

Proof. 1. Let $x \in \cup_{i \in I} G_i (G_i \in \mathcal{T}_\gamma)$. Then, there exists $i_0 \in I$, such that $x \in G_{i_0}$. Since $G_{i_0} \in \mathcal{T}_\gamma$, there exists $G \in \mathcal{T}$ such as $x \in G$ and $\gamma(G) \subseteq G_{i_0} \subseteq \cup_{i \in I} G_i$. It means $\cup_{i \in I} G_i \in \mathcal{T}_\gamma$.

2. That the intersection of the two $\gamma-$open sets is not always a $\gamma-$open set, as shown by the following example.

□

Example 1. Let $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}.$$

In X, we define $\gamma-$operation in this way:

$$\gamma(G) = \begin{cases} G, b \in G \\ clG, b \notin G \end{cases} . \tag{2}$$

We define the $\mathcal{T}_\gamma$ family, by proving which of the sets from the $\mathcal{T}$ family, satisfy the terms of Definition 2.

We consider the set $\{a\}$. Sets $\{a\}, \{a,b\}$ and $\{a,c\}$, are open sets that contain the single element a, of the set $\{a\}$. On the other hand, we have:
$\gamma(\{a\}) = cl\{a\} = \{a,c\} \nsubseteq \{a\}, \gamma(\{a,b\}) = \{a,b\} \nsubseteq \{a\}, \gamma(\{a,c\})$
$= \{a,c\} \nsubseteq \{a\}$. It means $\{a\} \notin \mathcal{T}_\gamma$.

Similarly, proving to all open sets of given space, we find that:

$$\mathcal{T}_\gamma = \{\emptyset, \{b\}, \{a,b\}, \{a,c\}, X\}. \tag{3}$$

We notice that $\{a,b\} \in \mathcal{T}_\gamma$ and $\{a,c\} \in \mathcal{T}_\gamma$, but $\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{T}_\gamma$.

Remark 1. The statement 2. of Theorem 4 shows that $\mathcal{T}_\gamma$ is not always the topology in X.

Based on Theorem 5 and De-Morgan laws, it follows:

Theorem 6. *Let $(X, \mathcal{T})$ be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a $\gamma-$operation in this space, then:*

1. *If for any $i \in I, F_i$ is the $\gamma-$closed set in X, then $\cap_{i\in I} F_i$ is the $\gamma-$closed set in X,*

2. *If F_1, F_2 are $\gamma-$closed sets in X, then $F_1 \cup F_2$, is not always $\gamma-$closed set.*

The statement 2. of Theorem 6 is proved by Example 1, according to which $\{c\}^c = \{a,b\} \in \mathcal{T}_\gamma$ and $\{b\}^c = \{a,c\} \in \mathcal{T}_\gamma$. So, sets $\{c\}$ and $\{b\}$, are $\gamma-$closed, but their union $\{b,c\}$, is not $\gamma-$ closed because $\{b,c\}^c = \{a\} \notin \mathcal{T}_\gamma$.

Definition 7. ([11]) Let $(X, \mathcal{T})$ be the topological space and $A \subseteq X$. The point $x \in A$ is called the point of $\gamma-$interior of set A, if there exists an open neighborhood U of point x, so $\gamma(U) \subseteq A$. The set of all $\gamma-$interiors points of set A is called $\gamma-$interior and is symbolically denoted by $int_\gamma(A)$. This means:

$$int_\gamma(A) = \{x \in A : x \in U \in \mathcal{T}, \gamma(U) \subseteq A\} \subseteq A.$$

By Definition 7, it turns out that $int_\gamma(A) \subseteq intA$. Also from Definitions 2 and 7, it turns out that A is a $\gamma-$open, then and only when $int_\gamma(A) = intA$.

Definition 8. ([11]) Let $(X, \mathcal{T})$ be the topological space and $A \subseteq X$. The point $x \in A$ is called the point of $\gamma-$ closure of set A, if $\gamma(U) \cap A \neq \emptyset$ for each open neighborhood U of point x. The set of all $\gamma-$closures points of set A is called $\gamma-$closure and is symbolically marked by $cl_\gamma(A)$. It means:

$$cl_\gamma(A) = \{x \in A : \gamma(U) \cap A \neq \emptyset,$$

$$\text{for each open neighborhood } U \text{ of point } x\}.$$

By Definition 8, it turns out that $clA \subseteq cl_\gamma(A)$ and $A \subseteq cl_\gamma(A)$. Also from Definitions 3 and 8, it turns out that A is a $\gamma-$closed, then and only when $A = cl_\gamma(A)$.

Definition 9. ([1]) The set $U \subseteq X$ is called the $\gamma-$ neighborhood of point $x \in X$, if there exists the $\gamma-$ open set V, such that $x \in V \subseteq U$.

Definition 10. ([3]) Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space and $\gamma : \mathcal{T} \rightarrow \mathcal{P}(X)$ a $\gamma-$operation in $(X, \mathcal{T})$. The function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})$ is called the $\gamma-$continuous if the inverse image of any $V-$open set in Y is the $\gamma-$open set in X.

Definition 11. ([5]) Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space. The function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})$ is called open if the image $f(G)$ of each open set G in X is open set in Y.

Definition 12. ([5]) Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space. The function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})$ is called closed if the image $f(F)$ of each open set F in X is open set in Y.

3. Main results

3.1. $\gamma-$open functions and $\gamma-$closed functions

Definition 13. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space and γ a $\gamma-$operation in space Y. The function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}^{'})$ is called the $\gamma-$open function if the image $f(G)$ of any G open set in X is the $\gamma-$open set in Y.

Definition 14. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space and γ a $\gamma-$operation in space Y. The function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^{'})$ is called the $\gamma-$closed function if the image $f(F)$ of any F closed set in X is the $\gamma-$closed set in Y.

By Definitions 13 and 14, the following simple statement follows:

Statement 1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological space and γ a $\gamma-$operation in space Y.*

1. *If the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^{'})$ is $\gamma-$open, then it is also open;*
2. *If the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^{'})$ is $\gamma-$closed, then it is also closed.*

Proof. 1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^{'})$ be $\gamma-$open function and G open set in X. Then from Definition 13, $f(G)$ is $\gamma-$open set in Y and as such is open set in Y. It means f is an open function.

2. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^{'})$ be $\gamma-$open function and F closed set in X. Then from Definition 14, $f(F)$ is $\gamma-$closed set in Y and as such is open set in Y. It means f is a closed function.

□

Next, by means of different examples we will see what is the relation among the $\gamma-$open, $\gamma-$closed and $\gamma-$continuous functions.

Example 2. Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$$

and $\gamma-$operation in X, given with: $\gamma(G) = clG, \forall G \in \mathcal{T}$. The closed sets in X are: $\emptyset, \{b, c\}, \{a, b\}, \{c\}, \{b\}$ and X. By definition of $\gamma-$operation we have:

$$\gamma(\emptyset) = cl\emptyset = \emptyset, \gamma(\{a\}) = cl\{a\} = \{a\}, \gamma\{c\} = cl\{c\} = \{c\},$$

$$\gamma(\{a, b\}) = cl\{a, b\} = \{a, b\}, \gamma\{a, c\} = cl\{a, c\} = X, \gamma(X) = X.$$

We will consider which of the family members $\mathcal{T}$ meet the terms of Definition 1. For example the set $\{a\}$ is not $\gamma-$open because open sets that contain its only element a are: $\{a\}, \{a, b\}, \{a, c\}$, and X, but none of the sets $\gamma(\{a\}), \gamma(\{a, b\}), \gamma(\{a, c\})$ and $\gamma(X)$ do not contained in $\{a\}$. By proving equally to all open sets in X, we conclude that $\mathcal{T}_\gamma = \{\emptyset, \{c\}, \{a, b\}, X\}$.

Let be $Y = \{1, 2\}$ with the topology $\mathcal{T}^{'} = \{\emptyset, \{1\}, Y\}$. In space Y let $\gamma^{'}-$operation be given in this way: $\gamma^{'}(G) = G, \forall G \in \mathcal{T}^{'}$. Since $\gamma^{'}(G) = G, \forall G \in$

$\mathcal{T}^{'}$. it comes out that $\mathcal{T}^{'}_{\gamma} = \mathcal{T}^{'} = \{\emptyset, \{1\}, Y\}$. We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x \in \{a, c\} \\ 2, x \in \{b\} \end{cases} .$$

From the definition of function f we have:

$$f(\emptyset) = \emptyset, f(\{a\}) = \{1\}, f(\{c\}) = \{1\}, f(\{a, b\}) = \{1, 2\} = Y,$$

$$f(\{a, c\}) = \{1\}, f(X) = Y.$$

It means that the image of any open set in X, is the $\gamma^{'}-$open set in Y, so f is the $\gamma^{'}-$open function.

Note that $\{c\}$ is a closed set in X, because $\{c\}^c = \{a, b\} \in \mathcal{T}$, but $f(\{c\}) = \{1\}$ is not a $\gamma^{'}-$closed set in Y, so f is not $\gamma^{'}-$closed function.

We also note that $\{1\}$ is $\gamma^{'}-$open in Y, but $f^{-1}(\{1\}) = \{a, c\} \notin \mathcal{T}_{\gamma}$. It means that f is not a $\gamma-$continuous function.

Example 3. Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$$

and let be given $\gamma-$ operation in X, with: $\gamma(G) = clG, \forall G \in \mathcal{T}$ (see Example 2). We have $\mathcal{T}_{\gamma} = \{\emptyset, \{c\}, \{a, b\}, X\}$.

Let be $Y = \{1, 2, 3\}$ with the indiscrete topology $\mathcal{T}^{'} = \{\emptyset, Y\}$. Since the only open sets in $\gamma^{'}$ are $\emptyset$ and Y, for whatever $\gamma^{'}-$operation in Y we have $\mathcal{T}^{'}_{\gamma} = \{\emptyset, Y\}$.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x = a \\ 2, x = b \\ 3, x = c \end{cases} .$$

It is clear that $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. It means that the inverse image of any open set in Y, is $\gamma-$open set in X. It means that f is the $\gamma-$continuous function.

On the other hand for example $\{a\}$ is a closed set in X, but $f(\{a\}) = \{1\} \notin \mathcal{T}^{'}_{\gamma}$. It means that f is not $\gamma^{'}-$open function.

As well $\{b, c\}$ is closed set in X, but $f(\{b, c\}) = \{1, 2\}$ and $\{1, 2\}$ is not $\gamma^{'}-$closed in Y. It means that f is not a $\gamma^{'}-$closed function.

Example 4. Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}.$$

Closed sets in X are: $\emptyset, \{b, c\}, \{a, c\}, \{c\}$ and X.

Let be $Y = \{1, 2\}$ with the topology $\mathcal{T}' = \{\emptyset, \{1\}, Y\}$ and let be given γ'−operation in Y with $\gamma'(G) = G, \forall G \in \mathcal{T}'$. We have $\mathcal{T}'_\gamma = \{\emptyset, \{1\}, Y\}$. γ'−closed sets in Y are: $\emptyset, \{2\}$ and Y.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x \in \{a\} \\ 2, x \in \{b, c\} \end{cases}.$$

We have $f(\emptyset) = \emptyset, f(\{b, c\}) = \{2\}, f(\{a, c\} = Y, f(\{c\}) = \{2\}, f(X) = Y$. It means that the image of each closed set in X, is γ'−closed set in Y. So f is γ'−closed function.

On the other hand $\{b\}$ is open set in X, but $f(\{b\}) = \{2\}$ is not γ'−open set in Y, consequently f is not γ'−open function.

Example 5. Let be $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, X\}$$

and let be $\gamma(G) = G, \forall G \in \mathcal{T}$. It means $\mathcal{T}_\gamma = \{\emptyset, \{a\}, X\}$. Closed sets in X are: $\emptyset, \{b, c\}$ and X.

Let be now $Y = \{1, 2\}$ with the topology $\mathcal{T}' = \{\emptyset, \{1\}, Y\}$ and $\gamma'(G) = G, \forall G \in \mathcal{T}'$. It means $\mathcal{T}'_\gamma = \{\emptyset, \{1\}, Y\}$. γ'−closed sets in Y are: $\emptyset, \{2\}, Y$.

We define the function $f : X \to Y$ with

$$f(x) = \begin{cases} 1, x \in \{c\} \\ 2, x \in \{a, b\} \end{cases}.$$

We have $f(\emptyset) = \emptyset, f(\{b, c\}) = Y, f(X) = Y$. It means that the image of each closed set in X, is the γ'−closed set in Y. So f is γ'−closed function. On the other hand $\{1\}$ is open set in Y, but $f^{-1}(\{1\}) = \{c\}$ which is not a γ−open set in$X$. It means that f is not a γ− continuous function.

The last four examples show that the meanings: γ−open function, γ−closed function, γ−continuous function are independent of each other.

3.2. Some properties of $\gamma-$open and $\gamma-$closed functions

The following theorems provide some statements equivalent to Definition 13, respectively Definition 14.

Theorem 15. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^{'})$ be two topological spaces, γ a $\gamma-$operation in space Y and $f : X \to Y$. The following statements are equivalent:*

1. *f is the $\gamma-$open.*
2. *$\emptyset, X \in \mathcal{T}_\gamma$ ($\emptyset$ and X are $\gamma-$closed sets)*
3. *For each open set B of base in $X, f(B)$ is $\gamma-$open set in Y.*
4. *For each $x \in X$ and each neighborhood U of the point x in X there exists $\gamma-$neighborhood V of the point $f(x)$ in Y such that $V \subseteq f(U)$.*

Proof. (1) $\Rightarrow$ (2) Let f be the $\gamma-$open function. It means that for each open set G in $X, f(G)$ is $\gamma-$open in Y. From $intA \subseteq A \Rightarrow f(intA) \subseteq f(A) \Rightarrow int_\gamma f(intA) \subseteq int_\gamma f(A)$. Since $intA$ is the open set in X and f the $\gamma-$open function, it turns out that $f(intA)$ is $\gamma-$open set in Y, and hence $int_\gamma f(intA) = f(intA) \Rightarrow f(intA) \subseteq int_\gamma f(A)$.

(2) $\Rightarrow$ (3) Let B be the open set of base in X. Then $intB = B \Rightarrow f(B) = f(intB) \subseteq int_\gamma f(B) \subseteq f(B)$. It means that $int_\gamma f(B) = f(B) \Rightarrow f(B)$ is $\gamma-$open in Y.

(3) $\Rightarrow$ (4) Let be $x \in X$ and U neighborhood of point x in X. Then there exists the open set G in X such that $x \in G \subseteq U$ and there exists an open set of base B such that $x \in B \subseteq G \subseteq U$. Note that $V = f(B)$, according to (3), V is $\gamma-$open in Y and as such is the $\gamma-$ neighborhood of its point $f(x) \in V \subseteq f(U).(B \subseteq U \Rightarrow f(B) = V \subseteq f(U))$.

(4) $\Rightarrow$ (1) Let G be open set in X. Then G is the neighborhood of each point $x \in G$. According to (4), for every $x \in G$, there exists the $\gamma-$neighborhood V_x of the point $f(x)$ in Y, such that $V_x \subseteq f(G)$. From the definition of $\gamma-$neighborhood for every $x \in G$ there exists the $\gamma-$open set H_x in Y, such that $f(x) \in H_x \subseteq V_x \subseteq f(G)$. This means $f(G) = f(\cup_{x\in G}\{x\}) = \cup_{x\in G}\{f(x)\} \subseteq \cup_{x\in G}H_x \subseteq f(G)$. It means $f(G) = \cup_{x\in G}H_x$. So $f(G)$ is the $\gamma-$open set as the union of the $\gamma-$open sets. Finally f is the $\gamma-$open function.

□

Theorem 16. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological spaces and γ a $\gamma-$operation in space Y and $f : X \to Y$. The function $f : X \to Y$ is $\gamma-$closed if and only if for each $A \subseteq X$ is valid: $cl_Y f(A) \subseteq f(clA)$.*

Proof. Let $f : X \to Y$ be the $\gamma-$closed function. Then $f(clA)$ is $\gamma-$closed set in Y, because clA is closed set in X. From $A \subseteq clA \Rightarrow f(A) \subseteq f(clA) \Rightarrow cl_\gamma f(A) \subseteq cl_\gamma f(clA) = f(clA)$, because $f(clA)$ is $\gamma-$closed in X. It means $cl_\gamma f(A) \subseteq f(clA)$.

Conversely: For each $A \subseteq X$, let $cl_\gamma f(A) \subseteq f(clA)$ and let be $F \subseteq X$ the closed set in X. From the assumption $f(F) \subseteq cl_\gamma f(F) \subseteq f(clF) = f(F)$, that because f is closed at X. It means $f(F) = cl_\gamma f(F)$. So $f(F)$ is the $\gamma-$closed set in Y and consequently f is the $\gamma-$closed function. □

Theorem 17. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological spaces and γ a $\gamma-$operation in space Y. The function $f : X \to Y$ is $\gamma-$closed if and only if for each $B \subseteq Y$ and any open set $G \subseteq X$ such that $f^{-1}(B) \subseteq G$, there exists the $\gamma-$open set U in Y, such that $B \subseteq U$ and $f^{-1}(U) \subseteq G$.*

Proof. Let $f : X \to Y$ be the $\gamma-$closed function. Let be further $B \subseteq Y$ and $G \subseteq X$ the open set in X, such that $f^{-1}(B) \subseteq G$. Then $f(X \setminus G)$ is $\gamma-$closed set in Y, because $X \setminus G$ is closed in X and f is $\gamma-$closed. The set $U = Y \setminus f(X \setminus G)$ is $\gamma-$open in Y. Since $f^{-1}(B) \subseteq G \Rightarrow X \setminus G \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \Rightarrow f(X \setminus G) \subseteq f(f^{-1}(Y \setminus B)) \subseteq Y \setminus B \Rightarrow Y \setminus f(X \setminus G) \supseteq Y \setminus (Y \setminus B) = B$. It means $B \subseteq U$. As well $f^{-1}(U) = f^{-1}(Y \setminus f(X \setminus G)) = X \setminus f^{-1}(f(X \setminus G)) \subseteq X \setminus (X \setminus G) = G$. It means $f^{-1}(U) \subseteq G$ and $B \subseteq U$.

Conversely: Let F be any closed set in X. We have to prove that $f(F)$ is $\gamma-$closed in Y. The set $B = Y \setminus f(F)$ satisfies the condition $f^{-1}(B) = f^{-1}(Y \setminus f(F)) = X \setminus f^{-1}(f(F)) \subseteq X \setminus F$. Note $X \setminus F = G$. G is open set in X such that $f^{-1}(B) \subseteq G$. From the assumption exists the $\gamma-$open set U in Y, such that $B \subseteq U$ and $f^{-1}(U) \subseteq G$. It means $B = Y \setminus f(F) \subseteq U$ and $f^{-1}(U) \cap (X \setminus G) = \emptyset \Rightarrow f(f^{-1}(U) \cap (X \setminus G)) = f(\emptyset) = \emptyset \Rightarrow U \cap f(X \setminus G) = \emptyset$. That is, $U \subseteq Y \setminus f(X \setminus G) = Y \setminus f(F)$. So we have $Y \setminus f(F) \subseteq U \subseteq Y \setminus f(F)$. That is, $U = Y \setminus f(F) \Rightarrow f(F) = Y \setminus U$ is $\gamma-$closed set in Y, because U is $\gamma-$open in Y. Thus f is $\gamma-$closed. □

Theorem 18. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^{'})$ be two topological spaces and γ a $\gamma-$operation in space Y. The function $f : X \to Y$ is $\gamma-$open if and only if for each $B \subseteq Y$ and any open set $F \subseteq X$ such that $f^{-1}(B) \subseteq F$, there exists the $\gamma-$closed set S in Y, such that $B \subseteq S$ and $f^{-1}(S) \subseteq F$.*

Proof. Let $f : X \to Y$ be the $\gamma-$open function. Let be further $B \subseteq Y$ and $F \subseteq X$ the closed set in X, such that $f^{-1}(B) \subseteq F$. Then $f(X \setminus F)$ is $\gamma-$open set in Y, because $X \setminus F$ is open in X and f is $\gamma-$open. The set $S = Y \setminus f(X \setminus F)$ is $\gamma-$closed in Y. Since $f^{-1}(B) \subseteq F \Rightarrow X \setminus F \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \Rightarrow f(X \setminus F) \subseteq f(f^{-1}(Y \setminus B)) \subseteq Y \setminus B \Rightarrow Y \setminus f(X \setminus F) \supseteq Y \setminus (Y \setminus B) = B$. It means $B \subseteq S$. As well $f^{-1}(S) = f^{-1}(Y \setminus f(X \setminus F)) = X \setminus f^{-1}(f(X \setminus F)) \subseteq X \setminus (X \setminus F) = F$. It means $f^{-1}(S) \subseteq F$ and $B \subseteq S$.

Conversely: Let G be any open set in X. We have to prove that $f(G)$ is $\gamma-$open in Y. The set $B = Y \setminus f(G)$ satisfies the condition $f^{-1}(B) = f^{-1}(Y \setminus f(G)) = X \setminus f^{-1}(f(G)) \subseteq X \setminus G$. Note $X \setminus G = F$. F is closed set in X such that $f^{-1}(B) \subseteq F$. From the assumption exists the $\gamma-$closed set S in Y, such that $B \subseteq S$ and $f^{-1}(S) \subseteq F$. It means $B = Y \setminus f(G) \subseteq S$ and $f^{-1}(S) \cap (X \setminus F) = \emptyset \Rightarrow f(f^{-1}(S) \cap (X \setminus F)) = f(\emptyset) = \emptyset \Rightarrow S \cap f(X \setminus F) = \emptyset$. It means $S \subseteq Y \setminus f(X \setminus F) = Y \setminus f(G)$. So we have $Y \setminus f(G) \subseteq S \subseteq Y \setminus f(G)$. It means $S = Y \setminus f(G) \Rightarrow f(F) = Y \setminus S$ is $\gamma-$open set in Y, because S is $\gamma-$closed in Y. Thus f is $\gamma-$open. □

The following theorems are related to the composition of $\gamma-$open ($\gamma-$closed) functions.

Theorem 19. *Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ the functions of topological spaces. Let be further γ_2 and γ_3, the $\gamma-$operations in Y and Z spaces, respectively. Then*

1. *If f and g are γ_2, respectively γ_3- open functions, then their composition $g \circ f : X \to Z$ is γ_3-open function.*
2. *If f and g are γ_2, respectively γ_3-closed functions, then their composition $g \circ f : X \to Z$ is γ_3-closed function.*
3. *If f is open function and g is γ_3-open functions, then their composition $g \circ f : X \to Z$ is γ_3-open function.*
4. *If f is closed function and g is γ_3-closed functions, then their composition $g \circ f : X \to Z$ is γ_3-closed function.*

Proof. 1. Let f and g be γ_2, respectively γ_3-open functions and let G be any open set in X. Then from $(g \circ f)(G) = g(f(G))$ and since f is γ_2-open, it follows that $f(G)$ is γ_2-open set in Y and as such it is open set in Y. As g is γ_3-open function it follows that $(g \circ f)(G) = g(f(G))$ is γ_3-open set in Z. So the composition $g \circ f : X \to Z$ is γ_3-open function.

2. Let f and g be γ_2, respectively γ_3-closed functions and let F be any closed set in X. Then from $(g \circ f)(F) = g(f(F))$ and since f is γ_2-closed, it follows that $f(F)$ is γ_2-closed set in Y and as such it is closed set in Y. As g is γ_3-closed function it follows that $(g \circ f)(G) = g(f(G))$ is γ_3-closed set in Z. So the composition $g \circ f : X \to Z$ is γ_3-closed function.

3. Let f be any open function and g be γ_3-open functions and G let be any open set in X. Then from $(g \circ f)(G) = g(f(G))$ and since G is open set in X and f open function, it follows that $f(G)$ is open set in Y. As g is γ_3-open function it follows that $(g \circ f)(G) = g(f(G))$ is γ_3-open set in Z. It means $g \circ f : X \to Z$ is γ_3-open function.

4. Let f be any closed function and g be γ_3-closed functions and F let be any closed set in X. Then from $(g \circ f)(G) = g(f(G))$ and since F is closed set in X and f closed function, it follows that $f(F)$ is closed set in Y. As g is γ_3-closed function it follows that $(g \circ f)(G) = g(f(G))$ is γ_3-closed set in Z. It means $g \circ f : X \to Z$ is γ_3-closed function.

□

Theorem 20. *Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ the functions of topological spaces. Let be further γ_3 a $\gamma-$operations in Z spaces. Then*

1. *If the composition $g \circ f : X \to Z$ is γ_3-open function and f is the surjective and continuous function, then the function g is γ_3-open.*

2. *If the composition $g \circ f : X \to Z$ is γ_3-closed function and f is the surjective and continuous function, then the function g is γ_3-closed.*

Proof. 1. Let G be any open set in Y and f the surjective and continuous function. Since f is surjective, there is a set $H \subseteq X$ such that $f^{-1}(G) = H$. From the continuity of the function f it follows that H is the open set in X. Since $g \circ f : X \to Z$ is γ_3-open, then $(g \circ f)(H)$ is γ_3-open in Z. Further $(g \circ f)(H) = g(f(H)) = g(f(f^{-1}(G)))$ =(since f is a surjective)= $g(G)$. It means $g(G)$ is γ_3-open set and consequently g is the γ_3-open function.

2. Let F be any closed set in Y and f the surjective and continuous function. Since f is surjective, there is a set $S \subseteq X$ such that $f^{-1}(F) = S$. From the continuity of the function f it follows that S is the closed set in X. Since $g \circ f : X \to Z$ is γ_3-closed, then $(g \circ f)(S)$ is γ_3-closed in Z. Further $(g \circ f)(S) = g(f(S)) = g(f(f^{-1}(F)))$ =(since f is a surjective)= $g(F)$. It means $g(F)$ is γ_3-closed set and consequently g is the γ_3-closed function.

□

Theorem 21. *Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2), g : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ the functions of topological spaces. Let be further γ_2 and γ_3 the $\gamma-$operations in Y and Z spaces, respectively. Then*

1. *If the composition $g \circ f : X \to Z$ is γ_3-open function and g is injective and γ_2-continuous function, then the function f is γ_2-open.*

2. *If the composition $g \circ f : X \to Z$ is γ_3-closed function and g is injective and γ_2-continuous function, then the function f is γ_2-closed.*

Proof. 1. Let $G \subseteq X$ be any open set in X. As $g \circ f : X \to Z$ is γ_3-open function, then $(g \circ f)(G) = g(f(G))$ is γ_3-open set in Z and as such it is open set in Z. From the γ_2-continuity of the function g, it follows that $g^{-1}(g(f(G)))$ is γ_2-open set in Y. As g is injective, it follows that $g^{-1}(g(f(G))) = f(G)$. It means $f(G)$ is γ_2-open in Y. Consequently f is γ_2-open function.

2. Let $F \subseteq X$ be any closed set in X. As $g \circ f : X \to Z$ is γ_3-closed function, then $(g \circ f)(G) = g(f(G))$ is γ_3-closed set in Z and as such it is closed set in Z. From the γ_2-continuity of the function g, it follows that $g^{-1}(g(f(F)))$ is γ_2-closed set in Y. As g is injective, it follows that $g^{-1}(g(f(F))) = f(F)$. It means $f(F)$ is γ_2-closed in Y. Consequently, f is γ_2-closed function.

□

Theorem 22. *1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and γ a $\gamma-$operation in space Y. Further let A be open subspace in X. If the function $f : X \to Y$ is the $\gamma-$open function then also the retract $f_{|A} : A \to Y$ is $\gamma-$open function.*

2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and γ a $\gamma-$operation in space*

Y. Further let A be closed subspace in X. If the function $f : X \to Y$ is the $\gamma-$closed function then also the retract $f_{|A} : A \to Y$ is $\gamma-$closed function.

Proof. 1. Let $G \subseteq A$ be any open set in A. As A is open in X it follows that G is open in X. As f is $\gamma-$open function, then $f(G)$ is $\gamma-$open set in Y and $f_{|A}(G) = f(G)$. It means $f_{|A}(G)$ is $\gamma-$open set in Y. Consequently $f_{|A} : A \to Y$ is $\gamma-$open function.

2. Let $F \subseteq A$ be any open set in A. As A is closed in X it follows that F is closed in X. As f is $\gamma-$closed function, then $f(F)$ is $\gamma-$closed set in Y and $f_{|A}(F) = f(F)$. It means $f_{|A}(F)$ is $\gamma-$closed set in Y. Consequently $f_{|A} : A \to Y$ is $\gamma-$closed function.

□

The following theorem shows in which case: $\gamma-$open function and $\gamma-$closed function are equivalent.

Theorem 23. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and γ a $\gamma-$operation in space Y. Further let $f : X \to Y$ be the bijective function of topological spaces. Then the following statements are equivalent:*

1. *f is $\gamma-$open function.*
2. *f is $\gamma-$closed function.*

Proof. (1) $\Rightarrow$ (2) Let f be the $\gamma-$open function and F any closed set in X. Then $X \setminus F$ is open set in X, from where $f(X \setminus F)$ is $\gamma-$open set in Y. As f is bijective, we have: $f(X \setminus F) = Y \setminus f(F)$. That is, $Y \setminus f(F)$ is $\gamma-$open in Y, and then $f(F)$ is $\gamma-$closed in Y. Consequently f is $\gamma-$closed function. (2) $\Rightarrow$ (1) Let f be the $\gamma-$closed function and G any open set in X. Then $X \setminus G$ is closed set in X, from where $f(X \setminus G)$ is $\gamma-$closed set in Y. As f is bijective, we have: $f(X \setminus G) = Y \setminus f(G)$. That is, $Y \setminus f(G)$ is $\gamma-$closed in Y, and then $f(G)$ is $\gamma-$open in Y. Consequently f is $\gamma-$open function. □

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