

BUSINESS ENVIRONMENTS UNDER SUB-QUEUED PREFERENCES

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Abstract: Business environments including agricultural business environments are known to have subgroups of desks, counters, workers and customers of distinct stationary preferences, allocations and characteristics. Is it possible then to carry out analysis with these agents embedded in a system? That is, can one differentiate basic agents such as subgroup preferences in analysis? This paper shows that the answer is Yes.

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1. Introduction

Generally, the use of group theory in queuing systems is scarce in the literature. This stand may not be unconnected with the popularity of bulk models, [1]. In bulk customer modeling, distinct characteristics of simple subgroups within a bulk is generally ignored, [3] and [7]. Suppose an agro-industrialist faces a customer related problem in a certain subgroup of a mega agricultural market where other subgroups are non null. Unless every subgroup leads to similar subgroups within the centralizer of an involution containing the problematic subgroup, then bulk analysis is less meaningful, [5].

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There are several instances in agricultural product markets, mega food stores, banks and other business environments where customer subgroups are not contained in the centralizers of an involution. For instance, the case of customers of a product market with fish selling and meat selling subgroups. If no fish customer is a meat customer in a given time interval, then the fish selling subgroup is clearly distinct from the meat selling subgroup. The case of customers of a withdrawal counter and that of currency exchange counter of a bank is another example. If no withdrawal customer requiring foreign exchange service, then the foreign exchange counter is nowhere the center of an involution containing the withdrawal counter. In the two cases exemplified above, both the fish selling subgroup and the foreign exchange counter cannot represent the meat selling subgroup and the withdrawal counter respectively. For more on this subject, see [6].

Thus, an analysis under null customer intersection assumption such as the one proposed in this work is vital for agro-allied and other financial systems, industries and environments for bettering performance.

2. Basic Assumptions

Consider a market environment with N customers queued over r business points such that $\{N\} = \{n_1\} \otimes \{n_2\} \otimes \{n_3\} \otimes \dots \otimes \{n_r\}$ with the property that $\{n_1\} \cap \{n_2\} \cap \{n_3\} \cap \dots \cap \{n_r\} = \{\}$. Suppose that customers arrive $\{n_r\} \in \{N\}$ according to a Poisson process at a rate of λ to receive service. The service time of customers is assumed to follow the exponential distribution with service parameter μ_r . For reasons to do with stability, we suppose that the occupation rate of the system $\rho = \frac{\lambda}{\sum_{i=1}^r \mu_i} < 1$.

3. Results

Lemma 1. *Given that $\{N\} = \{n_1\} \otimes \{n_2\} \otimes \{n_3\} \otimes \dots \otimes \{n_r\} : \{n_1\} \cap \{n_2\} \cap \{n_3\} \cap \dots \cap \{n_r\} = \{\}$. Then the stationary probability P_N that there are N customers in $\{N\}$ with r sub-queued preferences is given by*

$$P_N = \prod_{r=1}^R \rho_{n_r}^{n_r} (1 - \rho_{n_r}). \quad (1)$$

Proof. The case of two subgroups $\{c\}$ and $\{n\}$ of $\{N\}$ suffices. Given $\{N(t) = c(t)n(t), \zeta(t)\}_{t \geq 0}$, where $N(t)$ denotes the number of customers in the system at

time t and $\zeta(t)$ is the past service time of an arbitrary customer in $\{N\}$. Upon completion of service of an arbitrary customer, $\{N(t), \zeta(t)\}_{t \geq 0}$ is a Markov process. Assuming that the service time of customers is continuous and that the system is empty at time $t = 0$. Then one can apply the supplementary variable technique on $\{N(t), \zeta(t)\}_{t \geq 0}$, [2].

Let P be a probability measure on $\{N(t), \zeta(t)\}_{t \geq 0}$ such that¹

$$P[\text{cad}\{N(t), \zeta(t)\} = 0]; \text{cad}\{c\} = \text{cad}\{n\} = 0. \quad (2)$$

$$P[\text{cad}\{N(t), \zeta(t)\} = c]; \text{cad}\{c\} = c, \text{cad}\{n\} = 1. \quad (3)$$

$$P[\text{cad}\{N(t), \zeta(t)\} = cn]; \text{cad}\{c\} = c, \text{cad}\{n\} = n. \quad (4)$$

Suppose the stability condition $\lambda < \mu$ for all arrivals in $\{N\}$ holds. Then as $t \rightarrow \infty$, $P[\text{cad}\{N(t), \zeta(t)\} = 0]$, $P[\text{cad}\{N(t), \zeta(t)\} = c]$ and $P[\text{cad}\{N(t), \zeta(t)\} = cn]$ converge to $P[\text{cad}\{N, \zeta\} = 0] = P_0$, $P[\text{cad}\{N, \zeta\} = c] = P_c$ and $P[\text{cad}\{N, \zeta\} = cn] = P_{cn}$ respectively, see Medhi [4]. Let $P[N = n|c]$ denote the probability that there are $n \in \{N\}$ customers given when there are $c \in \{N\}$ fixed customers. Under the rate-equality principle for all arrivals in $\{N\}$ such that $\{c\} \cap \{n\} = \{\}$, the following difference equations are satisfied:

$$\lambda_c P_0 = \mu_c P_1 : c = 0. \quad (5)$$

$$(\lambda_c + \mu_c) P_c = \lambda_c P_{c-1} + \mu_{c+1} : c > 0. \quad (6)$$

$$\lambda_n P(n = 0|c) = \mu_n P(n = 1|c) : n = 0. \quad (7)$$

$$(\lambda_n + \mu_n) P(n|c) = \lambda_n P(n - 1|c) + \mu_n P(n + 1|c) : n > 0. \quad (8)$$

By the Markov property of the system as claimed and adding (5) and (6) for $c = 0, 1, 2, 3, \dots$, we have

$$P_c = \left(\frac{\lambda_c}{\mu_c} \right)^c P_0. \quad (9)$$

¹The cardinality of a group denoted by (cad) is the number of customers, elements or objects in the group

The zero state probability P_0 that clears $c \in \{c\}$ customers is obtained from the normalization condition that

$$\sum_{c=0}^{\infty} P_c = P_0 + P_1 + P_2 + \dots = 1. \quad (10)$$

So that

$$P_c = \left(\frac{\lambda_c}{\mu_c} \right)^c \left(1 - \frac{\lambda_c}{\mu_c} \right). \quad (11)$$

Also, adding (7) and (8) for $n = 0, 1, 2, 3, \dots$ when there are $c \geq 0$ customers, we have

$$P(n|c) = \left(\frac{\lambda_n}{\mu_n} \right)^n P(n=0|c). \quad (12)$$

□

Lemma 2. *Under two customer sub-queued preferences, the expected number of customers in $\{n\}$ when there are $c \in \{c\}$ customers is given by*

$$E[n|c] = cP_c \cdot [E[N]_{M/M/1}]. \quad (13)$$

Proof. Denote by $V(z)$ the probability generating function (PGF) of the number of customers in the group $\{N\}$ generally. From the definition of PGF for a fixed c and $n = 0, 1, 2, 3, \dots$, we have

$$V(z) = P_0 + P_{1,c}z^c + P_{2,c}z^{2c} + \dots \quad (14)$$

In view of (1), (14) will have the representation that

$$V(z) = P_0(1 + \rho_n z^c + (\rho_n z^c)^2 + \dots). \quad (15)$$

Equation (15) is an infinite geometric series with common ratio $\rho_n z^c$. Hence

$$V(z) = \frac{P_0}{1 - \rho_n z^c}. \quad (16)$$

Differentiating (16) with respect to z yields

$$V'(z) = \frac{cP_0\rho_n z^{(c-1)}}{(1 - \rho_n z^c)^2}. \quad (17)$$

The lemma holds upon evaluating (17) at $z = 1$ and simplifying the said equation. □

Lemma 3. Suppose $N \rightarrow K$ for some $K \in \mathbb{R}^+$. Then the expected number of customers $n \in \{n\}$ given $c \in \{c\}$ in a business environment with two sub-queued preferences is given by

$$E[n|c] = \frac{c\rho_n\rho_c^c(1-\rho_c)[(1-\rho_n^K) - K\rho_n^{K-1}(1-\rho_n)]}{(1-\rho_n)}. \quad (18)$$

Proof. Denote by $U(z)$ the PGF of the number of customers $n \in \{n\}$ when there are $c \in \{c\}$ customers in the system. In view of (14), we have

$$U(z) = P_0 + P_c z^c + P_{2c} z^{2c} + \dots + P_{cK} z^{cK}. \quad (19)$$

$$U(z) = \rho_c^c(1-\rho_c)(1-\rho_n)[1 + \rho_n z^c + (\rho_n z^c)^2 + \dots + (\rho_n z^c)^K]. \quad (20)$$

Application of basic properties of geometric series leads to

$$U(z) = P_0 \left[\frac{1 - (\rho_n z^c)^K}{1 - \rho_n z^c} \right]. \quad (21)$$

Let the numerator and the denominator of (21) be ψ and ϕ respectively. Differentiating the said equation with respect to z yields

$$U'(z) = \frac{-cK P_0 \phi(z) \rho_n^K z^{cK-1} + \psi(z) c \rho_n z^{c-1}}{(1 - \rho_n z^c)^2}. \quad (22)$$

The lemma follows after evaluating (22) at $z = 1$ upon simplification. $\square$

4. Remarks

Remark 1. Suppose a subgroup $\{n\} = \{\}$ in the presence of another customer subgroup $\{c\}$. Then the probability P_N in (1) is that of the classical $M/M/1$ queuing system with a single sub-queued preference.

Intuitively, given that $\{n\} = \{\}$, then the stability condition of the system is saddled on the subgroup $\{c\}$. Consequently, $\{N\} = \{c\} \cup \{n\} = \{c\} \cup \{\} = \{c\}$. Putting $n = 0$ in (1) yields

$$P_N = \rho_c^c(1-\rho_c)(1-\rho_0), \quad (23)$$

where $\rho_{n=0} = \rho_0$ is the probability that no arrival occurring at the rate λ_n in the system takes place. Finally, the remark holds if one lets $\rho_0 = 0$ in (23) trivially. Under this condition, there is an equivalence relation between (23) at $\rho_0 = 0$ and that of the classical $M/M/1$ queuing system.

Remark 2. In a business environment with two sub-queued customer preferences, there exists a direct relationship between the size of the subgroup $\{n\}$ and the stationary probability P_c .

This is in view of Lemma 2. Thus, the maximizer of elements of $\{n\}$ is any real valued function that increases the tendency P_c in $(0, 1)$.

5. Scope for Future Work

There is a scope in extending our results to some special cases of the problem solved in this work. For instance, when customers within specified subgroups have tendencies of reneging, balking or shunting or any combination of these known processes. The authors are grateful to all literature sources used.

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