International Journal of Applied Mathematics

Volume 32 No. 4 2019, 579-585
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v32i4.3

ON THE PRICE VECTOR FIELDS OF A DIFFERENTIATED OLIGOPOLY

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Abstract: We apply topological results to the price dynamics of firms in an oligopoly by first giving an analytical treatment to the case of a duopoly and then making some general homotopically invariant observations for N greater than two.

AMS Subject Classification: 91B24, 91B52, 91B74
Key Words: price vector fields, oligopoly equilibria, oligopoly dynamics

1. Introduction

An industry is considered as an oligopoly if the number of firms therein is small enough as to cause "mutual dependence" recognized, so that in a differentiated oligopoly where the products are similar but not identical, price decisions are made in a game-theoretical framework with collusion and competition coexisting (cf. e.g., [5] for evolutionary game dynamics). Classic oligopoly equilibria (for the general construct of fixed points, see, e.g., [2], [4]) have been associated with such names as Cournot-Nash (cf. e.g., [1]), Bertrand, Edgeworth, Stackelberg, and Chamberlin. Our focus here will be on the price vector fields that may characterize an oligopoly. In the sequel we will first construct a price vector field for a duopoly to derive some consequent observations and next treat the case of three firms or more in broad topological terms. Finally we will conclude this paper with some summary remarks.

Received: April 26, 2019

2. Analysis

We begin with a model of a duopoly, where $N = 2$. For expository purposes, we will use (x, y) to stand for prices of firms X and Y.

Let $a > 0$ and consider the orbit

$$(x - a)^2 + y^2 = a^2. \tag{1}$$

Then we may have

$$x_+ (t) \quad : \quad = a \cos\left(-\pi + 2\pi \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz\right) + a, \text{ and} \tag{2}$$

$$y_\mp (t) \quad : \quad = a \sin\left(-\pi + 2\pi \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz\right), \tag{3}$$

with

$$\frac{dx_+}{dt} = -y\sqrt{2\pi} e^{-0.5t^2}, \text{ and} \tag{4}$$

$$\frac{dy_\mp}{dt} = (x - a)\sqrt{2\pi} e^{-0.5t^2}], \text{where} \tag{5}$$

$$\begin{aligned} t &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi - \arccos\left(\frac{x-a}{a}\right)\right)\right] \quad \forall y \leq 0; \\ &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{x-a}{a}\right)\right)\right] \quad \forall y \geq 0. \end{aligned} \tag{6}$$

This is an orbit that approaches $(0, 0)$ as $t \to -\infty$, but as t increases, $(x(t), y(t))$ moves along the lower half of the circle, to pass through $(2a, 0)$ and continue the motion along the upper half of the circle; in particular, at $(a, -a)$ we have $\arccos 0 = \frac{\pi}{2}$ so that $t = \text{norminv}\left(\frac{1}{4}\right)$ is the standard normal variable z that accumulates an area of $\frac{1}{4}$ and thereby

$$x_+ = a \cos\left(-\pi + \frac{2\pi}{4}\right) + a = a, \text{ and} \tag{7}$$

$$y_\mp = a \sin\left(-\pi + \frac{2\pi}{4}\right) = -a. \tag{8}$$

At $(2a, 0)$ we have $t = \text{norminv}\left(\frac{1}{2\pi}(\pi)\right) = 0$, with cumulative area $\frac{1}{2}$; at (a, a) we have $t = \text{norminv}\left(\frac{1}{2\pi}\left(\pi + \frac{\pi}{2}\right)\right)$, with cumulative area $\frac{3}{4}$; at $(2a, 0)$ we have $t = \text{norminv} \to \infty$, we have $\pi + \arccos\left(\frac{x-a}{a}\right) \to 2\pi$ and the motion approaches $(0, 0)$. As such, $(0, 0)$ is a globally asymptotically stable equibibrium.

This orbit describes a dynamics where X being the leader of the two firms intitiated a price increase from the existing equilibrium and Y reacted by a price decrease; this interaction lasted until $(a,-a)$. Then Y began to collude with X by increasing its price as well until $(2a,0)$. The third segment continued from $(2a,0)$ to (a,a), when Y kept increasing its price but X did not go along, instead it began to reduce its price. The last segment from (a,a) back to $(0,0)$ showed a "price war" between X and Y, decreasing their prices to their original equilibrium. In summary, the four arcs of the circle consists of, in time evolution, $(+,-)$, $(+,+)$, $(-,+)$, and $(-,-)$. Here we remark that: (1) Actual price movements in markets are descrete, not continuous. The circular orbit here represents *potential* observations, in analogy to quantum states being observed in isolated mode despite the underlying wavefunction being continuous. In fact, here we have prescribed the Gaussian densities to the above circular orbit. (2) Clearly the orbit of "real interactions" does not have to be that of a circle, but here is where homotopy invariance becomes relevant. That is, topological values of manifolds are invariant under smooth transformations of the manifold, e.g., the index of an equilibrium.

We next reflect the above right half-plane onto $\{(x,y) \mid x \leq 0\}$, i.e.,

$$x_{-}(t) := -x_{+}(t), \tag{9}$$

with $y_{\mp}(t)$ remaining the same. Then we must modify the time parameter t into

$$\begin{aligned} t &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi - \arccos\left(\frac{x+a}{a}\right)\right)\right] \ \forall y \leq 0; & (10) \\ &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{x+a}{a}\right)\right)\right] \ \forall y \geq 0. & (11) \end{aligned}$$

This left half-plane represents that X, the leader, initiates a price decrease with the same kind of subsequent dynamics as that of the right half-plane, i.e., in time evolution, $(-,-)$, $(-,+)$, $(+,+)$, and $(+,-)$ over the four equal arcs of any circular orbit.

By taking the union of the above two half-planes we obtain a vector field over $\mathbb{R}^2$ that possesses one unique globally asymptotically stable equilibrium $(0,0)$, which has index 2.

While the above depicted a situation where one of the duopoly, X, led the dynamics, it is entirely possible that X and Y are of equal stature, as measured by, say, market shares. Then we consider a $90° -$ *rotation* of the above

$$U_{L,R} \equiv L \cup R, \text{ with} \tag{12}$$

$$\begin{aligned} L &\equiv \{(x,y) \mid x \le 0\} \text{ and} \\ R &\equiv \{(x,y) \mid x \ge 0\} \end{aligned}$$

into

$$\begin{aligned} U_{B,T} &\equiv B \cup T, \text{ with} \\ B &\equiv \{(x,y) \mid y \le 0\} \text{ and} \\ T &\equiv \{(x,y) \mid y \ge 0\} \end{aligned} \tag{13}$$

with a subsequent field superposition of

$$U := U_{L,R} + U_{B,T}. \tag{14}$$

This $90° - \textit{rotation}$ clearly can be made in two ways, counterclockwise and clockwise; however, there exists a restriction on the time parameter t, i.e., the requirement of

$$U(t) = U_{L,R}(t) + U_{B,T}(t).$$

Now by design the flows in $U_{L,R}$ are in the lower half-plane during $t \in (-\infty, 0]$ so that $R(t \le 0) \cap B(t \le 0)_{clockwise}$ would fail to approach the equilibrium $(0,0)$ as it must be approached over $t \in [0,\infty)$; likewise, $L(t \le 0) \cap B(t \le 0)_{counterclockwise}$ would fail to approach the equilibrium $(0,0)$ also. As such, we will only consider case (i) $R \cap T_{clockwise}$ and (ii) $L \cap T_{counterclockwise}$, $\forall t \in [0,\infty)$. Patentheticaly of course if we had modeled flows in $U_{L,R}$ beginning from the upper half-plane, then we would have the opposite symmetrical results.

Hence for $R \cap T_{clockwise}$ we have $T_{clockwise} = \{(x_{\mp}(t), y_{+}(t))\}$, with

$$x_{\mp}(t) = a\cos\left(-\pi + 2\pi \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz\right), \text{ and} \tag{15}$$

$$y_{+}(t) = a\sin\pi + 2\pi \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz + a, \tag{16}$$

so that

$$\frac{dx_{\mp}}{dt} = -(y-a)\sqrt{2\pi}e^{-0.5t^2}, \text{ and} \tag{17}$$

$$\frac{dy_{+}}{dt} = x\sqrt{2\pi}e^{-0.5t^2}, \text{ where} \tag{18}$$

$$t = \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{y-a}{a}\right)\right)\right]. \tag{19}$$

Thus we have the following field superposition by Equations (4), (5), (17), and (18)

$$\frac{d}{dt}\begin{pmatrix} x_{+} + x_{\mp} \\ y_{\mp} + y_{+} \end{pmatrix} = \sqrt{2\pi}e^{-0.5t^2}\begin{pmatrix} -y-(y-a) \\ (x-a)+x \end{pmatrix}$$

$$= \sqrt{2\pi}e^{-0.5t^2}\left[2\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} a \\ -a \end{pmatrix}\right], \qquad (20)$$

where

$$\begin{aligned} t &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{x-a}{a}\right)\right)\right] \text{ from Equation (6)} \\ &\equiv \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{y-a}{a}\right)\right)\right] \text{ from Equation (19)} \\ &\quad \text{implies that} \\ x &= y \qquad (21) \end{aligned}$$

and that a new equilibrium is established at

$$(x, y) = \left(\frac{a}{2}, \frac{a}{2}\right), \qquad (22)$$

for which we have the characteristic equation

$$\begin{aligned} 0 &= \det\begin{pmatrix} \lambda & 2 \\ -2 & \lambda \end{pmatrix} = \lambda^2 + 4 \qquad (23) \\ &\Longrightarrow \lambda = \pm 2i \text{ and hence} \qquad (24) \\ \left(\frac{a}{2}, \frac{a}{2}\right) &= \text{a center.} \qquad (25) \end{aligned}$$

This corresponds to a situation where X and Y simultaneously make an identical price increase by a, then gravitating toward $\left(\frac{a}{2}, \frac{a}{2}\right)$ with potential rotations around $\left(\frac{a}{2}, \frac{a}{2}\right)$, indeed a plausible event to a duopoly (cf. [3]).

For bookdeeping we have for the vector field $\mathbf{V}(R \cap T_{clockwise})$ a divergence equal to 0, hence the flows being incompressible, and $curl(\mathbf{V}) = (0, 0, 4)^T$.

Next we consider $L \cap T_{counterclockwise}$, where

$$\begin{aligned} T_{counterclockwise} &= \{(x_{\pm}(t), y_{+}(t))\}, \text{ with} \\ x_{\pm}(t) &= a\cos\left(-\pi + 2\pi\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-0.5z^2}dz\right), \qquad (26) \\ y_{+}(t) &= a\sin\left(-\pi + 2\pi\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-0.5z^2}dz\right) + a, \qquad (27) \end{aligned}$$

so that

$$\begin{aligned} \frac{dx_{\pm}}{dt} &= -(y-a)\sqrt{2\pi}e^{-0.5t^2} \text{ and} \qquad (28) \\ \frac{dy_{+}}{dt} &= x\sqrt{2\pi}e^{-0.5t^2}, \text{ where by Equation (19)} \qquad (29) \end{aligned}$$

$$t = \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{y-a}{a}\right)\right)\right] \tag{30}$$

and the field superposition is, by Equations (4), (9), (28), (29), (11), and (30),

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x_- + x_\pm \\ y_\mp + y_+ \end{pmatrix} &= \sqrt{2\pi}e^{-0.5t^2}\begin{pmatrix} y-(y-a) \\ (x-a)+x \end{pmatrix} \\
&= \sqrt{2\pi}e^{-0.5t^2}\begin{pmatrix} a \\ 2x-a \end{pmatrix}, \text{ with} \qquad (31) \\
t &= \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{y-a}{a}\right)\right)\right] \\
&\equiv \text{norminv}\left[\frac{1}{2\pi}\left(\pi + \arccos\left(\frac{x+a}{a}\right)\right)\right], \\
&\quad \text{implying that} \\
x+a &= y-a. \qquad (32)
\end{aligned}$$

As such, $\{(x,y) \mid y-x=2a\}$ approaches $(0,0)$ as $t \to \infty$ and $(0,0)$ remains to be the unique globally asymptotically stable equilibrium. Here we have

$$\begin{aligned}
div\left(V\left(L \cap T_{counterclockwise}\right)\right) &= 0 \text{ and} \\
curl\left(V\left(L \cap T_{counterclockwise}\right)\right) &= (0,0,2)^T.
\end{aligned}$$

We now consider the case of $N = 3$ by modeling $(p_1, p_2, p_3(p_1, p_2))$ as a vector field over S^2. Since the Euler characteristic $\chi(S^2) = 2$ and by the Poincaré-Hopf theorem the sum of indices of equlibria equals χ, an application of the Borsuk-Ulam theorem then leads to the existence of a pair of antipodal equilibria, one unstable and the other globally asymptotically stable.

Lastly we consider the case of $N \geq 4$. Assume that

$$\{(p_1, p_2, \cdots, p_N)\} \equiv M^N$$

form a compact $N - manifold$. Then M^N is triangulable and

$$\chi\left(M^N\right) = \sum_{j=0}^{N} (-1)^j \cdot F_j, \text{ where}$$

$F_j =$ the number of $j-$dimensional "faces": $j = 0$ for vertices, $j = 1$ for edges, and $j = 2$ for faces in triangles. Then it follows that the number of equilibria must be at least $\chi\left(M^N\right)$.

3. Summary remark

In this paper we applied vector fields to oligopolistic pricing dynamics. While our models were specific, by homotopy invariance our topological results can be generalized to any smooth deformation of the vector fields. In our analyses above there was a free parameter $a > 0$, we envision that future studies may assign a probability measure on $a \in \mathbb{R}$ so that the vector fields carry probabilities analogous to quantum waves.

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