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SOME SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS FOR OPERATORS ON HILBERT SPACE

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Abstract: The main object of the present paper is to investigate some results concerning a sufficient and necessary condition, coefficient estimates and distortion theorem for the class $\mathcal{T}_\delta^\lambda(\alpha, A)$. Furthermore, some applications of the fractional calculus for operator on Hilbert space are also considered.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in the unit disk $\mathbb{U}$.

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Then a function $f(z) \in \mathcal{S}$ is said to be *starlike of order* α $(0 \leq \alpha < 1)$ in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leq \alpha < 1; z \in \mathbb{U}). \tag{2}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α in $\mathbb{U}$.

A function $f(z) \in \mathcal{S}$ is said to be *convex of order* α $(0 \leq \alpha < 1)$ in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (0 \leq \alpha < 1; z \in \mathbb{U}). \tag{3}$$

We denote by $\mathcal{K}(\alpha)$ the class of all functions in $\mathcal{S}$ which are convex of order α in $\mathbb{U}$.

Let a, b and c be complex numbers with $c \neq 0, -1, -2, \cdots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 1 & (k = 0) \\ \eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}). \end{cases}$$

The hypergeometric function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{U}$ and if a or b is a negative integer, then it reduces to a polynomial.

For functions $f_j(z) \in \mathcal{A}$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2),$$

we define the *Hadamard product (or convolution)* of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \qquad (z \in \mathbb{U}).$$

For the purpose to define the Srivastava-Attiya transform, we recall here the general Hurwitz-Lerch Zeta function, which is defined in [15] by the following series:

$$\Phi(z, \lambda, \delta) := \frac{1}{\delta^\lambda} + \sum_{k=1}^{\infty} \frac{z^k}{(k + \delta)^\lambda}$$

$$(\delta \in \mathbb{C} \setminus \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}; \lambda \in \mathbb{C} \text{ when } z \in \mathcal{U}; \\ \text{Re}(\lambda) > 1 \text{ when } |z| = 1).$$

For the properties and characteristics of the Hurwitz-Lerch Zeta function and other related special functions, see for example [4], [9] and [16].

Recently, Srivastava and Attiya [14] have introduced the linear operator $\mathcal{L}_{\lambda,\delta} : \mathcal{A} \to \mathcal{A}$, defined in terms of the Hadamard product by

$$\mathcal{L}_{\lambda,\delta} f(z) = \mathcal{G}_{\lambda,\delta}(z) * f(z) \qquad (\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \lambda \in \mathbb{C}; z \in \mathbb{U}), \tag{4}$$

where

$$\mathcal{G}_{\lambda,\delta}(z) = (1+\delta)^\lambda \left[\Phi(z, \lambda, \delta) - \delta^{-\lambda}\right] \qquad (z \in \mathbb{U}). \tag{5}$$

The operator $\mathcal{L}_{\lambda,\delta}$ is now popularly known in the literature as the *Srivastava-Attiya operator*. Various class-mapping properties of the operator $\mathcal{L}_{\lambda,\delta}$ (and its variants) are discussed in the recent works of Srivastava and Attiya [14], Liu [8], Murugusundaramoorthy [10], Yuan and Liu [19], Yunus et al. [20] and others.

It is easy to observe from (1) and (4) that

$$\mathcal{L}_{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\delta}{k+\delta}\right)^\lambda a_k z^k. \tag{6}$$

We note that:

(i) $\mathcal{L}_{0,b} f(z) = f(z)$;

(ii) $\mathcal{L}_{1,0} f(z) = \mathcal{L} f(z) = \int_0^z \frac{f(t)}{t} dt \quad (f \in \mathcal{A})$ (see Alexander [1]);

(iii) $\mathcal{L}_{m,1} f(z) = \mathcal{I}^m f(z) \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \cdots\})$ (see Flett [5]);

(iv) $\mathcal{L}_{\gamma,1} f(z) = \mathcal{Q}^\gamma f(z) \quad (\gamma > 0)$ (see Jung et al. [6]);

(v) $\mathcal{L}_{m,0} f(z) = \mathcal{L}^m f(z) \quad (m \in \mathbb{N}_0)$ (see Sălăgean [12]).

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions whose nonzero coefficients are negative. That is, an analytic and univalent function f is in $\mathcal{T}$ if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{7}$$

We also denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the subclasses of $\mathcal{T}$ that are, respectively, starlike of order α and convex of order α. See Silverman [13] for further information on them. And let $\mathcal{T}_\delta^\lambda(\alpha)$ denote the class of functions of the form (7) satisfying the condition:

$$\text{Re}\left(\frac{z(\mathcal{L}_{\lambda,\delta} f)'(z)}{\mathcal{L}_{\lambda,\delta} f(z)}\right) > \alpha \quad (\lambda \in \mathbb{R}; \delta > -1; 0 \leq \alpha < 1; z \in \mathbb{U}). \tag{8}$$

Clearly, we have $\mathcal{T}_\delta^0(\alpha) = \mathcal{T}^*(\alpha)$ and $\mathcal{T}_0^{-1}(\alpha) = \mathcal{C}(\alpha)$ $(0 \le \alpha < 1)$.

Finally, let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For a complex valued function f analytic on a domain E of the complex plane containing the spectrum $\sigma(A)$ of A we denote $f(A)$ as Riesz-Dunford integral [2, p.568], that is,

$$f(A) := \frac{1}{2\pi i}\int_C f(z)(zI - A)^{-1}dz, \tag{9}$$

where I is the identity operator on $\mathcal{H}$ and C is positively oriented simple closed rectifiable contour containing $\sigma(A)$. Also $f(A)$ can be defined by the series $f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}A^k$ which converges in the norm topology, [3]. If $f(z)$ is defined by (1), we also have

$$\mathcal{L}_{\lambda,\delta} f(A) = \sum_{k=1}^{\infty} \left(\frac{1+\delta}{k+\delta}\right)^{\lambda} a_k A^k \qquad (a_1 = 1). \tag{10}$$

Throughout this paper, A^* shall always denote the conjugate operator of A.

By using arguments similar to [13, Theorem 2] with (8), we prove the following lemma.

Lemma 1. *Let $f(z)$ of the form (7) be analytic in $\mathbb{U}$, $\lambda \in \mathbb{R}$, $\delta > -1$, and $0 \le \alpha < 1$. Then the following statements are equivalent:*

(i) $f \in \mathcal{T}_\delta^\lambda(\alpha)$;

(ii) $\left|\dfrac{z(\mathcal{L}_{\lambda,\delta} f(z))'}{\mathcal{L}_{\lambda,\delta} f(z)} - 1\right| \le 1 - \alpha \qquad (z \in \mathbb{U})$;

(iii) $\displaystyle\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} B_k(\lambda,\delta) a_k \le 1$,

where

$$B_k(\lambda,\delta) = \left(\frac{1+\delta}{k+\delta}\right)^{\lambda}. \tag{11}$$

Proof. In view of the definition of $\mathcal{T}_\delta^\lambda(\alpha)$, we obtain

$$f \in \mathcal{T}_\delta^\lambda(\alpha) \Leftrightarrow \mathcal{L}_{\lambda,\delta} f \in \mathcal{T}^*(\alpha),$$

and so, Lemma 1 follows immediately from the result [13, Theorem 2]. □

From Lemma 1, we define a new class $\mathcal{T}_\delta^\lambda(\alpha, A)$ as following.

Definition 1. Let $\mathcal{T}_\delta^\lambda(\alpha, A)$ denote the class of functions of the form (7) satisfying the condition

$$\|A(\mathcal{L}_{\lambda,\delta}f)'(A) - \mathcal{L}_{\lambda,\delta}f(A)\| \leq (1-\alpha)\|\mathcal{L}_{\lambda,\delta}f(A)\|, \tag{12}$$

where $\lambda \in \mathbb{R}$, $\delta > -1$, $0 \leq \alpha < 1$, and all operators A with $\|A\| < 1$, $A \neq \theta$ (θ denotes the zero operator on $\mathcal{H}$).

The following definition is given below for some operators of generalized fractional calculus defined by Kim et al. [7] (see also [11] and [17]).

Definition 2. For an invertible operator A, *the fractional integral operator* $\mathcal{I}_{0,A}^{a,b,c}$ is defined by

$$\mathcal{I}_{0,A}^{a,b,c}f(A) = \frac{1}{\Gamma(a)}\int_0^1 A^{-b}{}_2F_1(a+b,-c;a;1-t)f(tA)(1-t)^{a-1}dt, \tag{13}$$

where $a > 0$ and $b, c \in \mathbb{R}$.

The fractional derivative operator $\mathcal{D}_{0,A}^{a,b,c}$ is defined by

$$\mathcal{D}_{0,A}^{a,b,c}f(A) = \frac{1}{\Gamma(1-a)}g'(A), \tag{14}$$

where

$$g(z) = \int_0^1 z^{-b}{}_2F_1(b-a+1,-c;1-a;1-t)f(tz)(1-t)^{-a}dt,$$

$0 < a < 1$ and $b, c \in \mathbb{R}$. In both (13) and (14), $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}(|z|^{\epsilon}) \quad (z \to 0)$$

for $\epsilon > \max\{0, b-c\} - 1$, and the multiplicity of $(1-t)^{a-1}$ is in (13) (and that of $(1-t)^{-a}$ in (14)) removed by requiring $\log(1-t)$ to be real when $1-t > 0$.

In this article, we provide some results concerning a sufficient and necessary condition, coefficient estimates and the distortion theorem for the class $\mathcal{T}_{\delta}^{\lambda}(\alpha, A)$. Also, we consider several applications of fractional calculus for operators on Hilbert space.

2. Some results for the class $\mathcal{T}_{\delta}^{\lambda}(\alpha, A)$

We begin by proving an equivalent condition for the class $\mathcal{T}_{\delta}^{\lambda}(\alpha, A)$ due to Lemma 1 as following.

Lemma 2. *Let $f(z)$ be in the class $\mathcal{T}_{\delta}^{\lambda}(\alpha, A)$ for all proper contraction A with $A \neq \theta$ if and only if*

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} B_k(\lambda,\delta) a_k \leq 1, \tag{15}$$

where $B_k(\lambda,\delta)$ *is given by* (11), $\lambda \in \mathbb{R}$, $\delta > -1$ *and* $0 \leq \alpha < 1$. *The result is sharp for the function*

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} z^k \qquad (k \geq 2). \tag{16}$$

Proof. Assume that the inequality (15) holds. By using (10) and (11), we have

$$\begin{aligned}
&\|A(\mathcal{L}_{\lambda,\delta} f)'(A) - \mathcal{L}_{\lambda,\delta} f(A)\| - (1-\alpha)\|\mathcal{L}_{\lambda,\delta} f(A)\| \\
&= \|A - \sum_{k=2}^{\infty} k B_k(\lambda,\delta) a_k A^k - A + \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k A^k\| \\
&\qquad - (1-\alpha)\|A - \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k A^k\| \\
&= \|\sum_{k=2}^{\infty} (k-1) B_k(\lambda,\delta) a_k A^k\| - (1-\alpha)\|A - \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k A^k\| \\
&\leq \sum_{k=2}^{\infty} (k-1+1-\alpha) B_k(\lambda,\delta) a_k - (1-\alpha) \leq 0.
\end{aligned}$$

Hence $f(z) \in \mathcal{T}_\delta^\lambda(\alpha, A)$. For the converse, assume that

$$\|A(\mathcal{L}_{\lambda,\delta} f)'(A) - \mathcal{L}_{\lambda,\delta} f(A)\| \leq (1-\alpha)\|\mathcal{L}_{\lambda,\delta} f(A)\|.$$

Then

$$\|\sum_{k=2}^{\infty} (k-1) B_k(\lambda,\delta) a_k A^k\| \leq (1-\alpha)\|A - \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k A^k\|.$$

Choose $A = eI$ $(0 < e < 1)$. We obtain

$$\frac{\sum_{k=2}^{\infty} (k-1) B_k(\lambda,\delta) a_k e^k}{e - \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k e^k} \leq 1-\alpha. \tag{17}$$

Upon clearing the denomination in (17) and letting $e \to 1$, we have

$$\sum_{k=2}^{\infty} (k-1) B_k(\lambda,\delta) a_k \leq (1-\alpha)\{1 - \sum_{k=2}^{\infty} B_k(\lambda,\delta) a_k\},$$

which yields the required condition (15). It is evident that the function (16) is an extreme one for Lemma 2. □

Corollary 1. *Let $f(z)$ be in the class $\mathcal{T}_\delta^\lambda(\alpha, A)$. Then*

$$a_k \leq \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} \qquad (k \geq 2),$$

where $B_k(\lambda,\delta)$ is given by (11), $\lambda \in \mathbb{R}$, $\delta > -1$ and $0 \leq \alpha < 1$. The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} z^k \qquad (k \geq 2).$$

Theorem 1. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} z^k \qquad (k \geq 2).$$

Then $f(z) \in \mathcal{T}_\delta^\lambda(\alpha, A)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{18}$$

where $\mu_k \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If we set

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z),$$

then

$$f(z) = z - \sum_{k=2}^{\infty} \mu_k \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} z^k.$$

Thus we obtain

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)B_k(\lambda,\delta)}{1-\alpha} \mu_k \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

Hence $f(z) \in \mathcal{T}_\delta^\lambda(\alpha, A)$. For the converse, we assume that $f(z)$ given by (7) is in the class $\mathcal{T}_\delta^\lambda(\alpha, A)$. From Corollary 1 we have

$$a_k \leq \frac{1-\alpha}{(k-\alpha)B_k(\lambda,\delta)} \qquad (k \geq 2).$$

We may set

$$\mu_k = \frac{(k-\alpha)B_k(\lambda,\delta)}{1-\alpha} a_k \qquad (k \geq 2)$$

and $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$. Hence we conclude that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z),$$

which completes the proof of Theorem 1. □

Theorem 2. *If $f(z) \in \mathcal{T}_\delta^\lambda(\alpha, A)$ for $\lambda \geq 0$, $\delta > -1$, $0 \leq \alpha < 1$ and $\|A\| < 1$, $A \neq \theta$, then*

$$\|A\| - \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^2 \leq \|f(A)\| \leq \|A\| + \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^2$$

and

$$1 - \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\| \leq \|f'(A)\| \leq 1 + \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|.$$

Proof. From Lemma 2, we see that

$$\frac{2-\alpha}{1-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda} \sum_{k=2}^\infty a_k \leq \sum_{k=2}^\infty \frac{k-\alpha}{1-\alpha} B_k(-\lambda, \delta) a_k \leq 1$$

for $\lambda \geq 0$ and $\delta > -1$ which gives

$$\sum_{k=2}^\infty a_k \leq \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda.$$

Therefore we have

$$\|f(A)\| \geq \|A\| - \|A\|^2 \sum_{k=2}^\infty a_k \geq \|A\| - \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^2$$

and

$$\|f(A)\| \leq \|A\| + \|A\|^2 \sum_{k=2}^\infty a_k \leq \|A\| + \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^2.$$

By noting the relation

$$\frac{k(2-\alpha)}{2(1-\alpha)}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda} \leq \frac{k-\alpha}{1-\alpha} B_k(-\lambda, \delta) \qquad (k \geq 2), \tag{19}$$

we have

$$\sum_{k=2}^\infty \frac{k(2-\alpha)}{2(1-\alpha)}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda} a_k \leq \sum_{k=2}^\infty \frac{k-\alpha}{1-\alpha} B_k(-\lambda, \delta) a_k \leq 1,$$

that is

$$\sum_{k=2}^\infty k a_k \leq \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda.$$

Thus

$$\|f'(A)\| \geq 1 - \|A\| \sum_{k=2}^\infty k a_k \geq 1 - \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|$$

and

$$\|f'(A)\| \leq 1 + \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|.$$

This completes the proof of Theorem 2. □

3. Some results for fractional calculus operators

By using Definition 2, we prove the following theorem.

Theorem 3. *Let* $\max\{b-c, b, -c-a\} < 2$, $2a > b(a+c)$, $\lambda \geq 0$ *and* $\delta > -1$. *If* $f(z) \in \mathcal{T}_{\delta}^{\lambda}(\alpha, A)$ $(0 \leq \alpha < 1)$, *then*

$$\begin{aligned} \|\mathcal{I}_{0,A}^{a,b,c} f(A)\| \; &\leq \; \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)}\|A\|^{1-b} \\ &\quad + \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^{2-b} \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{I}_{0,A}^{a,b,c} f(A)\| \; &\geq \; \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)}\|A\|^{1-b} \\ &\quad - \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^{2-b} \end{aligned}$$

for $a > 0$, $b, c \in \mathbb{R}$ *and all invertible operator* A *with* $(A^{\frac{1}{q}})^{*}A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q}})^{*}$ $(q \in \mathbb{N})$, $\|A\| \leq 1$ *and* $r_{sp}(A)r_{sp}(A^{-1}) \leq 1$, *where* $r_{sp}(A)$ *is the radius of spectrum of* A.

Proof. Consider the function

$$\begin{aligned} F(A) &= \frac{\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(2-b-c)} A^{b}\mathcal{I}_{0,A}^{a,b,c} f(A) \\ &= A - \sum_{k=2}^{\infty} \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(k+1-b)\Gamma(a+k+1+c)\Gamma(2-b+c)} a_k A^k \\ &= A - \sum_{k=2}^{\infty} b_k A^k, \end{aligned}$$

where

$$b_k = \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(k+1-b)\Gamma(a+k+1+c)\Gamma(2-b+c)} a_k.$$

Further, for convenience, we put

$$\Phi(k) = \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(k+1-b)\Gamma(a+k+1+c)\Gamma(2-b+c)} \qquad (k \geq 2).$$

Then, by the constraints of the hypotheses, we see that $\Phi(k)$ is non-increasing for integers $k \geq 2$ and we have $0 < \Phi(k) < 1$. By virtue of Lemma 2, we obtain

$$\begin{aligned} \frac{2-\alpha}{1-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda}\sum_{k=2}^{\infty} b_k &\leq \sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}B_k(-\lambda,\delta)b_k \\ &\leq \sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}B_k(-\lambda,\delta)a_k \leq 1, \end{aligned}$$

which gives

$$\sum_{k=2}^{\infty} b_k \leq \frac{1-\alpha}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda} \qquad \text{and} \qquad F(z) \in \mathcal{T}_\delta^\lambda(\alpha, A).$$

Therefore we have

$$\begin{aligned} \|\mathcal{I}_{0,A}^{a,b,c} f(A)\| \leq{}& \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)}\|A\|\|A^{-b}\| \\ &+ \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^2\|A^{-b}\| \end{aligned} \tag{20}$$

and

$$\begin{aligned} \|\mathcal{I}_{0,A}^{a,b,c} f(A)\| \geq{}& \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)}\|A\|\|A^{-b}\| \\ &- \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^2\|A^{-b}\|. \end{aligned} \tag{21}$$

By the equation (7) of [18, p.307],

$$\|A^b\| = \|A\|^b \qquad (b > 0).$$

Since $A^*A = AA^*$, $\|A\| = r_{sp}(A)$. So,

$$1 = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| = r_{sp}(A)r_{sp}(A^{-1}) \leq 1.$$

Thus $\|A^{-1}\| = \|A\|^{-1}$. Therefore

$$\|A^b\| = \|A\|^b \tag{22}$$

for all real b. By applying (20), (21) and (22), we evidently completes the proof of Theorem 3. □

Theorem 4. *Let* $\max\{b-c-1, b, -2-c+a\} < 1$, $c+1 < (1-b)(2-a+c)$, $b(2-a+c) \leq 2(1-\alpha)$, $\lambda \geq 0$ *and* $\delta > -1$. *If* $f(z) \in \mathcal{T}_\delta^\lambda(\alpha, A)$ $(0 \leq \alpha < 1)$, *then*

$$\begin{aligned}\|\mathcal{D}_{0,A}^{a,b,c} f(A)\| \ &\leq \ \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \\ &\quad + \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^{1-b}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{D}_{0,A}^{a,b,c} f(A)\| \ &\geq \ \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \\ &\quad - \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)}\left(\frac{1+\delta}{2+\delta}\right)^\lambda \|A\|^{1-b}\end{aligned}$$

for $a > 0$, $b, c \in \mathbb{R}$ *and all invertible operator* A *with* $(A^{\frac{1}{q}})^* A^{\frac{1}{q}} = A^{\frac{1}{q}} (A^{\frac{1}{q}})^*$ $(q \in \mathbb{N})$, $\|A\| \leq 1$ *and* $r_{sp}(A) r_{sp}(A^{-1}) \leq 1$, *where* $r_{sp}(A)$ *is the radius of spectrum of* A.

Proof. Consider the function

$$\begin{aligned}G(A) &= \frac{\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(2-b-c)} A^{b+1} \mathcal{D}_{0,A}^{a,b,c} f(A) \\ &= A - \sum_{k=2}^{\infty} \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} a_k A^k \\ &= A - \sum_{k=2}^{\infty} c_k A^k,\end{aligned}$$

where

$$c_k = \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} a_k.$$

Further, for convenience, we put

$$\Psi(k) = \frac{\Gamma(k+1-b+c)\Gamma(k)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} \qquad (k \geq 2).$$

Then, by the constraints of the hypotheses, we see that $\Psi(k)$ is non-increasing for the integers $k \geq 2$ and we have $0 < \Psi(k) < 1$, that is,

$$0 < \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} < k.$$

Therefore, by applying (19) and Lemma 2, we obtain

$$\begin{aligned}\frac{2-\alpha}{2(1-\alpha)}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda}\sum_{k=2}^{\infty}c_k &= \sum_{k=2}^{\infty}\frac{2-\alpha}{2(1-\alpha)}\left(\frac{1+\delta}{2+\delta}\right)^{-\lambda}k\Psi(k)a_k \\ &\leq \sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}B_k(-\lambda,\delta)\Psi(k)a_k \\ &\leq \sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}B_k(-\lambda,\delta)a_k \leq 1,\end{aligned}$$

which gives

$$\sum_{k=2}^{\infty}c_k \leq \frac{2(1-\alpha)}{2-\alpha}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}.$$

Hence, by using same arguments with the proof of Theorem 3, we have

$$\begin{aligned}&\|\mathcal{D}_{0,A}^{a,b,c}f(A)\| \\ &\leq \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} + \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{1-b}\sum_{k=2}^{\infty}c_k \\ &\leq \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \\ &\qquad + \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^{1-b}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{D}_{0,A}^{a,b,c}f(A)\| &\geq \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \\ &\quad - \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)}\left(\frac{1+\delta}{2+\delta}\right)^{\lambda}\|A\|^{1-b}\end{aligned}$$

This evidently completes the proof of Theorem 4. □

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