OSCILLATORY INTEGRAL OPERATORS AND THEIR COMMUTATORS IN MODIFIED WEIGHTED MORREY SPACES WITH VARIABLE EXPONENT

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Abstract: In this paper first we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators and their commutators in the modified weighted Morrey spaces with variable exponent $\widetilde{\mathcal{L}}_{\omega}^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ are unbounded sets. After that we prove the boundedness of these operators on the spaces $\widetilde{\mathcal{L}}_{\omega}^{p(\cdot),\lambda}(\Omega)$.

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1. Introduction

The variable exponent analysis is a popular topic which attract many researchers. This topic is mainly focused on the Lebesgue and Sobelev spaces with variable order of integrability and operator theory in these spaces. The study of these spaces has been stimulated by problems of various fields, influenced by many applications, for instance, mechanics of the continuum medium, elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions, see [6], [23]. In particular, various results on non-weighted and weighted boundedness in Lebesgue spaces with variable

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exponents $p(x)$ have been proved for maximal, singular and fractional type operators, we refer to surveying papers [8] and [25].

Morrey spaces in its classical version were introduced in [21] in relation to the study of partial differential equations and the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10], [21]. These spaces were widely investigated during the last decades, including the study of classical operators of harmonic analysis, maximal, singular, and potential operators and their generalizations were studied.

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] in the Euclidean setting in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. P. Hästö in [14] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [16].

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [22], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [5].

A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x) = \int_\Omega K(x,y)f(y)dy, \tag{1}$$

$$Sf(x) = \int_\Omega e^{P(x,y)}K(x,y)f(y)dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega \times \Omega$. Lu and Zhang [20] used L^2-boundedness of T to get L^p- boundedness of S with $1 < p < \infty$.

The commutator generated by the operator S for a given measurable function b is formally defined by

$$[S,b]f = S(bf) - bS(f).$$

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int_{\{y\in\Omega:|x-y|\geq\varepsilon\}} K(x,y)f(y)dy.$$

In this paper first we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators and their commutators in the modified Morrey spaces with variable exponent $\widetilde{\mathcal{L}}^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ are unbounded sets. After that we prove the boundedness of these operators on the spaces $\widetilde{\mathcal{L}}^{p(\cdot),\lambda}(\Omega)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c, C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent weighted Lebesgue and Morrey spaces

Let Ω be an unbounded set in $\mathbb{R}^n$ and $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{3}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent. In the one-dimensional case $n = 1$ we deal with the interval $(0,1]$ and the standart Lebesgue measure. (see e.g. [9]). We refer to [3] for definition and fundamental properties of Banach function spaces.

Let $\mathcal{P}(\Omega)$ be the set of bounded measurable functions $p : \Omega \to [1, \infty)$; $\mathcal{P}^{log}(\Omega)$ be the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x - y| \leq \frac{1}{2} \quad x, y \in \Omega, \tag{4}$$

where $A = A(p) > 0$ does not depend on x, y.

By $\mathcal{A}^{log}(\Omega)$ we denote the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (4); by $\mathbb{P}^{log}(\Omega)$ we denote the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$; for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}_\infty^{log}(\Omega)$, $\mathbb{P}_\infty^{log}(\Omega)$, $\mathcal{A}_\infty^{log}(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n. \tag{5}$$

where $p_\infty = \lim_{x \to \infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [6], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\theta_p(x,r)}, \quad x \in \Omega, \ p \in \mathbb{P}_\infty^{log}(\Omega), \tag{6}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, \ r \leq 1, \\ \frac{n}{p(\infty)}, \ r \geq 1 \end{cases}$.

A locally integrable function $\omega : \Omega \to (0, \infty)$ is called a weight. We say that $\omega \in A_p(\Omega)$, $1 < p < \infty$, if there is a constant $C > 0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega(x)dx \right) \left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega^{1-p'}(x)dx \right)^{p-1} \leq C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1(\Omega)$ if there is a constant $C > 0$ such that $M\omega(x) \leq C\omega(x)$ almost everywhere.

By ω we always denote a weight, i.e. a positive, locally integrable function with domain Ω. The weighted Lebesgue space $L_\omega^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L_\omega^{p(\cdot)}(\Omega)} = \|f\omega\|_{L^{p(\cdot)}(\Omega)}.$$

Let us define the class $A_{p(\cdot)}(\Omega)$ to consist of those weights ω for which

$$\sup_B |B|^{-1}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))} < \infty.$$

The extrapolation theorem (Lemma 1 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [4]. Here we use the form in [6], see Theorem 7.2.1 and Theorem 7.2.3 in [6].

Lemma 1. [6]) *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $1 < p_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_{p_0}$,*

$$\int_\Omega |f(x)|^{p_0}\omega(x)dx \leq C_0 \int_\Omega |g(x)|^{p_0}\omega(x)dx.$$

Then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C\|g\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [7]. From Theorem 4.8 and Remark 4.6 of [7] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1.   ([7, Theorem 4.8]) *Let $\Omega \subset \mathbb{R}^n$ be a unbounded open set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$, $[t]_1 = \min\{1,t\}$. The variable modified Morrey space $\widetilde{\mathcal{L}}^{p(\cdot),\lambda(\cdot)}(\Omega)$ and variable modified weighted Morrey space $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\widetilde{\mathcal{L}}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1}\|f\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)},$$

respectively.

Definition 2. We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x \in \Omega, \, r > 0} |B(x, r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy,$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x \in \Omega, \, r > 0} |B(x, r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - C| dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x, t)|^{-1} \int_{\widetilde{B}(x,t)} f(y) dy$.

Definition 3. We define the $BMO_{p(\cdot),\omega}(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p(\cdot),\omega}} = \sup_{x \in \Omega, \, r > 0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)}) \chi_{\widetilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}.$$

Theorem 4. ([18, Theorem 4.4]) *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}_\infty^{log}(\Omega)$ *and* ω *be a Lebesgue measurable function. If* $\omega \in A_{p(\cdot)}(\Omega)$, *then the norms* $\|\cdot\|_{BMO_{p(\cdot),\omega}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 2. ([15]) *Let* $b \in BMO(\Omega)$. *Then there is a constant* $C > 0$ *such that*

$$\left| b_{\widetilde{B}(x,r)} - b_{\widetilde{B}(x,t)} \right| \leq C \|b\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C *is independent of* b, x, r, *and* t.

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [11].

Theorem 5. *Let* v_1, v_2 *and* w *be weights on* $(0, \infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t > 0} v_2(t) H_w^* g(t) \leq C \sup_{t > 0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. Oscillatory integral operators in $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$

In this section we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators S and the commutators generated by S, $[b, S]$, in the modified weighted Morrey spaces with variable exponent $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ are unbounded sets. After that we prove the boundedness of the operators S and $[b, S]$ on the spaces $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$ by the help of these Calderón-Zygmund-type integral inequalities.

Lemma 3.   ([19]) *Let K be a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$. Then for any real polynomial $P(x, y)$ and $\omega \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that*

$$\|Sf\|_{L_\omega^p(\Omega)} \leq C\|f\|_{L_\omega^p(\Omega)}.$$

Theorem 1.   *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_\infty^{log}(\Omega)$. Then the operator S is bounded on the space $L_\omega^{p(\cdot)}(\Omega)$.*

Proof. From Lemmas 1 and 3, we obtain that the operator S is bounded on the space $L_\omega^{p(\cdot)}(\Omega)$. $\qquad\qquad\square$

The following theorem gives the Calderón-Zygmund-type integral inequality for oscillatory integral operators S.

Theorem 2.   *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $f \in L_\omega^{p(\cdot)}(\Omega)$. Then the following inequality*

$$\|Sf\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}, \quad (1)$$

holds, where C does not depend on f, $x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \ f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \ f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \ t > 0, \quad (2)$$

and have

$$\|Sf\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq \|Sf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorem 1 we obtain

$$\|Sf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq \|Sf_1\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C\|f_1\|_{L_\omega^{p(\cdot)}(\Omega)},$$

so

$$\|Sf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}$$

we get

$$\|Sf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}. \quad (3)$$

To estimate $\|Sf_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \leq C \int_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x - z| \leq t$, $|z - y| \geq 2t$ imply $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$. Therefore we get

$$|Sf_2(z)| \leq C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x - y|^{-n}|f(y)|dy,$$

To estimate Sf_2, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x - y|^{-n}|f(y)|dy$$
$$\leq C t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \quad (4)$$

With this aim we choose $\delta > 0$ and proceed as follows

$$
\int_{\Omega \setminus \widetilde{B}(x,t)} |x - y|^{-n} |f(y)| dy
$$

$$
\leq \delta \int_{\Omega \setminus \widetilde{B}(x,t)} |x - y|^{-n+\delta} |f(y)| dy \int_{|x-y|}^{\infty} s^{-\delta-1} ds
$$

$$
\leq C \int_t^{\infty} s^{-n} \frac{ds}{s} \int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |f(y)| dy
$$

$$
\leq C \int_t^{\infty} s^{-n} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\chi_{\widetilde{B}(x,s)}\|_{L_\omega^{p'(\cdot)}(\Omega)} \frac{ds}{s}
$$

$$
\leq C \int_t^{\infty} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{5}
$$

Hence by inequality (5), we get

$$
\|Sf_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C \|\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} \int_t^{\infty} s^{-\theta_p(x,s)} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}
$$

$$
= C \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^{\infty} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{6}
$$

From (3) and (6) we arrive at (1). $\qquad\square$

In the following theorem we prove the boundedness of the operators S on the spaces $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$.

Theorem 3. Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $0 \leq \lambda(x) < n$.

Then the singular integral operator S is bounded from the space $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$ to the space $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$.

Proof. Let $f \in \widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$. Then the following equality

$$
\|Sf\|_{\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)} = \sup_{x \in \Omega, \, t > 0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|Sf\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} \tag{7}
$$

holds. We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)}$ in (7) by means of Theorems 2, 5 and obtain

$$
\|Sf\|_{\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)} \leq C \sup_{x \in \Omega, \, t > 0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}
$$

$$\times \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}$$

$$\leq C \sup_{x \in \Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{L}_\omega^{p(\cdot),\lambda}(\Omega)}.$$

$\square$

The following theorem gives the Calderón-Zygmund-type integral inequality for the commutators $[b, S]$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $b \in BMO(\Omega)$. Then the following inequality*

$$\|[b,S]f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$$

$$\times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s} \qquad (8)$$

holds for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (2) and have

$$\|[b,S]f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq \|[b,S]f_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} + \|[b,S]f_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}.$$

Since $[b, S]$ is bounded on the space $L_\omega^{p(\cdot)}(\Omega)$, we obtain

$$\|[b,S]f_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq \|[b,S]f_1\|_{L_\omega^{p(\cdot)}(\Omega)}$$

$$\leq C\|b\|_* \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} = C\|b\|_* \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,2t))}, \qquad (9)$$

where C does not depend on f. From (9) we easily obtain

$$\|[b,S]f_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$$

$$\times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s} \qquad (10)$$

from the fact that $\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,2t))}$ on the right-hand side of (9) is dominated by the right-hand side of (10). To estimate $\|[b,S]f_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|[b,S]f_2(z)| \leq C \int_{\Omega \setminus B(x,2t)} |b(z) - b(y)| \frac{|f(y)|\, dy}{|y - z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x - z| \leq t$, $|z - y| \geq 2t$ imply $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$. Therefore

$$|[b, S]f_2(z)| \leq C \int_{\Omega \backslash \widetilde{B}(x,2t)} |x - y|^{-n}|b(z) - b(y)| \, |f(y)| dy.$$

To estimate $[b, S]f_2$, first we need to prove the following auxiliary inequality

$$\int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n}|b(z) - b(y)||f(y)|dy$$

$$\leq C\|b\|_* \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{11}$$

To estimate $[b, S]f_2(z)$, we observe that for $z \in \widetilde{B}(x,t)$ we have

$$\int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n}|b(z) - b(y)||f(y)|dy$$

$$\leq \int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n}|b(y) - b_{\widetilde{B}(x,t)}||f(y)|dy$$

$$+ \int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n}|b(z) - b_{\widetilde{B}(x,t)}||f(y)|dy = I_1 + I_2.$$

To this end, we choose $\delta > 0$, by Theorem 4 and Lemma 2 we obtain

$$I_1 = \int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n}|b(y) - b_{\widetilde{B}(x,t)}||f(y)|dy$$

$$\leq \delta \int_{\Omega \backslash \widetilde{B}(x,t)} |x - y|^{-n+\delta}|b(y) - b_{\widetilde{B}(x,t)}||f(y)|dy \int_{|x-y|}^\infty s^{-\delta-1}ds$$

$$\leq C \int_t^\infty s^{-n-1} \int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |b(y) - b_{\widetilde{B}(x,t)}||f(y)|dyds$$

$$\leq C \int_t^\infty s^{-n-1}\|b(\cdot) - b_{\widetilde{B}(x,s)}\|_{L_{\omega^{-1}}^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$+ C \int_t^\infty s^{-n-1}|b_{\widetilde{B}(x,t)} - b_{\widetilde{B}(x,s)}| \int_{\widetilde{B}(x,s)} |f(y)|dyds$$

$$\leq C\|b\|_* \int_t^\infty s^{-n-1}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$+ C\|b\|_* \int_t^\infty s^{-n-1}\ln \frac{s}{t}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$\leq C\|b\|_* \int_t^\infty \left(1 + \ln\frac{s}{t}\right) \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}.$$

To estimate I_2, by (4), we have

$$I_2 = |b(z) - b_{\widetilde{B}(x,t)}| \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n} |f(y)| dy$$

$$\leq C|B(x,t)|^{-1} \int_{\widetilde{B}(x,t)} |b(z) - b(y)| dy \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}$$

$$\leq C M_b \chi_{B(x,t)}(z) \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s},$$

where C does not depend on x, t.

Hence by inequality (11), we get

$$\|[b,S]f_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \lesssim \|\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} \|b\|_*$$

$$\times \int_t^\infty \left(1 + \ln\frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}$$

$$= \|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty \left(1 + \ln\frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}. \quad (12)$$

From (10) and (12) we arrive at (8).

$\square$

In the following theorem we prove the boundedness of the operators $[b, S]$ on the spaces $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and $0 \leq \lambda(x) < n$. Then the singular integral operator $[b, S]$ is bounded from the space $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$ to the space $\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$.*

Proof. Let $f \in \widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)$. We have

$$\|[b,S]f\|_{\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|[b,S]f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorems 5 and 4 we obtain

$$\|[b,S]f\|_{\widetilde{\mathcal{L}}_\omega^{p(\cdot),\lambda}(\Omega)}$$

$$\leq C\|b\|_* \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \times$$

$$\int_t^\infty \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}^{-1} \frac{ds}{s}$$

$$\leq C\|b\|_* \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} = C\|b\|_* \|f\|_{\widetilde{\mathcal{L}}^{p(\cdot),\lambda}_\omega(\Omega)}$$

which completes the proof. $\qquad\square$

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