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IDEALS IN SEMIRING WITH INVOLUTION

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Abstract: In this paper, we study the notion of $*$-prime ideal in semiring with involution and shown that if M is a non-void $*$-m-system in a semiring with involution and if I is a $*$-ideal of R with $I \cap M = \phi$, then there exists a $*$-prime ideal P of R such that $I \subseteq P$ and $P \cap M = \phi$. We also introduce the notion of $*$-k-prime ideal and we have shown that if P is a $*$-k-ideal of a semiring R with involution, then P is semiprime if and only if P is $*$-k-prime.

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Key Words: semiring, involution, prime ideals, m-system and $*$-m-system

1. Introduction

The concept of semirings was introduced by H.S. Vandiver in 1935, and it has been studied by several authors. Throughout this paper R denotes a semiring.

A semiring R is a non-empty set R together with two binary operation $+$ and . such that:

i) $< R, + >$ is a commutative monoid with identity denoted by 0_R or simply 0,

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ii) $< R, . >$ is a semigroup,
iii) For every $r,\ s,\ t \in R,\ r(s+t) \ = \ rs+rt$ and $(s+t)r \ = \ sr+tr$,
iv) For every $r \in R,\ r0 \ = \ 0r \ = \ 0$.

Recall from [3] that a semiring with involution is an algebra $R = < R, +, ., * >$ such that $< R,\ +,\ . >$ is a semiring, and the following identities are satisfied $(a+b)^* = a^* + b^*$; $(ab)^* = b^*a^*$; $(a^*)^* = a$.

For any nonempty set S, we define $S^* = \{s^* : s \in S\}$. Observe that involution of every non-zero element is non-zero. A non-empty subset I of a semiring R is called a left (resp. right) ideal of R if $a+b \in I,\ ra \in I$ (resp. $ar \in I$) for all $a,\ b \in I$ and for all $r \in R$. If I is both left and right ideal of R, then I is called an ideal of R. Following [2], we say that an ideal I of R is said to be $*$-ideal if $I^* \subseteq I$. Clearly if I is $*$-one sided ideal of R, then I is a $*$-ideal of R. Observe that if K is an ideal of R, then $K^*K,\ KK^*,\ K \cap K^*$ and $K+K^*$ are $*$-ideals of R and K^* is also an ideal of R. An ideal P is said to be prime if whenever A, B are ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. Following [2], we say that a $*$-ideal P of R is said to be $*$-prime if whenever $A,\ B$ are $*$-ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. Observe that if P is a prime and $*$-ideal of R, then P is a $*$-prime ideal of R. The following example shows that there exists a $*$-prime ideal of R which is not prime.

Example 1.1. Consider the ring Z_6 and commutative semiring $B = B(3,2)$ (F.E. Alarcon and D. Polkoska [1]).

Let $R = Z_6 \oplus Z_6 \oplus B$ be a semiring. Define $*$-on R via $(a_1, a_2, b_1)^* = (a_2, a_1, b_1)$. Let $A = \{0, 3\}$. Then $P = (A, A, B)$ is a $*$-prime ideal but not a prime ideal, since if $I = \{0, 2, 4\}$ and if $B = (I, 0, 0)$ and $C = (0, I, 0)$, then $BC \subseteq P$ but neither B nor C is included in P and hence P is not prime.

But the notion of $*$-semiprime ideal and semiprime ideal are coincide. Indeed, if I is a $*$-semiprime ideal of R and J is an ideal of R with $J^2 \subseteq I$. Then $(J^*)^2 \subseteq I$ and $(J+J^*)^2 = J^2 + JJ^* + J^*J + (J^*)^2 \subseteq J + I$ which imply $(J+J^*)^4 \subseteq I$, so $J \subseteq J + J^* \subseteq I$.

In 1956, M. Henriksen [4] defined a more restricted class of ideals in semirings, which he called k-ideal. A left (resp. right) ideal I of R is called left (resp. right) k-ideal if $a \in I$ and $x \in R$ and if $a + x \in I$, then $x \in I$. If I is both left and right k-ideal of R, then I is k-ideal of R. Clearly intersection of k-ideals of R is again k-ideal of R and I is a k-ideal of R if and only if I^* is a k-ideal of R. A $*$-k-ideal I is a k-ideal and $I^* \subseteq I$. If I is a k-ideal of R, then $I \cap I^*$ is a $*$-k-ideal of R. For subsets A, B of R, we denote $(A : B)_l = \{r \in R/rB \subseteq A\}$ and $(A : B)_r = \{r \in R/Br \subseteq A\}$. For any $a \in R$, $< a >$ the principle ideal of R generated by a. One can easily prove that $< a >= \{na + sa + at + \sum_i s_i a t_i / \ n \in N^+, \ s, \ t, \ s_i, \ t_i \in R\}$ and $< a^* >=< a >^*$.

2. Main Results

Lemma 2.1. *Let R be a semiring.*

i) If A and B are left (resp. right) ideals of R, then $(A : B)_l$ (resp. $(A : B)_r$) is an ideal of R.

ii) If A and B are left (resp. right)- k-ideal of R, then $(A : B)_l$ (resp. $(A : B)_r$) is a k-ideal of R.

Lemma 2.2. *Let R be a semiring with involution and let P be a $*$-ideal of R. Then P is a $*$-prime ideal of R if and only if whenever $AB \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$ with either A or B is a $*$-ideal.*

Proof. Let P be a $*$-prime ideal of R. Without loss of generality, let us assume that A is an ideal of R and B is a $*$-ideal of R such that $AB \subseteq P$. Then $BA^* \subseteq P$ and $(A^*B)^2 = A^*BA^*B \subseteq P$ which imply $A^*B \subseteq P$. Thus $(A+A^*)B \subseteq P$. By assumption, we have $(A+A^*) \subseteq P$ or $B \subseteq P$. Hence $A \subseteq P$ or $B \subseteq P$. The converse is obvious. □

Theorem 2.3. *Let R be a semiring with involution and P be a $*$-ideal of R. Then the following conditions are equivalent:*

(i) *P is a $*$-prime ideal.*

(ii) *If $a, b \in R$ such that $aRb \subseteq P$; $a^*Rb \subseteq P$, then $a \in P$ or $b \in P$.*

(iii) *If $< a >$ and $< b >$ are principal ideals of R such that $< a >< b >\subseteq P$; $< a^* >< b >\subseteq P$, then $a \in P$ or $b \in P$.*

(iv) *If U and V are right ideals in R such that $UV \subseteq P$; $U^*V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.*

(v) *If U and V are left ideals in R such that $UV \subseteq P$; $U^*V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.*

Proof. $(i) \Rightarrow (ii)$ Suppose $aRb \subseteq P$ and $a^*Rb \subseteq P$. By Lemma 2.1, we have $< a > R < b >\subseteq P$ and $< a^* > R < b >\subseteq P$. Then $R(< a > + < a >^*)RR < b > R \subseteq R < a > R < b > R + R < a^* > R < b > R \subseteq P$. By Lemma 2.2, we have $R(< a > + < a^* >)R \subseteq P$ or $R < b > R \subseteq P$.

If $R(< a > + < a^* >)R \subseteq P$, then $(< a > + < a^* >)^3 \subseteq P$. Hence $a \in P$.

Otherwise $R < b > R \subseteq P$. Then $< b >^3 \subseteq P$ implies $b \in < b > \subset P$.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (v)$ Let U and V be left ideals of R such that $UV \subseteq P$ and $U^*V \subseteq P$. Suppose $U \nsubseteq P$. Then there exists $u \in U$ such that $u \notin P$. Let $v \in V$. Then $< u >< v > \subseteq UV + RUV \subseteq P$ and $< u^* >< v > \subseteq U^*V + RU^*V \subseteq P$. By assumption, we have $< u > \subseteq P$ or $< v > \subseteq P$, but $u \notin P$. Hence $V \subseteq P$.

$(v) \Rightarrow (i)$ It is clear. □

A non-empty set M of elements of a semiring R is said to be m-system if $a, b \in M$, there exists $x \in R$ such that $axb \in M$. A non-empty set M of elements of a semiring R is said to $*$-m-system if $a, b \in M$, there exists $x \in R$ such that $axb \in M$ or $a^*xb \in M$. Obviously every m-system is a $*$-m-system. Also P is a $*$- prime ideal if and only if its complement is a $*$-m-system.

The following example shows that there exists a $*$-m-system of R that is not an m-system of R.

Example 2.4. Let R be a semiring of non-negative integers where $a+b = max\{a,\ b\}$ and $ab = min\{a,\ b\}$. Let Z_6 be the ring of integer of modulo 6. Then $S = Z_6 \oplus Z_6 \oplus R \oplus R$ is a semiring. Define $*$-on R via $(a_1,\ a_2,\ b_1,\ b_2)^* = (a_2,\ a_1,\ b_2,\ b_1)$.

Let $M = \{(i,\ j,\ m,\ n)/\ i \neq j;\ i,\ j \neq 0;\ m,\ n\ <\ 3\}$. Clearly M is a $*$-m-system but not a m-system because $(2,\ 3,\ 1,\ 2)x(3,\ 2,\ 1,\ 2) \notin M$ for all $x \in S$.

Theorem 2.5. *Let M be a non-void $*$-m-system in R and I be a $*$-ideal of R with $I \cap M = \phi$. Then I is contained in a $*$-prime ideal $P \neq R$ with $P \cap M = \phi$.*

Proof. Let $A = \{J\ /\ J\ is\ a\ *-ideal\ of\ R\ with\ I \subseteq J\ and\ J \cap M = \phi\}$. Clearly $A \neq \phi$. By Zorn's lemma, A contains a maximal element (say) P with $P \subseteq I$ and $P \cap M = \phi$. Let A, B be $*$-ideals of R such that $AB \subseteq P$. Suppose $A \nsubseteq P$ and $B \nsubseteq P$. Then there exists $a \in A$ and $b \in B$ such that $a, b \notin P$. Now $P \subset P + (< a > + < a >^*)$ and $P \subset P + (< b > + < b >^*)$ which gives $(P+ < a > + < a >^*) \cap M \neq \phi$ and $(P+ < b > + < b >^*) \cap M \neq \phi$. Then there exists $x \in (P+ < a > + < a >^*) \cap M$ and $y \in (P+ < b > + < b^* >) \cap M$ such that $xty \in M$ or $x^*ty \in M$ for some $t \in R$. Clearly $xty \in (P+ < a > + < a >^*)(P+ < b > + < b >^*)$ and $x^*ty \in (P+ < a > + < a >^*)(P+ < b > + < b >^*)$. Now $(P+ < a > + < a >^*)(P+ < b > + < b >^*) \subseteq P+ < a >< b > + < a >< b >^* + < a >^*< b > + < a >^*< b >^* \subseteq P + AB \subseteq P$. Then

$P \cap M \neq \phi$, a contradiction. Hence P is a $*$-prime ideal of R contains I. □

3. $*$-k-Prime Ideal

In this section, we continue our investigation of interrelations between various types of ideals in semiring with involution. Also, we introduce the notions of $*$-k-prime and $*$-m_k-system.

From [7], if I is any additive subsemigroup of R, then $\overline{I} = \{a \in R \ / \ a + x \in I \ for \ some \ x \in I\}$ is called k-closure of I. Observe that $I \subseteq \overline{I}$, $\overline{I^*} = \overline{I}^*$ and $\overline{\overline{I}} = \overline{I}$. It is easy to verify that if I is an ideal of R, then I is k-ideal if and only if $I = \overline{I}$. If I is an ideal of R, then $\overline{I}$ is an ideal of R. Observe that $\overline{< a >}$ is a principal k-ideal generated by a. Following [5], an ideal P is said to be k-prime if whenever A, B are k-ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. A $*$-ideal P of R is said to be $*$-k-prime if whenever $A,\ B$ are $*$-k-ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. From [5], a non-empty set M of elements of a semiring R is said to be m_k-system if $a, b \in M$, there exists $x \in \overline{< a >}$ and $y \in \overline{< b >}$ such that $xy \in M$. A non-empty set M of elements of a semiring R is said to be $*$-m_k-system if $a, b \in M$, there exists $x \in \overline{< a > + < a^* >}$ and $y \in \overline{< b > + < b^* >}$ such that $xy \in M$ or $x^*y \in M$. Observe that every m_k-system is a $*$-m_k-system. In Example 2.4 M is a $*$-m_k-system not an m_k-system. It is easy to see that if P is a $*$-ideal in R, then P is $*$-k-prime if and only if R/P is $*$-m_k-system. Also if P is an ideal of R, then P is k-prime if and only if R/P is an m_k-system. Let I be a additive subsemigroup of R and let $L(I) = \{x \ \in I \ / \ Rx \subseteq I\}$ and $H(I) = \{y \ \in L(I) \ / \ yR \ \subseteq L(I)\}$. Clearly $L(I)$ is a left ideal of R.

Lemma 3.1. *Let R be a semiring. If I is any additive subsemigroup of R, then $H(I)$ is the (unique) largest ideal of R contained in I.*

Proof. Clearly $H(I) \subseteq I$. From [10, Proposition 4], we have $H(I)$ is the largest ideal of R contained in I. □

It is well-known [7] that if I is an ideal of R, then $\overline{I}$ is the smallest k-ideal containing I.

Lemma 3.2. *Let R be a semiring. If I is a additive subsemigroup of R with $I = \overline{I}$, then I is an k-ideal of R or $H(I)$ is a k-ideal of R and it is the largest k-ideal contained in I.*

Proof. By Lemma 3.1, we have $H(I)$ is the largest ideal of R contained in I. Clearly $H(I) \subseteq \overline{H(I)}$ and $\overline{H(I)}$ is an ideal of R. Let $x \in \overline{H(I)}$. Then $x + h \in H(I)$ for some $h \in H(I)$. Since $H(I) \subseteq I$, we have $x + h \in I$ for some $h \in I$. Since $I = \overline{I}$, we have $x \in I$. Thus $\overline{H(I)} \subseteq I$. By Lemma 3.1, we have $\overline{H(I)} = I$ or $H(I) = \overline{H(I)}$. Hence I is a k-ideal of R or $H(I)$ is a k-ideal of R. □

Theorem 3.3. *Let R be a semiring and let P be a k-ideal of R. Then P is a prime ideal if and only if P is a k-prime ideal.*

Proof. If P is a prime then P is k-prime. Let A and B be ideals of R such that $AB \subseteq P$. From Lemma 2.1, we have $\overline{A}\ \overline{B} \subseteq P$. Then by assumption, we have $A \subseteq P$ or $B \subseteq P$. Hence P is a prime ideal. □

Theorem 3.4. *Let R be a semiring with involution and P be a $*$-k-ideal of R. Then P is $*$-prime if and only if P is $*$-k-prime.*

Theorem 3.5. *Let R be a semiring with involution and let P be a $*$-k-ideal of R. Then P is semiprime if and only if P is $*$-k-semiprime.*

Proof. If P is a semiprime ideal, then clearly P is $*$-k-semiprime.

Conversely, let P be a $*$-k-semiprime ideal, and let J be any ideal of R with $J^2 \subseteq P$. Also $(J^*)^2 \subseteq P$. Then $(J + J^*)^4 \subseteq P$. Since $(P : (J + J^*)^2)_l$ and $(P : (J + J^*)^2)_r$ are k-ideals of R, we have $\overline{(J + J^*)^2}\ \overline{(J + J^*)^2} \subseteq P$. By assumption, we have $\overline{(J + J^*)^2} \subseteq P$. Then $(J + J^*)^2 \subseteq P$. Again by using $(P : (J + J^*))_l$ and $(P : (J + J^*))_r$, we have $(\overline{J + J^*})^2 \subseteq P$. Then $J + J^* \subseteq P$. Thus $J \subseteq P$. Hence P is a semiprime ideal. □

Lemma 3.6. *Let R be a semiring with involution. If P is a k-prime and $*$-ideal of R, then P is $*$-k-prime.*

The converse of Lemma 3.6 is not true, in general as the following example shows.

Example 3.7. Consider the ring $A = Z_4$ of modulo 4 and semiring $B = B(4, 2)$ (F.E. Alarcon and D. Polkoska [5]).

+	0	1	2	3
0	0	1	2	3
1	1	2	3	2
2	2	3	2	3
3	3	2	3	2

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

Here $R = A \oplus A \oplus B \oplus B$ is a semiring. Define $*$-on R via $(a_1, a_2, b_1, b_2)^* = (a_2, a_1, b_2, b_1)$. Let $A_1 = \{0, 2\}$, $P = (A, A, A_1, A_1)$, $I = (A, A, B, 0)$ and $J = (A, A, 0, B)$. Then P is a $*$-k-prime ideal of R but not a k-prime ideal because of $IJ \subseteq P$ but neither $I \subseteq P$ nor $J \subseteq P$.

Theorem 3.8. *Let R be a semiring with involution and let P be a $*$-k-ideal of R. Then P is a $*$-k-prime ideal if and only if whenever $AB \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$ with either A or B is a $*$-k-ideal of R.*

Proof. Let P be a $*$-k-prime ideal of R. Without loss of generality, let us assume that A is a $*$-k-ideal of R and B is an ideal of R and $AB \subseteq P$. Then $B^*A \subseteq P$. Thus $(AB^*)^2 \subseteq P$. By Theorem 3.5, we have $AB^* \subseteq P$. Then $A(B + B^*) \subseteq P$. Since $(P : A)_r$ is a k-ideal of R, we have $A\overline{(B + B^*)} \subseteq P$. By assumption, we have $A \subseteq P$ or $\overline{(B + B^*)} \subseteq P$. Hence $A \subseteq P$ or $B \subseteq P$. Converse is clear. □

Theorem 3.9. *Let Q be a $*$-ideal of a semiring R with involution and let M be a $*$-m_k-system of R such that $\overline{Q} \cap M = \phi$. Then there exists a $*$-prime ideal $P \neq R$ such that $Q \subseteq P$ with $\overline{P} \cap M = \phi$.*

Proof. Let $A = \{J \ / \ J \ is \ *-ideal \ of \ R \ such \ that \ Q \subseteq J \ and \ \overline{J} \cap M = \phi\}$. Clearly $A \neq \phi$. By Zorn's Lemma, A contains a maximal element (say) P with $Q \subseteq P$ and $\overline{P} \cap M = \phi$. Let A and B be $*$-ideals of R such that $AB \subseteq P$. Suppose $A \nsubseteq P$ and $B \nsubseteq P$. Then there exists $a \in A$ and $b \in B$ with $a, \ b \notin P$. Thus $P \subset P + < a > + < a^* >$ and $P \subset P + < b > + < b^* >$. By maximality of P, we have $\overline{P + < a > + < a^* >} \cap M \neq \phi$ and $\overline{P + < b > + < b^* >} \cap M \neq \phi$. Then there exists $x \in \overline{P + < a > + < a^* >}$ and $y \in \overline{P + < b > + < b^* >}$ such that $x_1 y_1 \in M$ or $x_1^* y_1 \in M$ for some $x_1 \in \overline{< x > + < x^* >}$ and $y_1 \in \overline{< y > + < y^* >}$. Since $x \in \overline{P + < a > + < a^* >}$ and $y \in \overline{P + < b > + < b^* >}$, we have

$$x_1 y_1 \in (\overline{P + < a > + < a^* >})(\overline{P + < b > + < b^* >})$$

and

$$x_1^* y_1 \in (\overline{P + < a > + < a^* >})(\overline{P + < b > + < b^* >}).$$

Let $s \in (\overline{P+ < a > + < a^* >})(\overline{P+ < b > + < b^* >})$. Then $s = \sum_{i=1}^{n} t_i t_i^{'}$ for some $t_i \in \overline{P+ < a > + < a^* >}$ and $t_i^{'} \in \overline{P+ < b > + < b^* >}$. Thus $t_i + x_i \in (P+ < a > + < a^* >)$ and $t_i^{'} + x_i^{'} \in (P+ < b > + < b^* >)$ for $x_i \in (P+ < a > + < a^* >)$ and $x_i^{'} \in (P+ < b > + < b^* >)$ for each i. Clearly $(P+ < a > + < a^* >)(P+ < b > + < b^* >) \subseteq P$ and $x_i x_i^{'} \in P \subseteq \overline{P}$. Now Consider $x_i t_i^{'} + x_i x_i^{'} = x_i(t_i^{'} + x_i^{'}) \in (P+ < a > + < a^* >)(P+ < b > + < b^* >) \subseteq P$. Then $x_i t_i^{'} \in \overline{P}$ since $x_i x_i^{'} \in P$. Similarly, we can get $t_i x_i^{'} \in \overline{P}$.

Since $\overline{P}$ is an ideal of R, we have $t_i x_i^{'} + x_i t_i^{'} + x_i x_i^{'} \in \overline{P}$. Now $t_i t_i^{'} + x_i t_i^{'} + t_i x_i^{'} + x_i x_i^{'} = (t_i + x_i)(t_i^{'} + x_i^{'}) \in (P+ < a > + < a^* >)(P+ < b > + < b^* >) \subseteq P \subseteq \overline{P}$. Then $t_i t_i^{'} \in \overline{\overline{P}} = \overline{P}$ for each i. Thus $s \in \overline{P}$. Hence $(\overline{P+ < a > + < a^* >})(\overline{P+ < b > + < b^* >}) \subseteq \overline{P}$. So $x_1 y_1$ and $x_1^* y_1 \in \overline{P}$, a contradicts to $\overline{P} \cap M = \phi$. Hence P is a $*$-prime ideal of R contains Q. □

Theorem 3.10. *Let Q be a $*$-ideal of semiring with involution of R, and let M be a $*$-m_k-system of R such that $\overline{Q} \cap M = \phi$. Then there exists a $*$-k-prime ideal $P \neq R$ such that $Q \subseteq P$ with $\overline{P} \cap M = \phi$.*

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