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LOCALIZED TRANSFUNCTIONS

Jason Bentley[1], Piotr Mikusiński[2 §]

[1,2] Department of Mathematics
University of Central Florida
Orlando – 32816, USA

Abstract: Generalized functions, called transfunctions, are defined as maps between spaces of measures on measurable spaces (X, Σ_X) and (Y, Σ_Y). Measurable functions $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ can be identified with transfunctions via the push forward operator $f_{\#}(\mu)(B) = \mu(f^{-1}(B))$. In this paper we introduce the notion of localization of transfunctions that gives an insight into which transfunctions arise from continuous functions or measurable functions or are close to such functions. We also introduce the notion of a graph of a transfunction and describe what it tells us about the transfunction. In our investigation of transfunctions, we are motivated by applications that include Monge-Kantorovich transportation problems and population dynamics.

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1. Introduction

Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces with sets of finite measures $\mathcal{M}_X$ and $\mathcal{M}_Y$, respectively. A *transfunction* is any function $\Phi : \mathcal{M}_X \to \mathcal{M}_Y$, [8].

One can think of transfunctions as maps where the inputs and outputs are "probability clouds" rather than points. While this intuitive interpretation is useful, we are not restricting the domain and range of a transfunction to

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§Correspondence author

probability measures. In fact there are situations where it is natural to consider transfunctions on signed measures or vector measures.

Another way to think of transfunctions is as follows. A family $\{\Phi_t : t \geq 0\}$ of transfunctions from $\mathcal{M}_X$ to $\mathcal{M}_X$ can be viewed instead as a family $\{\Phi_\bullet(\mu) : \mu \in \mathcal{M}_X\}$ of measure-valued functions on $[0, \infty)$. Then the measure-valued function $\Phi_\bullet(\mu)$ can describe the evolution of μ as time grows, while the transfunction Φ_t can represent an overall rule on how measures will change from time 0 to time t. When $\{\Phi_t : t \geq 0\}$ form a (C_0) semigroup, many results follow as regular finite measures form Banach spaces, [7].

While formally a transfunction is a map $\Phi : \mathcal{M}_X \to \mathcal{M}_Y$ we are interested in its properties as a "generalized function" from X to Y. To emphasize this point of view we will use the notation $\Phi : X \rightsquigarrow Y$ when the context is clear.

Every measurable function is a transfunction. More precisely, if (X, Σ_X) and (Y, Σ_Y) are measurable spaces and $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ is a measurable function, then the push forward operator $f_\# : \mathcal{M}_X \to \mathcal{M}_Y$ defined by $f_\#(\mu)(B) = \mu(f^{-1}(B))$ is a transfunction. We will say that the transfunction Φ corresponds to f, or simply Φ is f, if $\Phi = f_\#$.

While every measurable function is a transfunction, we are obviously interested in transfunctions that do not necessarily correspond to measurable functions. In this paper we investigate the following general questions:

- Under what conditions will a tranfunction Φ be a measurable function?
- Under what conditions will a tranfunction Φ be a continuous function?
- If a tranfunction Φ is not a function, under what conditions is Φ "close" to a measurable or continuous function?

We also introduce the notion of a graph of a transfunction, which is related to the above questions and gives us additional intuition about the nature of transfunctions. The main tool in our investigation is the idea of localization of transfunctions which is introduced in Section 3.

Our long term goal is to investigate to what extent the tools developed for functions can be extended to transfunctions and to use transfunctions to describe and solve problems arising from applications. In our study of theoretical properties of transfunctions we are motivated by applications to specific problems, including the two areas described below.

Transfunctions provide a natural framework for population dynamics models. A population can be described as a measure μ on $X \subseteq \mathbb{R}^2$ which contains information about the size of the population and its spatial distribution. A transfunction captures the dynamics of the population over one unit of time. For example, the transfunction

$$\Phi(\mu) = g \cdot ((f_{\#}\mu) * \kappa) := \int_{\square} g \; d((f_{\#}\mu) * \kappa),$$

models how a population μ will migrate via a function $f : X \to X$ to become $f_{\#}\mu$, disperse by convolution with measure κ from territorial behavior/offspring to become $(f_{\#}\mu) * \kappa$, and grow/shrink locally via environmental factors $g : X \to [0, \infty)$ (which accounts for food, water, shelter, predators, etc) to become $g \cdot ((f_{\#}\mu) * \kappa)$ after some set amount of time.

The discrete logistic growth model with location-dependent growth rate can be described by the transfunction

$$\Phi(\mu) = \mu + r\left(1 - \frac{d\mu}{d\nu}\right) \cdot \mu,$$

where μ is a given population, ν is a measure describing the (non-uniform) carrying capacity, $r > 0$ is the population growth rate, and $\frac{d\mu}{d\nu}$ is the Radon-Nikodym derivative of μ with respect to ν. To account for dispersive behavior, one may instead substitute $\mu * \gamma$ into the formula of Φ, where γ is a measure of dispersal. See [4] for comparison/contrast.

A tree population can be described by an l^1-valued vector measure

$$\mu(S) = (\mu_0(S), \mu_1(S), \mu_2(S), \dots),$$

where $\mu_j(S)$ represents the number of trees in S that are j years old. Note that for every $j \in \mathbb{N}_0$, μ_j is a positive measure. A basic discrete model for the tree population dynamics can be described as a transfunction

$$\Phi(\mu) = AV\mu + e_0 \sum_{j=0}^{\infty} p_j \mu_j * \gamma_j,$$

where A is the right shift operator (aging operator) on l^1, $V \in l^\infty$ is the survivalship of trees at various ages (applied component-wise), p_j represents the fecundity of j-year-old trees, γ_j represents the distribution of seeds from one tree of age j at the origin, and $e_0 = (1, 0, 0, \dots)$. Altogether, Φ describes how a tree population changes over the course of one year.

Now we briefly describe an application of transfunctions to problems related to the Monge-Kantorovich transportation problem, [1], [9].

Let $(X, \Sigma_X) = (Y, \Sigma_Y) = (\mathbb{R}^d, \mathcal{R}^d)$ be the usual measurable space with the Lebesgue measure λ. Let $\mathcal{M}_\lambda$ denote the space of finite measures on $\mathcal{R}^d$ that are absolutely continuous with respect to λ. Let $\rho^X, \rho^Y \in \mathcal{M}_\lambda$ be prior and posterior probability measures. Let $c : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be a cost function.

We consider the collection $\mathcal{P}$ of all transport plans – that is, all measures on the product σ-algebra of Σ_X and Σ_Y – that have ρ^X and ρ^Y as their marginals. The goal is to find a transport plan with minimum cost

$$\inf_{\mu \in \mathcal{P}} \left\{ \int_{X \times Y} c \, d\mu \right\}.$$

Since a transport plan "maps" a prior measure ρ^X to a posterior measure ρ^Y, it can be described in the framework of transfunctions. There are a few main advantages when using transfunctions. First, all transport plans with the same "instructions" but with different prior and posterior measures correspond to the same transfunction, see [3]. Second, while transport plans are by definition measure preserving, it may not be a reasonable assumption in some applications. Finally, while a transport plan optimizes how ρ^X is transformed into ρ^Y, it may be more natural to optimize how ρ^X is transformed into one of several acceptable measures. However, describing how a transfunction (not necessarily corresponding to a transport plan) will be optimal with respect to cost function c and prior/posterior measures ρ^X, ρ^Y is not as simple as with transport plans.

Now we discuss how transfunctions compare and contrast with fuzzy functions. While the intuition behind transfunctions is similar to that of fuzzy functions, the mathematical formalisms of these two approaches are very different.

For an arbitrary set X, by a fuzzy subset of X we mean any function $m : X \to [0, 1]$ (see, for example, [5]). Such a function m describes the degree of membership of an $x \in X$, with $m(x) = 0$ meaning that x is not in the set, $0 < m(x) < 1$ meaning that x is partially in the set, and $m(x) = 1$ meaning that x is fully in the set. Let X and Y be two sets and let $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ denote all fuzzy subsets of X and Y, respectively. A fuzzy function F from X to Y is simply a function mapping $\mathcal{F}(X)$ to $\mathcal{F}(Y)$.

Now assume that X and Y are measurable spaces. Since, in general, measures on X do not assign values to points in X, not every measure can be identified in a natural way with a fuzzy set. Consequently, transfunctions operate on objects that are not fuzzy sets, which means that there are transfunctions that cannot be identified with fuzzy functions. On the other hand, fuzzy functions operate on nonmeasurable membership functions and therefore they cannot be identified with transfunctions.

Finally, we describe a special situation when fuzzy functions can be identified with transfunctions. Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be topological measure spaces such that μ and ν are strictly positive measures on X and Y, respectively. Let F be a fuzzy function that maps measurable fuzzy sets $\mathcal{F}_\Sigma(X)$ to measurable fuzzy sets $\mathcal{F}_\Sigma(Y)$ such that μ-equivalent functions in $\mathcal{F}_\Sigma(X)$ are mapped via F to ν-equivalent functions in $\mathcal{F}_\Sigma(Y)$. Let $\mathcal{M}_\mu = \{m\mu : m \in \mathcal{F}_\Sigma(X)\}$, $\mathcal{M}_\nu = \{m\nu : m \in \mathcal{F}_\Sigma(Y)\}$, and let $\iota_\mu : \mathcal{F}_\Sigma(X) \to \mathcal{M}_\mu$ and $\iota_\nu : \mathcal{F}_\Sigma(X) \to \mathcal{M}_\nu$ be defined by $\iota_\mu(m) = m\mu$ and $\iota_\nu(m) = m\nu$, respectively. Then F is identified by the unique transfunction $\Phi : \mathcal{M}_\mu \to \mathcal{M}_\nu$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}_\Sigma(X) & \xrightarrow{F} & \mathcal{F}_\Sigma(Y) \\ \Big\downarrow{\scriptstyle \iota_\mu} & & \Big\downarrow{\scriptstyle \iota_\nu} \\ \mathcal{M}_\mu & \xrightarrow[\Phi]{} & \mathcal{M}_\nu \end{array}$$

commutes. We can also determine F from Φ by a similar commuting diagram.

2. Preliminaries

Unless otherwise specified, all instantiated measures shall be finite and positive. Occasionally, we may sum countably many measures together. When this occurs, the sum may be finite or infinite and we will not determine the finiteness of the measure whenever it is inconsequential to the argument at hand.

If μ is a positive or a vector measure on (X, Σ_X) and $A \in \Sigma_X$, then we say that A is a *carrier* of μ and write $\mu \sqsubset A$ if $|\mu|(A^c) = 0$, where $|\mu|$ denotes the variation measure of μ. If μ is a positive measure, then $\mu \sqsubset A$ is also equivalent to the simpler condition that $\mu(A^c) = 0$.

If $A \subseteq X$, $B \subseteq Y$, and $\Phi : \mathcal{M}_X \to \mathcal{M}_Y$ is a transfunction such that $\mu \sqsubset A$ implies $\Phi\mu \sqsubset B$ for every μ, we shall write $\Phi(A) \sqsubset B$.

Let μ be a measure on measurable space (X, Σ_X) and let $A \in \Sigma_X$. Then the *projection* of μ onto A, denoted as $\pi_A\mu$, is the measure defined as $\pi_A\mu(B) = \mu(B \cap A)$ for $B \in \Sigma_X$. If $\mathcal{M}_X$ is a space of measures on (X, Σ_X), then we say that $\mathcal{M}_X$ is *closed under projections* if $\mu \in \mathcal{M}_X$ implies that $\pi_A\mu \in \mathcal{M}_X$ for all $A \in \Sigma_X$.

If μ, ν are measures in $\mathcal{M}_X$, then they are called *orthogonal*, written as $\mu \perp \nu$, if there exists $A \in \Sigma_X$ such that $\mu \sqsubset A$ and $\nu \sqsubset A^c$. A countable sequence of measures $\{\mu_n\}_{n=1}^\infty$ is called *(pairwise) orthogonal* if $\mu_i \perp \mu_j$ for $i \neq j$.

If a sequence of measures $(\mu_i)_{i=1}^\infty$ satisfies $\sum_{i=1}^\infty ||\mu_i|| < \infty$, then we call the finite measure $\mu = \sum_{i=1}^\infty \mu_i$ the *bounded sum of* $(\mu_i)_{i=1}^\infty$. A bounded sum

$\mu = \sum_{i=1}^{\infty} \mu_i$ with $\{\mu_i\}_{i=1}^{\infty}$ being orthogonal will be called a *bounded orthogonal sum*.

Definition 1. Let (X, τ) be a topological space. A space of finite positive measures or of vector measures $\mathcal{M}_X$ on (X, Σ_X) is called *ample* if the following conditions hold:

(i) $\mathcal{M}_X$ is closed under projections;

(ii) $\mathcal{M}_X$ is closed under bounded orthogonal sums;

(iii) Every nonempty open set in X carries some nonzero measure in $\mathcal{M}_X$.

A measure λ on a topological measurable space (X, Σ_X) is called *strictly-positive* if $\lambda(U) > 0$ for every nonempty open set U in X. If λ is a finite strictly-positive measure on a topological measurable space (X, Σ_X) and $\mathcal{M}_\lambda = \{\pi_A \lambda : A \in \Sigma_X\}$, then $\mathcal{M}_\lambda$ is an ample space of finite measures. Certain spaces (e.g. 2nd-countable locally compact non-atomic Hausdorff spaces, compact groups) admit finite strictly-positive measures, hence they also admit ample spaces of measures, [2].

Ample spaces will be useful for transfunctions because we will decompose a measure into bounded orthogonal sums of projections and use local properties to determine the behavior of each projection. If the transfunctions are of a certain type, then summing the outputs will result in the output of the original measure.

In this paper we will assume that X and Y are second-countable topological spaces, that Σ_X and Σ_Y are collections of Borel subsets of X and Y, respectively, and that any transfunction $\Phi : \mathcal{M}_X \to \mathcal{M}_Y$ will be defined on an ample space $\mathcal{M}_X$ unless otherwise specified.

Definition 2. Let $\Phi : X \rightsquigarrow Y$ be a transfunction with $\mathcal{M}_X$ closed under bounded orthogonal sums.

(i) Φ is called *weakly monotone* if $\Phi\mu \leq \Phi(\mu + \nu)$ for each orthogonal pair of measures μ and ν.

(ii) Φ is called *weakly σ-additive* if $\Phi(\sum_{i=1}^{\infty} \mu_i) = \sum_{i=1}^{\infty} \Phi\mu_i$ for every bounded orthogonal sum $\sum_{i=1}^{\infty} \mu_i$ in $\mathcal{M}_X$.

(iii) Φ is called *strongly σ-additive* if $\Phi(\sum_{i=1}^{\infty} \mu_i) = \sum_{i=1}^{\infty} \Phi\mu_i$ for every bounded sum $\sum_{i=1}^{\infty} \mu_i$ in $\mathcal{M}_X$.

Notice that strong σ-additivity implies weak σ-additivity and weak σ-additivity implies weak monotonicity. The transfunction $f_{\#} : X \rightsquigarrow Y$ with $\mathcal{M}_X$ closed under bounded orthogonal sums is strongly σ-additive for any measurable function $f : X \to Y$.

The properties of weakly σ-additive transfunctions listed in the following proposition are often used in arguments.

Proposition 3. *Let $\Phi : X \rightsquigarrow Y$ be a weakly σ-additive transfunction. Let A, A', and the sequence $(A_i)_{i=1}^{\infty}$ be from Σ_X and let B, B', and the sequence $(B_j)_{j=1}^{\infty}$ be from Σ_Y.*

(i) *If $\Phi(A) \sqsubset B$, if $A' \subseteq A$ and if $B' \supseteq B$, then $\Phi(A') \sqsubset B'$;*

(ii) *If $\Phi(A_i) \sqsubset B_j$ for all $i, j \in \mathbb{N}$, then $\Phi(\cup_{i=1}^{\infty} A_i) \sqsubset \cap_{j=1}^{\infty} B_j$.*

(iii) *If $\Phi(A_i) \sqsubset B_i$ for all $i \in \mathbb{N}$, then $\Phi(\cap_{i=1}^{\infty} A_i) \sqsubset \cap_{j=1}^{\infty} B_j$ and $\Phi(\cup_{i=1}^{\infty} A_i) \sqsubset \cup_{j=1}^{\infty} B_j$.*

The following proposition will be useful in the characterization of transfunctions that correspond to continuous functions.

Proposition 4. *Let $\Phi : X \rightsquigarrow Y$ be a weakly σ-additive transfunction. Let U be open in X with open cover $\{S_i : i \in I\}$, and let B be measurable in Y. Then $\Phi(S_i) \sqsubset B$ for all $i \in I$ implies that $\Phi(U) \sqsubset B$. In particular, if μ is a measure on X and if $\Phi(\pi_{S_i}\mu) \sqsubset B$ for all $i \in I$, then $\Phi(\pi_U \mu) \sqsubset B$.*

A transfunction $\Phi : X \rightsquigarrow Y$ is said to *vanish on an open set* U if $\Phi(U) \sqsubset \varnothing$. Let $\mathcal{V}_\Phi$ denote the collection of all vanishing open sets of Φ.

Let $\Phi : X \rightsquigarrow Y$ be a weakly σ-additive transfunction. Then $\cup\mathcal{V}_\Phi$ is called the *null space of* Φ, denoted as $\operatorname{null}\Phi$. Its complement, $(\cup\mathcal{V}_\Phi)^c$, is called the *spatial support of* Φ, denoted as $\operatorname{supp}\Phi$.

Note that $\Phi\mu = \Phi\left(\pi_{\operatorname{supp}\Phi}\mu\right)$, which implies that Φ is essentially a transfunction between the subspace $\operatorname{supp}\Phi$ and Y, that is, $\Phi : \operatorname{supp}\Phi \rightsquigarrow Y$.

A transfunction $\Phi : X \rightsquigarrow Y$ is called *non-vanishing* if Φ has no non-empty vanishing sets, that is, if $\operatorname{supp}\Phi = X$. Furthermore, Φ is *norm-preserving* if $||\Phi\mu|| = ||\mu||$ for all μ on X.

3. Localized Transfunctions

In this and the following two sections we assume that X and Y are metric spaces. We use $B(z;\rho)$ to denote the open ball of radius ρ centered at z.

Definition 5. Let $x \in X$ and $\varepsilon > 0$. We say that a transfunction $\Phi : X \rightsquigarrow Y$ is *ε-localized at x* if there exist $\delta > 0$ and $y \in Y$ such that $\Phi(B(x,\delta)) \sqsubset B(y,\varepsilon)$. We say that Φ is *0-localized at x* if Φ is ε-localized at x for all $\varepsilon > 0$. If Φ is ε-localized at x for some $\varepsilon > 0$, then we say that Φ is *localized at x*. If Φ is localized at every point in some set $A \in \Sigma_X$, then we say that Φ is *localized on A*.

If y needs emphasis, we can say that Φ is *ε-localized at (x,y)*. If we need to emphasize δ, we can say that Φ is *(δ,ε)-localized at (x,y)*.

Note in the definition of 0-localization that the values for δ and y may depend on ε.

Definition 6. Let $A \subseteq X$. We say that that a transfunction $\Phi : X \rightsquigarrow Y$ is *uniformly localized on A* if there exist $\varepsilon > 0$ and $\delta > 0$ such that Φ is (δ,ε)-localized on A.

If δ and ε are to be emphasized, then we say that Φ is *uniformly (δ,ε)-localized on A*. If only ε is to be emphasized, then we say that Φ is *uniformly ε-localized on A*.

Definition 7. For a transfunction $\Phi : X \rightsquigarrow Y$ we define a function $E_\Phi : X \to [0,\infty]$ via

$$E_\Phi(x) = \inf\{\varepsilon : \Phi \text{ is } \varepsilon\text{-localized at } x\}.$$

The function E_Φ measures how localized Φ is be throughout X. Note that Φ is $(E(x)+\eta)$-localized at x for all $\eta > 0$ whenever $E(x) < \infty$ and that Φ is not localized at x when $E(x) = \infty$.

Definition 8. Let $A \subseteq X$ and let $f : X \to Y$ be function. We say that Φ is *ε-localized on A via f* or that Φ is *ε-close to f on A* if Φ is ε-localized at $(x, f(x))$ for all $x \in A$.

It is worth noting that transfunctions are not necessarily localized anywhere.

When verifying whether a transfunction is localized the following simple proposition is often useful.

Proposition 9. *Let f be a measurable function, let $\mu \in \mathcal{M}_X$ be a positive measure or a vector measure and let $A \in \Sigma_X$. Then $|f_\# \mu| \le f_\# |\mu|$. If μ is a positive measure, then $\mu \sqsubset f^{-1}(A)$ if and only if $f_\# \mu \sqsubset A$, and if μ is a vector measure, the forward implication holds.*

Now we consider some examples.

1. If $f : X \to Y$ is a continuous function, then for $\Phi = f_\#$ we have $E_\Phi = 0$.
2. Let $H : \mathbb{R} \to \mathbb{R}$ be the Heaviside function centered at 0 and let $\Phi = H_\#$. Since H is continuous everywhere except at 0, it follows that $E_\Phi(x) = 0$ for all $x \neq 0$. However, $E_\Phi(0) = 1/2$.
3. Consider the measurable function $g : \mathbb{R} \to \mathbb{R}$ via $g = \sum_{n=0}^{\infty} 2^n H_n$, where H_n is the Heaviside function centered at n, and define $\Phi = g_\#$. Then $E_\Phi(n) = 2^{n-1}$ for each $n \in \mathbb{N}$ and $E(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{N}$, meaning that Φ is localized on $\mathbb{R}$ but that $\sup_{x \in \mathbb{R}} E(x) = \infty$.
4. Let $A \in \Sigma_X$. Then the projection transfunction $\Phi = \pi_A$ is 0-localized via the identity function since every carrier of μ is also a carrier of $\pi_A \mu$.
5. Let $Y = \mathbb{R}$, $\Sigma_Y = \mathcal{B}(\mathbb{R})$, and let ν be a strictly positive finite measure on $\mathbb{R}$. The transfunction $\Phi : \mathbb{R} \rightsquigarrow \mathbb{R}$ defined via $\Phi(\mu) = \|\mu\| \nu$ is not localized anywhere.
6. Let $X = Y = \mathbb{R}^d$ and let λ^d be the Lebesgue measure on $\mathbb{R}^d$. For some $\varepsilon > 0$, define $\kappa = \pi_{B(0;\varepsilon)} \lambda^d$. The transfunction $\Phi : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ defined via $\Phi(\mu) = \mu * \kappa$, the convolution of measures μ and κ, is ε-localized on X.

If U denotes the set of points in X where a transfunction Φ is localized, then the function $E_\Phi|_U : U \to [0, \infty)$ does not have to be continuous. However, as the next proposition states, it does have to be upper-semi-continuous on U, implying that U is an open set.

Proposition 10. *Let $\Phi : X \rightsquigarrow Y$ be a transfunction and let U denotes the set of points in X where Φ is localized. Then the function $E_\Phi|_U$ is an upper-semi continuous function and U is an open set.*

Proof. Let $x \in U$ with $E(x) = \eta$ and let $\varepsilon > \eta$. Then there exists a $y \in Y$ and a $\delta > 0$ such that $\Phi(B(x;\delta)) \sqsubset B(y;\varepsilon)$. Choose an $x_1 \in B(x;\delta)$ different from x. Then there exists a $\delta_1 > 0$ such that $B(x_1;\delta_1) \subset B(x;\delta)$, so that $\Phi(B(x_1;\delta_1)) \sqsubset B(y;\varepsilon)$. Therefore, Φ is ε-localized at (x_1, y), yielding that $E_\Phi(x_1) \leq \varepsilon$ and that $x_1 \in U$. Since $x_1 \in B(x;\delta)$ was arbitrary, this means that $\sup E_\Phi(B(x;\delta)) \leq \varepsilon$ and that $B(x;\delta) \subseteq U$. Since ε was arbitrary, this means that $\limsup_{\delta\to 0} E_\Phi(B(x;\delta)) \leq \eta = E_\Phi(x)$, meaning that E is upper-semi-continuous. Since x was arbitrary, this means that U is open. □

4. 0-Localized Transfunctions

When $f : X \to Y$ is continuous, we know that $f_\#$ is weakly σ-additive, norm-preserving, and 0-localized on X. We will show that these three properties characterize transfunctions that correspond to continuous functions.

Proposition 11. *Let X be a metric space with an ample family of measures $\mathcal{M}_X$ and let Y be a complete metric space. For any $A \in \Sigma_X$ and for any non-vanishing transfunction $\Phi : X \rightsquigarrow Y$ which is 0-localized on A there is a unique continuous function $f : A \to Y$ such that Φ is 0-close to f on A.*

Proof. Since Φ is 0-localized on A, it follows that $E_\Phi(x) = 0$ for all $x \in A$. Then for any fixed $x \in A$, there are $\delta_n > 0$ and $y_n \in Y$ indexed by $n \in \mathbb{N}$ such that $\Phi(B(x,\delta_n)) \sqsubset B\left(y_n, \frac{1}{n}\right)$ for every $n \in \mathbb{N}$.

First we show that $d(y_m, y_n) \leq \frac{1}{m} + \frac{1}{n}$ for all $m, n \in \mathbb{N}$. Suppose $d(y_m, y_n) > \frac{1}{m} + \frac{1}{n}$ for some $m, n \in \mathbb{N}$. Then $B\left(y_m; \frac{1}{m}\right) \cap B\left(y_n; \frac{1}{n}\right) = \varnothing$. Since $\mathcal{M}_X$ is ample, there is a non-zero measure $\nu \sqsubset B(x;\delta_m) \cap B(x;\delta_n)$. But then $\Phi(\nu) \sqsubset B\left(y_m; \frac{1}{m}\right) \cap B\left(y_n; \frac{1}{n}\right) = \varnothing$, which is impossible since Φ is non-vanishing.

Since $d(y_m, y_n) \leq \frac{1}{m} + \frac{1}{n}$ for all $m, n \in \mathbb{N}$, (y_n) is a Cauchy sequence in the complete metric space Y. So there exists $y \in Y$ with $y_n \to y$. Furthermore, $B\left(y_n, \frac{1}{n}\right) \subseteq B\left(y, \frac{1}{n} + d(y_n, y)\right)$ for $n \in \mathbb{N}$. Indeed, y is the unique point in Y with this property and Φ is 0-localized at (x, y). Now we define $f : A \to Y$ by $f(x) = y$, where $y \in Y$ is the unique point such that Φ is 0-localized at (x, y). Clearly Φ is 0-localized on A via f.

We now show that f is continuous on A. Let $x_n \to x_0$ in A. Define $y_0 = f(x_0)$ and $y_n = f(x_n)$ for $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $\Phi(B(x_0,\delta)) \sqsubset B(y_0,\varepsilon)$. Let $N \in \mathbb{N}$ be such that $d(x_m, x_0) < \delta/2$ for $m \geq N$. For every $m \geq N$ there is a $\delta_m < \delta/2$ such that $\Phi(B(x_m,\delta_m)) \sqsubset B(y_m,\varepsilon)$.

Then $B(x_m, \delta_m) \subset B(x_0, \delta)$ implies that $\Phi(B(x_m, \delta_m)) \sqsubset B(y_0, \varepsilon) \cap B(y_m, \varepsilon)$. Consequently, $d(y_m, y_0) \leq 2\varepsilon$, since $\mathcal{M}_X$ is ample and Φ is non-vanishing. $\square$

Theorem 12. *Let X be a metric space with an ample family of measures $\mathcal{M}_X$, Y a complete metric space, and $\Phi : X \rightsquigarrow Y$ a non-vanishing transfunction. Then $\Phi = f_\#$ for some continuous function $f : X \to Y$ if and only if Φ is norm-preserving, weakly σ-additive, and 0-localized on X.*

Proof. We only need to show that if Φ is norm-preserving, weakly σ-additive, and 0-localized on X, then $\Phi = f_\#$ for some continuous function $f : X \to Y$. Let $f : X \to Y$ be the unique continuous function guaranteed by Proposition 11. We will show that $\Phi = f_\#$.

Let $V \subseteq Y$ be an open set. Define $U = f^{-1}(V)$. By Proposition 11, there exists an open ball cover $\{B(x; \delta_x) : x \in U\}$ of U such that $\Phi(B(x; \delta_x)) \sqsubset V$ for all $x \in X$. By Proposition 4, we have that $\Phi(U) \sqsubset V$. Similarly, if we define $W = f^{-1}\left(\left(\overline{V}\right)^c\right)$, which is also open, we have that $\Phi(W) \sqsubset \left(\overline{V}\right)^c$.

Next, we define $Z = f^{-1}(\partial V) = f^{-1}(\overline{V} \cap V^c)$, which is closed, and for each $n \in \mathbb{N}$, let $L_n = \cup_{y \in \partial V} B(y; 1/n)$, which is an open set. Since Φ is 0-localized, for each $x \in Z$ and $n \in \mathbb{N}$, there exists $\delta_{x,n} > 0$ such that $\Phi(B(x; \delta_{x,n})) \sqsubset L_n$. For $K_n = \cup_{x \in Z} B(x; \delta_{x,n})$ we have $\Phi(K_n) \sqsubset L_n$ for all $n \in \mathbb{N}$, by Proposition 4. Noting that $\cap_{n=1}^{\infty} K_n = Z$ and $\cap_{n=1}^{\infty} L_n = \partial V$, it is clear that $\Phi(Z) \sqsubset \partial V$, by Proposition 3.

Let $\mu \in \mathcal{M}_X$. Since $\pi_U \mu \sqsubset U$ and $\Phi(U) \sqsubset V$, we have that $\Phi(\pi_U \mu) \sqsubset V$ and we have via norm-preservation of Φ that

$$\Phi(\pi_U \mu)(V) = ||\Phi(\pi_U \mu)|| = ||\pi_U \mu|| = \mu(U) = f_\#(\mu)(V).$$

Since $\Phi(W) \sqsubset \left(\overline{V}\right)^c$ and $V \cap \left(\overline{V}\right)^c = \varnothing$, it follows that $\Phi(\pi_W \mu)(V) = 0$. Similarly, since $\Phi(Z) \sqsubset \partial V$ and $V \cap \partial V = \varnothing$, it follows that $\Phi(\pi_Z \mu)(V) = 0$. From the above we obtain

$$\Phi(\mu)(V) = \Phi(\pi_U \mu)(V) + \Phi(\pi_W \mu)(V) + \Phi(\pi_Z \mu)(V) = f_\#(\mu)(V).$$

Moreover, since $\Phi(\mu)$ and $f_\#(\mu)$ are finite measures which agree on open sets, they must agree on all sets in Σ_Y by the $\pi - \lambda$ Theorem. Finally, since $\mu \in \mathcal{M}_X$ is arbitrary, we have $\Phi = f_\#$. $\square$

Now we characterize transfunctions which correspond to measurable functions, but under stricter settings. First, we define restrictions of transfunctions.

Definition 13. Let $\Phi : X \rightsquigarrow Y$ be a transfunction, and let $A \subseteq X$ be measurable. Then the composition $\Phi \circ \pi_A$ is called the *restriction* of Φ to A.

Note that $\Phi \circ \pi_A = \Phi$ when $\operatorname{supp} \Phi \subseteq A$ and that $\Phi \circ \pi_B = 0$ when $B \subseteq \operatorname{null} \Phi$.

Theorem 14. *Let X be locally compact, let λ be a finite regular measure on X, and let $\mathcal{M}_X$ only have measures absolutely continuous with respect to λ. Let $\Phi : X \rightsquigarrow Y$ be a weakly σ-additive transfunction. Then Φ corresponds to a measurable function if and only if there exists a sequence of compact sets $\{F_n\}_{n=1}^{\infty}$ such that $\lambda(F_n^c) < \frac{1}{n}$ and that $\Phi \circ \pi_{F_n}$ is identified with some continuous function on F_n.*

Proof. The forward direction is a straight-forward consequence of Lusin's theorem, where the measurable and continuous functions are identified with the respective transfunctions.

We now prove the reverse direction. For each natural n, let $\Phi \circ \pi_{F_n}$ be identified with continuous function $f_n : F_n \to Y$. Let $i \neq j$. If $\lambda(F_i \cap F_j) > 0$, then there exists some compact subset $G_{i,j} \subseteq F_i \cap F_j$ such that $\lambda(G_{i,j}) = \lambda(F_i \cap F_j)$ and $\lambda(U \cap G_{i,j}) > 0$ whenever $U \cap G_{i,j} \neq \varnothing$ for open U, [6]. Otherwise, if $\lambda(F_i \cap F_j) = 0$, then define $G_{i,j} = \varnothing$.

For the latter case, $f_i = f_j$ is vacuously true on $G_{i,j}$. For the former case, let $x \in G_{i,j}$. Suppose that $f_i(x) \neq f_j(x)$. If we let $\varepsilon < d(f_i(x), f_j(x))/2$, this would imply by 0-localization of $\Phi \circ \pi_{F_i}$ and $\Phi \circ \pi_{F_j}$ the existence of $\delta > 0$ such that $\Phi(B(x;\delta) \cap G_{i,j}) \sqsubset B(f_i(x);\varepsilon) \cap B(f_j(x);\varepsilon) = \varnothing$. Choosing μ_0 to be the projection of λ onto $B(x;\delta) \cap G_{i,j}$, we observe that $\mu_0 \neq 0$ and that $\Phi(\mu_0) = \Phi \circ \pi_{F_i}(\mu_0) = 0$, which contradicts the norm-preservation of $\Phi \circ \pi_{F_i}$ on F_i. It follows that $f_i = f_j$ on $G_{i,j} \subseteq F_i \cap F_j$. Having i, j arbitrary, we have that outside the λ-null Borel set $N = (\cup_{i,j=1}^{\infty} (F_i \cap F_j - G_{i,j})) \cup (\cap_{i=1}^{\infty} F_i^c)$, the functions $(f_i)_{i=1}^{\infty}$ coincide, allowing them to be glued to a measurable function $h : X \to Y$, where $h(N) = \{y_0\}$ for some fixed $y_0 \in Y$.

We now show that $\Phi = h_\#$. Let $\mu \in \mathcal{M}_X$ and let $A_n = N^c \cap (F_n - \cup_{m<n} F_m)$. Since $\lambda(N) = 0$ and $\mu \ll \lambda$, $\mu(N) = 0$. This means that

$$\Phi(\mu)(B) = \Phi\left(\pi_N \mu + \textstyle\sum_{n=1}^{\infty} \pi_{A_n} \mu\right)(B) = \textstyle\sum_{n=1}^{\infty} \Phi(\pi_{A_n} \mu)(B)$$

$$= \textstyle\sum_{n=1}^{\infty} f_{n\#}(\pi_{A_n} \mu)(B) = \textstyle\sum_{n=1}^{\infty} \mu\left(A_n \cap h^{-1}(B)\right)$$

$$= \mu\left(N \cap h^{-1}(B)\right) + \mu\left(N^c \cap h^{-1}(B)\right) = h_{\#}(\mu)(B).$$

□

5. ε-Localized Transfunctions

When Φ is not indentifiable with a measurable function, under what condition is it "close" to a measurable function? We consider this question for uniformly localized transfunctions. Given that $\Phi : X \rightsquigarrow Y$ is uniformly ε-localized, can we find a measurable function $f : X \to Y$ such that Φ is uniformly ε-close to f? If we can find such a function, then it gives a rough idea of how the transfunction behaves. In our settings, we can always find such a measurable function: in fact, it can be chosen so that f is σ-simple. Can we choose a continuous f in this way? The answer is also affirmative, but it requires a more demanding setting.

Proposition 15. *Let X and Y be metric spaces, with X second-countable. Then every transfunction Φ which is uniformly ε-localized on X is uniformly ε-close to some measurable function $f : X \to Y$.*

Proof. Let $\Phi : X \rightsquigarrow Y$ be a uniformly (δ, ε)-localized transfunction on X. This means that for all $x \in X$, there exists some $y_x \in Y$ with $\Phi(B(x;\delta)) \sqsubset B(y_x;\varepsilon)$. This choice function $x \mapsto y_x$ will be used later. Note that the collection $\{B(x;\delta/3) : x \in X\}$ is an open cover of X. It follows from second-countability of X that there is a countable subcover, which shall be indexed as $\{B(x_n;\delta/3) : n \in \mathbb{N}\}$. For each natural n, let $y_n = y_{x_n}$ from the choice function above. Next we create a function $f : X \to Y$ given by $f(x) = y_n$ whenever $x \in B(x_n;\delta/3) - \cup_{m<n} B(x_m;\delta/3)$. It follows that f is a σ-simple function, and therefore is measurable. Furthermore, when $f(x) = y_n$, it follows that $x \in B(x;\delta/3) \subseteq B(x_n;\delta)$.

Therefore, it follows that $\Phi(B(x;\delta/3)) \sqsubset B(y_n;\varepsilon) = B(f(x);\varepsilon)$, which shows that Φ is uniformly $(\delta/3, \varepsilon)$-localized on X via f. □

We build upon the proof of Proposition 15 to develop the next theorem. First, we define left-translation-invariance of a metric on locally compact groups.

Definition 16. Let X be a locally compact group with identity e, and let d be a metric on X. Then d is *left-translation-invariant* if $d(x, y) = d(zx, zy)$

for all $x, y, z \in X$. When the metric is understood by context, the equivalent definition is that $xB(e;\varepsilon) = B(x;\varepsilon)$ for all $x \in X$ and $\varepsilon > 0$.

Theorem 17. *Let X be a second-countable metrizable locally compact group with left-translation-invariant metric and let Y be a normed space. Then every Φ which is uniformly ε-localized on X is uniformly ε-close to some continuous function $g : X \to Y$.*

Proof. Let e denote the identity of X and let 0_Y denote the zero in Y. Take from Proposition 15 the measurable $f : X \to Y$ as described in the previous proof with the same details. Then there exists $\alpha > 0$ such that for all $x \in X$, $B(x;\alpha)$ has compact closure. Let $x \in X$ be arbitrary. Since $\{B(x_n;\delta/3) : n \in \mathbb{N}\}$ covers $\overline{B(x;\alpha)}$, it follows that there is a finite subcover $\{B(x_n;\delta/3) : n \leq N_x\}$ for some natural number N_x depending on x. Therefore, $f(B(x;\alpha)) \subseteq \{y_n : n \leq N_x\} \subseteq B(0_Y;M_x)$ for some real M_x depending on x. Since x was arbitrary, this means that f is locally bounded.

Now let $\beta = \min\{\delta/3, \alpha/2\}$. Since X is a locally compact group, there exists a non-zero (uniformly) continuous function $\varphi : X \to [0,\infty)$ with compact support within $B(e;\beta)$. Now choose the unique appropriately scaled left Haar measure κ on X such that $\int \varphi(u^{-1})d\kappa(u) = 1$.

Now consider the function $g : X \to Y$ given by $g = f * \varphi$, the convolution of $f : X \to Y$ and $\varphi : X \to \mathbb{R}$ using the (vector-valued) integral

$$g(x) = f * \varphi(x) = \int f(t)\varphi(t^{-1}x)d\kappa(t) = \int f(xu)\varphi(u^{-1})d\kappa(u).$$

Note that the integral above is well-defined, because $t \mapsto \varphi(t^{-1}x)$ is zero outside of $xB(e;\beta) = B(x;\beta)$ and f is bounded and finitely-valued on the set $B(x;\beta)$ by an earlier argument. Also, the second equality holds due to left-invariance of κ and the substitution $u = x^{-1}t$ which yields $xu = t$ and $u^{-1} = t^{-1}x$.

We shall now show that g is continuous. Let $x \in X$ and let $\varepsilon > 0$ and choose some $\eta \in (0,\beta)$ with respect to uniform continuity of φ. Let x' be η-close to x in X: that is, let $x^{-1}x' \in B(e;\eta)$. This implies that $(t^{-1}x)^{-1}(t^{-1}x') = x^{-1}x' \in B(e;\eta)$ for all $t \in X$, so that $t^{-1}x$ and $t^{-1}x'$ are also η-close in X for all $t \in X$. Since $d(x,x') < \alpha/2$, it follows that $B(x';\alpha/2) \subseteq B(x;\alpha)$, which means that $f(B(x';\alpha/2)) \subseteq f(B(x;\alpha)) \subseteq B(0_Y;M_x)$. Therefore M_x bounds the vectors obtained by f in both $B(x;\beta)$ and $B(x';\beta)$. Then it follows that

$|\varphi(t^{-1}x) - \varphi(t^{-1}x')| < \varepsilon$ for all $t \in X$ and that

$$\begin{aligned} ||g(x) - g(x')|| &= \left|\left| \int f(t)[\varphi(t^{-1}x) - \varphi(t^{-1}x')] d\kappa(t) \right|\right| \\ &\leq 2 \cdot M_x \cdot \varepsilon \cdot \kappa(B(e;\beta)). \end{aligned}$$

Continuity of g follows since M_x only depends on x, $\kappa(B(e;\beta))$ is a constant, and ε was arbitrary.

To show that Φ is uniformly (β, ε)-localized via g, let $x \in X$ be arbitrary and let $\mu \sqsubset B(x;\beta)$. Recall that $B(x;\beta)$ is covered by $\cup_{m=1}^{N_x} B(x_m;\delta/3)$. Notice that for every x_m with $B(x_m;\delta/3) \cap B(x;\delta/3) \neq \varnothing$ we have that $B(x;\delta/3) \subseteq B(x_m;\delta)$ which implies that $\Phi(B(x;\delta/3)) \sqsubset B(y_m;\varepsilon)$.

If we denote $R = \{y_m : m \leq N_x \text{ and } B(x_m;\delta/3) \cap B(x;\delta/3) \neq \varnothing\}$, and if we denote $C = \mathrm{Conv}(R)$, the convex hull of R, this implies that

$$\Phi\mu \sqsubset \bigcap_{y \in R} B(y;\varepsilon) = \bigcap_{y \in C} B(y;\varepsilon).$$

If we can show that $g(x) \in C$, then it follows from above that Φ is (β, ε)-localized at $(x, g(x))$.

For each natural $m \leq N_x$, we define $A_m = B(e;\beta) \cap x^{-1} f^{-1}(y_m)$ which is empty if $y_m \notin R$ and we define $c_m = \displaystyle\int_{A_m} \varphi(u^{-1}) d\kappa(u)$ which is zero if $y_m \notin R$. Then $\sum_{m=1}^{N_x} c_m = \int \varphi(u^{-1}) d\kappa(u) = 1$, and by looking at the convolution function g, we see that $g(x)$ equals

$$\begin{aligned} &\int_{B(e;\beta)} f(xu)\varphi(u^{-1}) d\kappa(u) = \int_{B(e;\beta)} \left[\sum_{m=1}^{N_x} y_m \chi_{A_m}(u) \right] \varphi(u^{-1}) d\kappa(u) \\ &= \sum_{m=1}^{N_x} y_m \int_{A_m} \varphi(u^{-1}) d\kappa(u) = \sum_{m=1}^{N_x} c_m y_m \in C. \end{aligned}$$

Therefore, it follows that $\Phi\mu \sqsubset B(g(x);\varepsilon)$, meaning that Φ is uniformly ε-close to g. □

Corollary 18. *Give $\mathbb{R}^n$ and $\mathbb{R}^m$ the usual norms. Every uniformly ε-localized $\Phi : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is uniformly ε-close to some continuous function $g : \mathbb{R}^n \to \mathbb{R}^m$.*

For transfunctions not uniformly localized, there is a result analogous to Proposition 15 with appropriate modifications of its proof.

Proposition 19. *Let X and Y be metric spaces with X second-countable. Then every transfunction $\Phi : X \rightsquigarrow Y$ which is ε-localized on X is ε-close to some measurable function $f : X \to Y$.*

Proof. We use the same framework as the proof from Proposition 15. Each $x \in X$ has some associated $\delta_x > 0$ from definition of ε-localization at x. We form the cover $\{B(x; \delta_x/3) : x \in X\}$ of X which has a countable subcover $\{B(x_n; \delta_n) : n \in \mathbb{N}\}$, where $\delta_n = \delta_{x_n}$. We define $f(x) = y_n$ when $x \in B(x_n; \delta_n/3) - \cup_{m<n} B(x_m; \delta_m/3)$. For x with $f(x) = y_n$, we have that $x \in B(x; \delta_n/3) \subseteq B(x_n; \delta_n)$. This means for all x with $f(x) = y_n$, we have that $\Phi(B(x; \delta_n/3)) \sqsubset B(y_n; \varepsilon) = B(f(x); \varepsilon)$, meaning that Φ is ε-localized on X via f. □

Alternatively, we develop a proposition analogous to the statement that continuous functions on compact sets are uniformly continuous.

Proposition 20. *Let $\Phi : X \rightsquigarrow Y$ be a transfunction which is ε-localized on X. Define*

$$D_\varepsilon(x) := \sup\{\delta > 0 : \Phi \text{ is } (\delta, \varepsilon)\text{-localized at } x\}.$$

Then $D_\varepsilon : X \to (0, \infty)$ is continuous on X and if X is compact, then Φ is uniformly ε-localized on X.

Proof. Let $x_0 \in X$. Let $x \in B(x_0; D_\varepsilon(x_0))$. It must follow by definition of D_ε that

$$D_\varepsilon(x_0) - d(x, x_0) \leq D_\varepsilon(x) \leq D_\varepsilon(x_0) + d(x, x_0);$$

this is because $B(x; D_\varepsilon(x_0) - d(x, x_0)) \subseteq B(x_0; D_\varepsilon(x_0)) \subseteq B(x; D_\varepsilon(x_0) + d(x, x_0))$.

Therefore, $|D_\varepsilon(x) - D_\varepsilon(x_0)| \leq d(x, x_0) \to 0$ as $x \to x_0$. Hence, D_ε is continuous on X. If X is compact, then D_ε obtains its minimum, positive value on X; call that value δ_X. Then for any positive $\delta < \delta_X$, we have that $\delta < D_\varepsilon(x)$ for all $x \in X$, meaning that Φ is (δ, ε)-localized at every $x \in X$. This precisely means that Φ is uniformly (δ, ε)-localized on X. □

6. Graphs of Transfunctions

We introduce a concept analogous to the graph of a function and prove three theorems that shed some light on the nature of localized transfunctions.

Definition 21. Let $\Phi : X \rightsquigarrow Y$ be a transfunction, and let $\Gamma \subseteq X \times Y$ be measurable with respect to the product σ-algebra. We say that Γ *carries* Φ, denoted as $\Phi \sqsubset \Gamma$, if for every measurable rectangle $A \times B$

$$(A \times B) \cap \Gamma = \varnothing \quad \text{implies} \quad \Phi(A) \sqsubset B^c.$$

Similar to how carriers of a measure describe its support, the carriers of a transfunction describe its graph. This is a generalization of the concept of a graph of a function, as indicated by the following theorem.

Proposition 22. *For every measurable function $f : X \to Y$ the graph of f carries $f_\#$, that is,*

$$f_\# \sqsubset \{(x, f(x)) : x \in X\}.$$

Proof. If $(A \times B) \cap \{(x, f(x)) : x \in X\} = \varnothing$, then $A \cap f^{-1}(B) = \varnothing$, so for every $\mu \sqsubset A$,

$$f_\#(\mu)(B) = \mu(f^{-1}(B)) = \mu(A \cap f^{-1}(B)) = 0.$$ □

We also have the reverse situation: a subset of $X \times Y$ can generate a carried transfunction.

Proposition 23. *Let (X, Σ_X) be a measurable space and let (Y, Σ_Y, λ) be a finite measure space. If $\Gamma \subseteq X \times Y$ is a measurable set with respect to the product σ-algebra, then*

$$\Phi(\mu)(B) = (\mu \times \lambda)(\Gamma \cap (X \times B))$$

defines a strongly σ-additive transfunction from X to Y such that $\Phi \sqsubset \Gamma$.

Proof. If $U_1, U_2, \dots \in \Sigma_Y$ are disjoint, then

$$\begin{aligned}
\Phi(\mu)(\cup_{n=1}^\infty U_n) &= (\mu \times \lambda)(\Gamma \cap (X \times \cup_{n=1}^\infty U_n)) \\
&= (\mu \times \lambda)(\Gamma \cap \cup_{n=1}^\infty (X \times U_n)) = (\mu \times \lambda)(\cup_{n=1}^\infty (\Gamma \cap (X \times U_n))) \\
&= \sum_{n=1}^\infty (\mu \times \lambda)(\Gamma \cap (X \times U_n)) = \sum_{n=1}^\infty \Phi(\mu)(U_n),
\end{aligned}$$

so $\Phi(\mu)$ is a measure on Y. Strong σ-additivity of Φ follows from the equality $(\sum_{i=1}^\infty \mu_i) \times \lambda = \sum_{i=1}^\infty (\mu_i \times \lambda)$. Moreover, if $(A \times B) \cap \Gamma = \varnothing$ and $\mu \sqsubset A$, then

$$\Phi(\mu)(B) = (\mu \times \lambda)(\Gamma \cap (X \times B)) = (\mu \times \lambda)(\Gamma \cap (A \times B)) = 0.$$

□

Some localized transfunctions which are "close" to measurable functions turn out to be carried by what one might call "fat graphs". And if a transfunction has a "fat continuous graph", then it is localized. The following proposition makes these claims precise.

Proposition 24. *Let $f : X \to Y$ be measurable. If Φ is weakly σ-additive and ε-localized on X via f, then*

$$\Phi \sqsubset \Gamma := \bigcup_{x \in X} (\{x\} \times B(f(x), \varepsilon)) .$$

If f is continuous and $\Phi \sqsubset \Gamma$, then Φ is localized on X via f with $E_\Phi \leq \varepsilon$.

Proof. If Φ is weakly σ-additive and ε-localized on X via f, then for each $x \in X$ there is a $\delta_x > 0$ such that Φ is (δ_x, ε)-localized at $(x, f(x))$. If $(A \times B) \cap \Gamma = \varnothing$, then $B \cap \left(\bigcup_{a \in A} B(f(a), \varepsilon)\right) = \varnothing$ and thus $\bigcup_{a \in A} B(f(a), \varepsilon) \subseteq B^c$. Note that $\{B(a; \delta_a) : a \in A\}$ is an open cover of A with a countable subcover $\{B(a_n; \delta_n) : n \in \mathbb{N}\}$, where $\delta_n = \delta_{a_n}$. Let $A_n = B(a_n; \delta_n)$. Since $\Phi(A_n) \sqsubset B^c$ for each $n \in \mathbb{N}$, we have $\Phi(\cup_{n=1}^{\infty} A_n) \sqsubset B^c$, by Proposition 3. It then follows from $A \subseteq \cup_{n=1}^{\infty} A_n$ that $\Phi(A) \sqsubset B^c$, again by Proposition 3.

Now assume that f is continuous and $\Phi \sqsubset \Gamma$. Let $x \in X$ and $n \in \mathbb{N}$ be arbitrary. Then there exists a δ such that $f(B(x; \delta)) \subseteq B(f(x); 2^{-n})$ and it follows by definition of Γ and by our previous argument that

$$B(x; \delta) \times B(f(x); \varepsilon + 2^{-n})^c \cap \Gamma = \varnothing.$$

Since $\Phi \sqsubset \Gamma$, it follows that $\Phi(B(x; \delta)) \sqsubset B(f(x); \varepsilon + 2^{-n})$, resulting in Φ being localized on X via f. Moreover $E_\Phi(x) \leq \varepsilon + 2^{-n}$ for all $x \in X$ and $n \in \mathbb{N}$. Since x and n were arbitrary, we have $E_\Phi \leq \varepsilon$. □

References

[1] L. Ambrosio, Optimal transport maps in Monge-Kantorovich problem, In: *Proc. International Conference of Mathematicians, Beijing* **3** (2002), 131-140.

[2] J. Bentley, Construction of regular non-Aatomic strictly-positive measures in second-countable locally compact non-atomic Hausdorff apaces, *arXiv:1807.05437*.

[3] J. Bentley and P. Mikusiński, Markov operators, transport plans, and transfunctions, *arXiv:1810.08349*.

[4] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, 2nd Ed., Ser. Texts in Applied Mathematics, Springer (2012).

[5] J.J. Buckley and E. Eslami, Fuzzy functions, In: *An Introduction to Fuzzy Logic and Fuzzy Sets*. Ser. Advances in Soft Computing **13**, Springer-Verlag, Berlin - Heidelberg (2002).

[6] D. Cohn, *Measure Theory*, 2nd Ed., Birkhäuser, Basel (2013), p. 213.

[7] J. Goldstein, *Semigroups of Linear Operators & Applications*, 2nd Ed., Dover Publications, Inc. Mineola, New York (2017).

[8] P. Mikusiński, Transfunctions, *arXiv:1507.03441*.

[9] C. Villani, *Optimal Transport: Old and New*. Springer-Verlag, Berlin - Heidelberg (2009).