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A NOTE ON THE HOP DOMINATION NUMBER OF A SUBDIVISION GRAPH

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Abstract: Let $G = (V, E)$ be a graph with p vertices and q edges. A subset $S \subset V(G)$ is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called a hop domination number of G and is denoted by $\gamma_h(G)$. The subdivision graph $S(G)$ of a graph G is a graph obtained by subdividing every edge of G exactly once. In this paper, we obtain an upper bound on hop domination number of subdivision graph of any connected graph G in terms of number of edges q, the maximum degree $\Delta(G)$ and domination number $\gamma(G)$ of G. We also characterize the family of connected graphs attaining this bound.

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Key Words: hop domination number; subdivision graph; connected graph

1. Introduction

Throughout this paper, by a graph $G = (V, E)$ we mean a connected simple graph. We denote a graph G of order p and size q by a (p, q)-graph. By subdividing an edge $e = uv$ of a graph G we mean deleting the edge e and introducing a new vertex x and the edges ux and xv. For a graph G, the *subdivision graph* $S(G)$ is a graph obtained by subdividing every edge of G exactly once. The *distance* between two vertices u and v of a graph G is the

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length of the shortest path joining u and v in G and is denoted by $d(u,v)$. A graph G with exactly one cycle is called an *unicyclic graph*. A set $D \subset V$ is said to be a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D. D is said to be a *minimal dominating set* of G if no subset of it is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. In [4], Chartrand et al. studied the exact 2-step dominating sets in graphs.

Recently, Ayyaswamy and Natarajan ([2, 9]) initiated a study on a new domination parameter called hop domination number of a graph and characterized the family of trees and unicyclic graphs with equal hop domination number and total domination number. Ayyaswamy et al. ([1]) found some bounds on hop domination number of a tree. Henning et al. [8] obtained certain probabilistic bounds for this parameter. Farhadi et al. [5] discussed the complexity results of k-hop dominating set of a graph. Pabilona et al. [10, 11] studied connected hop domination and total hop domination on graphs under some binary operations. A vertex u of a graph G is said to hop dominate a vertex $v \in V(G)$ if $d(u,v) = 0$ or $d(u,v) = 2$. A subset $S \subset V(G)$ of a graph G is a *hop dominating set* (**hd-set**) of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. A path on n vertices is denoted by P_n and a cycle of length n is denoted by C_n. We denote a complete graph with n vertices by K_n and a complete bipartite graph with a bipartition (V_1, V_2) mn vertices by $K_{m,n}$. For other terminologies not defined here we refer to Chartrand and Lesniak ([3]) and Haynes et al. ([6, 7]). It is easy to verify that:

(i) $\gamma_h(P_n) = \gamma_h(C_n) = \begin{cases} 2r, & \text{if } n = 6r; \\ 2r+1, & \text{if } n = 6r+1; \\ 2r+2, & \text{if } n = 6r+s;\ 2 \le s \le 5. \end{cases}$

(ii) $\gamma_h(K_n) = n$.

(iii) $\gamma_h(K_{m,n}) = 2$.

(iv) $\gamma_h(W_n)$=3, where W_n is a wheel with $n-1$ spokes.

(v) $\gamma_h(P) = 2$, where P denotes the Peterson graph.

2. Main Results

Observation 1. $\gamma_h[S(P_n)] = \gamma_h(P_{2n-1})$.

Observation 2. $\gamma_h[S(C_n)] = \gamma_h(C_{2n})$.

Proposition 3. $\gamma_h[S(K_n)] = \lfloor \frac{n}{2} \rfloor + 1$, $n \geq 2$.

Proof. Let $V_1 = V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $V_2 = \{w_{ij} \in S(K_n) : w_{ij}$ is a vertex subdividing the edge v_iv_j in $S(K_n)\}$. Then $|V_2| = \frac{n(n-1)}{2} = nC_2$. Any one vertex v_i is enough to hop dominate all vertices of V_1 and we again require exactly $\lfloor \frac{n}{2} \rfloor$ vertices to hop dominate the nC_2 vertices of V_2. Thus $\{v_i\} \cup \{w_{j(j+1)} : j$ is odd; $1 \leq j \leq n\}$ is a γ_h-set of $S(K_n)$ and therefore $\gamma_h[S(K_n)] = \lfloor \frac{n}{2} \rfloor + 1$. □

Proposition 4. $\gamma_h[S(K_{m,n})] = 2 + \min\{m, n\}$.

Proof. Let (V_1, V_2) be the bipartition of $V(K_{m,n})$ with $|V_1| = m$ and $|V_2| = n$. Let $V_1 = \{v_i : 1 \leq i \leq m\}$ and $V_2 = \{u_j : 1 \leq j \leq n\}$.

Let $m \leq n$.

Let $V[S(K_{m,n})] = V_1 \cup V_2 \cup V_3$ where $V_3 = \{w_{ij} : w_{ij}$ is a vertex subdividing the edge v_iu_j in $S(K_{m,n})\}$. As V_1 and V_2 are independent sets, any γ_h-set of $S(K_{m,n})$ contains a vertex from each V_1 and V_2. One can observe that the set $D = \{w_{kk} : 1 \leq k \leq m\}$ is a minimum hd-set of V_3. Therefore $\gamma_h[S(K_{m,n})] = 2 + |D| = 2 + m = 2 + \min(m, n)$. Hence the result follows. □

Proposition 5. *For the Petersen graph P, $\gamma_h[S(P)] = 7$.*

Proof. Let us label the vertices of the outer cycle C_5 by v_1, v_2, v_3, v_4, v_5 and the inner cycle by u_1, u_2, u_3, u_4, u_5. Consider the three pairs (v_i, v_j), (v_i, u_j), (u_i, u_j). Only one of them forms an edge in P. Let w_{ij} be the vertex subdividing that edge. It is clear that the set $\{u_i, u_j\} \cup \{v_k\} \cup \{w_{i(i+1)} : i$ is odd, $1 \leq i \leq 4\} \cup \{w_{l(l+3)} : l = 1, 2\}$ is a γ_h-set of $S(P)$ where u_i and u_j are non adjacent vertices and the vertex v_k is adjacent to a vertex $u_k \in N(u_i) \cap N(u_j)$ in P. Hence $\gamma_h[S(P)] = 7$. □

Proposition 6. *For a wheel graph W_{n+1} with $n+1$ vertices, $\gamma_h[S(W_{n+1})] = \lfloor \frac{n}{3} \rfloor + 2$.*

Proof. Let the centre of W_{n+1} be v and let $v_1, v_2, \cdots, v_n$ be the vertices of the outer cycle C_n of W_{n+1}. Let the vertex which is adjacent to v and v_i in $S(W_{n+1})$ be denoted by u_i; $1 \leq i \leq n$ and let the vertex subdividing the edge v_iv_j in $S(W_{n+1})$ be denoted by w_{ij}. One can easily observe that the centre vertex v of W_{n+1} hop dominates the vertices $\{v_i : 1 \leq i \leq n\}$ and the vertex u_2 hop dominates the vertices $\{u_i : 1 \leq i \leq n\}$ and also the vertices w_{12} and w_{23}. Furthermore, the set $D \setminus \{w_{12}\}$ hop dominates the remaining $n-2$ vertices in $S(W_{n+1})$ where $D = \{w_{i(i+1)} : i \equiv 1 \ (mod\ 3)\}$. Therefore, $\{v\} \cup \{u_j\} \cup D$ is a γ_h-set of $S(W_{n+1})$. Thus $\gamma_h[S(W_{n+1})] = \left\lfloor \frac{n}{3} \right\rfloor + 2$. □

Theorem 7. *For any graph G, $\gamma(G) < \gamma_h[S(G)]$.*

Proof. Let D be a hop dominating set of $S(G)$. Let $D_1 = D \cap V(G)$ and $D_2 = D \setminus D_1$. Clearly D_1 is the only set hop dominating $V(G)$ since every vertex in D_2 is at odd distance from any vertex of $V(G)$. This shows that $D_1 \neq \emptyset$. Similarly $D_2 \neq \emptyset$. Further, if a vertex v is hop dominated by a vertex u in D_1, then $d(u, v) = 2$ in $S(G)$. This implies there is a path uwv in $S(G)$ and $uv \in E(G)$. Thus v is dominated by u and so $\gamma(G) \leq |D_1| < |D| = \gamma_h[S(G)]$. □

Theorem 8. *Let G be a (p, q)-graph. Let u be a vertex of maximum degree $\Delta(G)$ and v be a vertex in $N(u)$ such that $deg(v) = \max\limits_{y \in N(u)} \{deg(y)\}$. Then $\gamma_h[S(G)] \leq \gamma(G) + q - \Delta(G) - deg(v) + 2$.*

Proof. Let w_0 be the new vertex subdividing the edge uv and S' be a γ-set of G. Let E' be the set of all edges incident with u or v. Let $D_1 = \{w : w \in V[S(G)] \setminus V(G)$ is a vertex subdividing the edge $vv' \in E(G) \setminus E'\}$. Then $D = S' \cup \{w_0\} \cup D_1$ is a hop dominating set of $S(G)$ and hence $\gamma_h[S(G)] \leq |D| = \gamma(G) + 1 + (q - \Delta(G) - (deg(v) - 1)) = \gamma(G) + 2 - deg(v) + q - \Delta(G)$. □

Theorem 9. *Let T be a tree with q edges. Let u be a vertex of maximum degree $\Delta(T)$ and let $v \in N(u)$ be a vertex in T such that $deg(v) = \max\limits_{y \in N(u)} \{deg(y)\}$. Then $\gamma_h[S(T)] = \gamma(T) + q + 2 - (\Delta(T) + deg(v))$ if and only if the following conditions hold:*

(i) *every vertex $w \in N(u) \cup N(v) \setminus \{u, v\}$ is either a leaf or a weak support.*

(ii) *both $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ cannot contain weak support vertices.*

Proof. Assume that $\gamma_h[S(T)] = \gamma(T) + q + 2 - (\Delta(T) + deg(v))$. Let w_0 be a vertex subdividing the edge uv in $S(T)$. Let E_1 denote the set of all edges neither incident at u nor at v and let D_1 be the set of vertices subdividing the edges in E_1. In what follows hereafter, we call $S^{'}$ a γ-set of T.

(i) Let $w \in N(u) \cup N(v) \setminus \{u, v\}$. Suppose $w \in N(u) \setminus \{v\}$ be a vertex of degree $r \geq 3$. Let w_1 be a vertex subdividing one of the edges incident at w except the edge uw, say ww'. Let E_w denote the set of edges incident at w except the edge uw and let D_w be the set of vertices subdividing the edges in E_w. Then w_1 hop dominates all vertices of D_w. Therefore $S' \cup \{w_0, w_1\} \cup (D_1 \setminus D_w)$ is a hd-set of $S(T)$ and so,

$$\begin{aligned}
\gamma_h[S(T)] \leq & \gamma(T) + q + 2 - (\Delta(T) + deg(v) - 1) - r + 1 \\
&= \gamma(T) + q + 2 - (\Delta(T) + deg(v)) + r + 2 \\
&= \gamma(T) + q + 4 - (\Delta(T) + deg(v)) + r \\
&\leq \gamma(T) + q + 4 - (\Delta(T) + deg(v)) - 3, \text{ since } r \ \geq 3 \\
&= \gamma(T) + q + 1 - (\Delta(T) + deg(v)) \\
&< \gamma(T) + q + 2 - (\Delta(T) + deg(v)), \ a \text{ contradiction.}
\end{aligned}$$

Hence $deg(w) \leq 2$ for every $w \in N(u) \setminus \{v\}$.

If $deg(w) = 1$, then nothing to prove. So, let $deg(w) = 2$.

Now we show that w is a weak support vertex in T.

Suppose $y \in N(w) \setminus \{u, v\}$ is vertex such that $deg(y) \geq 2$.

Let E_u be the set of edge incident with u except the edge uv and D_u be the set of vertices subdividing the edges in E_u. Let E_v be the set of edges which are incident at v except the edge uv and D_v be the set of vertices subdividing the edges in E_v.Let E_2 be the set of edge which are not incident at u. Let D_2 be the set of vertices subdividing the edges in E_2. Let E_y be the set of edges incident at y except the edge wy and D_y be the set of vertices subdividing the edges in E_y. Then the vertex w_2 which subdivides the edge wy in $S(T)$ will hop dominate all vertices of D_y and the vertex w_0 hop dominates all vertices in $D_u \cup D_v$. Therefore, $S' \cup \{w_0\} \cup (D_2 \setminus (D_v \cup D_y))$ is a hd-set of $S(T)$. Hence,

$$\begin{aligned}
\gamma_h[S(T)] \leq & \gamma(T) + q + 1 - (\Delta(T) + deg(v) - 1) - deg(y) + 1 \\
&= \gamma(T) + q + 3 - (\Delta(T) + deg(v)) - deg(y) \\
&\leq \gamma(T) + q + 1 - (\Delta(T) + deg(v)), \text{ since } deg(y) \ \geq 2 \\
&< \gamma(T) + q + 2 - (\Delta(T) + deg(v)), \ a \text{ contradiction.}
\end{aligned}$$

Thus $deg(y) = 1$ for all $y \in N(w) \setminus \{u, v\}$. That is, w is a weak support vertex of T.

Similarly, one can prove that every vertex $w \in N(v) \setminus \{u\}$ is either a leaf of a weak support of T.

(ii) Suppose both $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ have weak support vertices in T.

Let $N'(u) = \{w \in N(u) \setminus \{v\} : deg(w) = 2\}$ and $N'(v) = \{w \in N(v) \setminus \{u\} : deg(w) = 2\}$. Let $N''(u) = \{w' : w'$ is the vertex subdividing the edge uw where $w \in N'(u)\}$ and $N''(v) = \{w' : w'$ is the vertex subdividing the edge vw where $v \in N'(v)\}$.

By our assumption $N'(u) \neq \emptyset$ and $N'(v) \neq \emptyset$. Clearly, $|N''(u) \cup N''(v)| = q - \Delta(T) - (deg(v) - 1)$ and so $S' \cup N''(u) \cup N''(v)$ is a hd-set of $S(T)$. Therefore, $\gamma_h[S(T)] \leq \gamma(T) + q - (\Delta(T) + deg(v) - 1) < \gamma(T) + q + 2 - (\Delta(T) + deg(v))$, a contradiction.

The converse is obvious. □

Theorem 10. *Let G be a connected (p, q)-graph having at least one cycle and let u and v be vertices as in Theorem 8. Then $\gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + deg(v))$ if and only if the following conditions hold:*

(i) *Every cycle C in G contains u or v or the edge uv and the length of C is at most 5.*

(ii) *If the longest cycle containing the edge uv in G is C_3, then*

- **(a)** *every vertex $w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$ is a leaf or weak support of degree 2 or a vertex of degree 2 in another cycle C_3 of G.*
- **(b)** *both $N(u) \setminus \{v\}$ and $N(v) \setminus \{u\}$ cannot contain weak support vertices in G.*

(iii) *If the longest cycle containing the edge uv in G is $C = C_4$, then every vertex $w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$ is a leaf or a vertex of degree 2 in C.*

(iv) *If the longest cycle in G is $C = C_5$, then*

- **(a)** *the edge uv is a chord of C*
- **(b)** *every vertex $w \in N(u) \cup N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$ is a leaf or a vertex of degree 2 in C.*

(v) *Every vertex $w \in N(u) \cap N(v)$ is of degree at most 3 and if $w \in N(u) \cap N(v)$ is of degree 3 in G, then there exists at most one edge not adjacent to uv in G.*

Proof. Assume that $\gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + deg(v))$.

Let E_1, D_1 and w_0 be as in Theorem 9.

Throughout this proof, $S^{'}$ denotes a γ-set of G.

(i) Suppose there exists a cycle C in G not containing u and v.

Let $V(C) = \{v_1, v_2, \cdots, v_k\}$. Let w_i, w_{i-1} and w_{i+1} be the vertices in D_1 subdividing the edges $v_{i-1}v_i$, $v_{i-1}v_{i-2}$ and v_iv_{i+1} in C, respectively. Then clearly the vertex w_i hop dominates the vertices w_{i-1} and w_{i+1} in $S(G)$. Therefore $S' \cup \{w_0\} \cup (D_1 \setminus \{w_{i-1}, w_{i+1}\})$ is a hd-set of $S(G)$. Hence

$$\begin{aligned}\gamma_h[S(G)] \leq & \gamma(G) + 1 + q - 2 - (\Delta(G) + deg(v) - 1)\\ &= \gamma(G) + q - (\Delta(G) + deg(v))\\ &< \gamma(G) + 2 + q - (\Delta(G) + deg(v)), \text{ } a \text{ contradiction.}\end{aligned}$$

Claim: Every cycle C in G is of length at most 5.

Suppose there exists a cycle C containing the edge uv in G such that the length k of $C \geq 6$. Let $V(C) = \{u = v_1, v = v_2, v_3, \cdots, v_k\}$. Then C contains at least three edges not incident at u or v. Let $v_{i-1}v_i, v_{i-2}v_{i-1}$ and v_iv_{i+1} be three edges in C not incident at u or v and let w_i, w_{i-1} and w_{i+1} be the vertices in $S(G)$ subdividing these edges respectively. Then w_i hop dominates w_{i-1} and w_{i+1}. Therefore $S' \cup \{w_0\} \cup (D_1 \setminus \{w_{i-1}, w_{i+1}\})$ is a hd-set of $S(G)$ so that

$$\begin{aligned}\gamma_h[S(G)] \leq & \gamma(G) + 1 + q - 2 - (\Delta(G) + deg(v) - 1)\\ &= \gamma(G) + q - (\Delta(G) + deg(v))\\ &< \gamma(G) + 2 + q - (\Delta(G) + deg(v)), \text{ } a \text{ contradiction.}\end{aligned}$$

Applying a similar argument given in Theorem 9 one can easily prove the conditions $(ii - a)$ and $(ii - b)$.

(iii) Let $C = \langle u, v, x, y \rangle$ be a longest cycle of length 4 containing the edge uv in G.

Claim: Every vertex $w \in N(u) \cup N(v) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$ is a leaf or a vertex of degree 2 in C. Let $w \in N(u) \cup N(v) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$.

Then either $w \in N(u) \setminus (N(u) \cap N(v) \cup \{u, v\})$ or $w \in N(v) \setminus N(u) \cap N(v) \cup \{u, v\}$.

Case 1: Let $w \in N(u) \setminus (N(u) \cap N(v) \cup \{u, v\})$. Then as discussed in Theorem 9, $deg(w) \leq 2$. If $deg(w) = 1$, then clearly w is a leaf. So assume that $deg(w) \neq 1$.

We claim that w is neither a weak support vertex of degree 2 in G nor a vertex of degree 2 in any other cycle of length 3 or 4 or 5.

Suppose w is a weak support vertex of degree 2 in G. Let z be the leaf adjacent to w in G. Let v_{uw} and v_{wz} be the vertices subdividing the edges uw and wz in $S(G)$. Then the vertex v_{uw} hop dominates all the vertices in D_u and the vertex v_{wz} in $S(G)$. Let w_1 be the vertex subdividing the edge vy in $S(G)$. Then w_1 hop dominates all the vertices in D_v and the vertex v_{xy} that subdivides the edge xy in $S(G)$.

Therefore $S^{'} \cup \{w_1, v_{wz}\} \cup D_2 \setminus (D_v \cup \{v_{xy}, v_{wz}\}$ is clearly a hd-set of $S(G)$. Hence

$$\begin{aligned}\gamma_h[S(G)] \leq& \gamma(G) + 2 + q - \Delta(G) - (deg(v) - 1) - 2\\ &= \gamma(G) + q - \Delta(G) - deg(v) + 1\\ &< \gamma(G) + 2 + q - (\Delta(G) + deg(v)), a \text{ contradiction.}\end{aligned}$$

Thus the vertex w cannot be a weak support of degree 2 in G. The other cases follow similarly.

Similarly, Case 2 can be argued for $w \in N(v) \setminus (N(u) \cap N(v) \cup \{u, v\})$.

Next we prove condition (iv). Let $C = C_5$ be a longest cycle of length 5 in G.

Claim: The edge uv is a chord of C in G.

Suppose the edge uv is not a chord of C.

Case 1: $u \in V(C)$ and $v \notin V(C)$.

Let $V(C) = \{u, w, x, y, z\}$. Then clearly the edges wx, xy and yz are in E_1. Let w_1, w_2 and w_3 be the vertices subdividing the edges wx, xy and yz respectively. Then $S^{'} \cup \{w_0\} \cup D_1 \setminus \{w_1, w_3\}$ is a hd-set of $S(G)$. Hence

$$\begin{aligned}\gamma_h[S(G)] \leq& \gamma(G) + 1 + q - 2 - (\Delta(G) + deg(v) - 1)\\ =& \gamma(G) + q - 1 - \Delta(G) - deg(v) + 1\\ =& \gamma(G) + q - \Delta(G) - deg(v)\\ <& \gamma(G) + 2 + q - (\Delta(G) + deg(v)),\ a \text{ contradiction.}\end{aligned}$$

Similarly we can prove that uv is not an edge in C_5.

One can prove the condition (b) of (iv) with similar arguments given in the proof of (iii).

(v) Suppose there exist two edges xw_1 and yw_2 in G such that $w_1, w_2 \in N(u) \cap N(v)$ and $deg(w_1) = deg(w_2) = 3$. Let E_u and E_v be the set of edges incident at u and v respectively except the edge uv. Let D_u be the set of vertices subdividing the edges in E_u and D_v be the set of vertices subdividing the edges in E_v. Let $w_1^{'}$ and $w_2^{'}$ be the vertices subdividing the edges uw_1 and vw_2 respectively. Let $x^{'}$ and $y^{'}$ be the vertices subdividing the edges xw_1 and

yw_2 respectively. Then $w_1^{'}$ hop dominates all vertices in D_u and the vertex $x^{'}$. Similarly, $w_2^{'}$ hop dominates all vertices in D_v and the vertex $y^{'}$ in $S(G)$. Therefore $S^{'} \cup \{w_1^{'}, w_2^{'}\} \cup (D_1 \setminus \{x^{'}, y^{'}\})$ is a hd-set of $S(G)$. Hence,

$$\begin{aligned}\gamma_h[S(G)] \leq & \gamma(G) + 2 + q - 2 - (\Delta(G) + deg(v) - 1) \\ = & \gamma(G) + q - \Delta(G) - deg(v) + 1 \\ < & \gamma(G) + 2 + q - (\Delta(G) + deg(v)), \ a \text{ contradiction.}\end{aligned}$$

Conversely, assume that the conditions (i) to (v) hold good.

Let w_0 be a vertex subdividing the edge uv. Then w_0 hop dominates all vertices subdividing the edges incident at u and v. As $deg(u) = \Delta(G)$ and by the choice of v, every hd-set of $S(G)$ contains w_0. Furthermore, as any two vertices of $V(G)$ in $S(G)$ are of even distance, every vertex $v \in V(G)$ can be hop dominated only by a vertex of $V(G)$ in $S(G)$. Therefore every γ_h-set of $S(G)$ contains $\gamma(G)$ vertices of $V(G)$. Let $e = wx \in E_1$. If $w \in N(u)$ and $x \notin N(v)$, then by condition (ii-a), w is a weak support vertex of degree 2 in G.

If $w \in N(u)$ and $x \in N(v)$, then by condition (iii), wx is an edge in C_4 that contains the edge uv. Thus in both cases either the vertex subdividing the edge uw or the vertex subdividing the edge wx is in every γ_h-set of $S(G)$.

If $w \in N(u) \cap N(v)$, then by condition (v) wx is the only edge not adjacent to uv in G. Therefore, one of the vertices subdividing the edges uv, vw and wx is in any γ_h-set of $S(G)$. If $w \notin N(u)$, then by condition (ii-b) the vertex w is a weak support vertex of degree 2 in $N(v) \setminus \{u\}$. Then the vertex subdividing the edge wx or vw is in every γ_h-set of $S(G)$.

Thus in all cases we see that for every edge in E_1 there corresponds a subdividing vertex in every γ_h-set of $S(G)$. Therefore every γ_h-set of $S(G)$ contains at least $\gamma(G) + 1 + |E_1|$ vertices. This implies

$$\begin{aligned}\gamma_h[S(G)] \geq & \gamma(G) + 1 + |E_1| \\ = & \gamma(G) + 1 + q - (\Delta(G) + deg(v) - 1) \\ = & \gamma(G) + 2 + q - (\Delta(G) + deg(v) - 1).\end{aligned}$$

But by Theorem 8, $\gamma_h[S(G)] \leq \gamma(G) + 2 + q - (\Delta(G) + deg(v))$.

Thus $\gamma_h[S(G)] = \gamma(G) + 2 + q - (\Delta(G) + deg(v))$. □

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