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# GLOBAL STABILITY ANALYSIS FOR LASSA FEVER TRANSMISSION DYNAMICS WITH OPTIMAL CONTROL APPLICATION

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**Abstract:** A mathematical model for transmission dynamics of Lassa fever with optimal control application is presented. The existence of region where the model is epidemiologically feasible is established with respect to the use of pesticide control measure. We use personal protection control measure and basic reproduction number in linear and nonlinear Lyapunov functions together with the Lasalle's invariant principle to show that disease free and endemic equilibria are globally asymptotically stable. The existence and uniqueness of an optimality system are discussed. A characterization of the optimal control via adjoint variables is established. The possible impact of using combinations of the three controls either one at a time or two at a time or three at a time on the spread of the disease is also examined.

**AMS Subject Classification:** 92B05, 93A30
**Key Words:** mathematical model, reservoir host, global stability, reproduction number, optimal control

## 1. Introduction

Lassa fever is an acute viral illness that occur in West Africa. The virus, a member of the virus family arenaviridae is a zonotic or animal borne. In

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other words, the well-known Lassa fever is mostly caused by the Lassa virus. The symptoms include flu-like illness characterized by fever, general weakness, cough, sore throat, headache, and gastrointestinal manifestations, [12]. According to the World Health Organization, 300 000 to 500 000 cases of Lassa fever and 5000 deaths occur yearly across West Africa, [13]. The major and most common lesion of Lassa fever in humans occurs in the liver, [1, 2, 4, 5]. There are a number of ways in which the virus can be transmitted or spread to humans. The Mastomys rodents shed the virus in urine and droppings. Therefore, the virus can be transmitted through direct contact with these materials, touching objects or eating food contaminated with these materials, or through cuts or sores. Because Mastomys rodents often live in and around homes and scavenge on human food remains or poorly stored food, transmission of this sort is common. Contact with the virus also may occur when a person inhales tiny particles in the air contaminated with rodent excretions. This is called aerosol or airborne transmission.

Finally, because Mastomys rodents are sometimes consumed as a food source, infection may occur via direct contact when they are caught and prepared for food. Lassa fever may also spread through person-to-person contact. This type of transmission occurs when a person comes into contact with virus in the blood, tissue, secretions, or excretions of an individual infected with the Lassa virus. The virus cannot be spread through casual contact (including skin-to-skin contact without exchange of body fluids). Person-to-person transmission is common in both village and health care settings, where, along with the above-mentioned modes of transmission, the virus also may be spread in contaminated medical equipment, such as reused needles (this is called nosocomial transmission).

To put this research into proper perspective, we briefly give an account of some existing literatures on mathematical study of Lassa fever. D. Okuonghae and R. Okuonghae [8] discussed a mathematical model for the transmission of Lassa fever. Steady states of their model were examined for epidemic and endemic situations. The results of their model show that in the interim control of the rodents carrying the virus and some isolation policy for the infected individuals are the best strategies against the spread of the disease.

Bawa et al. [7] developed a mathematical model for Lassa fever transmission dynamics in two interacting population. They obtained the basic reproduction number and stability of the disease free equilibrium was established. The results of their work suggest that every effort must be put in place by all concerned to prevent the virus infection by reducing reproduction number.

James et al. [9] formulated a mathematical model for Lassa fever transmission dynamics. They obtained the basic reproduction number which can be used to control the transmission dynamics of the disease and conditions for local stability of the disease free equilibrium was established.

Onuorah, Akinwande et al. [10] developed a mathematical model for Lassa fever as a six dimensional system of nonlinear ordinary differential equation with rigorous analyzes. The results of their analysis and numerical simulation show the effects of the control parameters on the various compartments of the model and conclude that if the basic reproduction number is low the disease will still continue to spread.

## 2. A Model for Optimal Control of Lassa Fever

We introduce the control functions $u_1(t)$, $u_2(t)$, and $u_3(t)$ for prevention, treatment and use of pesticide respectively on a time interval $[0, T]$. Known practices of prevention are use of rodent-proof container, use of infection control measure such as complete equipment sterilization, improving home hygiene and strict barrier nursing such as masks, gloves, gowns and goggles to prevent human to human contact. Ribavirin the antiviral drug is effective in the treatment of Lassa fever but only if administered early in the course of illness, [6].

Consequently, $1 - u_1(t)$ describes the failure of prevention effort for $t \geq 0$; $\alpha_1$ and $\alpha_2$ are the treatment rates of exposed and infected class respectively. We assumed that $0 \leq u_2(t) \leq 1$ to eliminate the case where the entire infected and exposed classes are treated effectively. Use of pesticide in and around homes can help reduce rodent population denoted by $1 - u_3(t)$. The model subdivides the total human population size at time t and discrete age $a_i$ denoted by $N_h(t, a_i)$ with $i = 0, 1, 2, ..., L$ and $a_L$ is the maximum age of humans in the population, into susceptible humans $S_h(t, a_i)$, exposed humans $E_h(t, a_i)$, infected humans $I_h(t, a_i)$ and recovered humans $R_h(t, a_i)$. Hence we have $N_h(t, a_i) = S_h(t, a_i) + E_h(t, a_i) + I_h(t, a_i) + R_h(t, a_i)$. A loss of individuals is as a result of infection and natural death $\mu_h(a_i)S_h(t, a_i)$. The exposed human gain individuals through infection and loses individual when they become infected $\epsilon_h(a_i)E_h(t, a_i)$ and to natural death $\mu_h(a_i)E_h(t, a_i)$. The infected human $I_h(t, a_i)$ gain individuals when exposed individuals becomes infected and loses individual when they die $\mu_h(a_i)I_h(t, a_i)$ and disease induced death $\delta_h(a_i)I_h(t, a_i)$. After some time, exposed and infected human recovers and moves to the recovered class $R_h(t, a_i)$. However recovered human has permanent immunity and never go back to susceptible class again. A loss of individuals is as a result of natural death $\mu_h R_h(t, a_i)$. It is assumed that the new birth of susceptible human $S_h(t,a_i)$ are susceptible, $\theta(e_j) \propto \frac{C_v}{K_v}$ and $\theta(e_j)$ is generated from urine and faeces of infectious rodents, where $C_v$ is the amount of virus in air and $K_v$ is the saturation of virus in air. Similarly, $\kappa(a_i)$ is generated from blood of infectious individuals, $\kappa(a_i) \propto \frac{A_v}{S_v}$, where $A_v$ is the amount of virus in needle and $S_v$ is the saturation of virus in needle.

In a similar manner, we subdivides the total rodent population size at time t and discrete age $e_j$ denoted by $N_r(t,e_j)$ with $j = 0,1,2,...,T$ and $e_T$ is the maximum age of rodents in the population, into susceptible rodents $S_r(t,e_j)$, exposed rodents $E_r(t,e_j)$ and infected rodents $I_r(t,e_j)$. Hence we have $N_r(t,e_j) = S_r(t,e_j)+E_r(t,e_j)+I_r(t,e_j)$. Susceptible rodent class $S_r(t,e_j)$ gain more individual into rodent population by input rate $\Lambda_r(e_j)$, while it loses rodents through infection, natural death $\mu_r(e_j)S_r(t,e_j)$, hunting $\delta_r(e_j)S_r(t,e_j)$. Transmission of Lassa virus to susceptible rodents occurs when they share unprotected storage of garbage, food stuff and water with infected rodents or from inhalation of aerosols from urine. When a susceptible rodent interacts with infectious rodent, the virus enters the rodent with probability $\beta(e_j)$ and therefore the susceptible go to the exposed class $E_r(t,e_j)$. The exposed rodent then becomes infectious and enters the class $I_r(t,e_i)$ after a given time. It is assumed that the recruitment rate of rodent is greater than rodent's number of death at initial time $(\Lambda_r(e_j) \geq \mu_r(e_j)N_r(0,e_j))$. In this study, it is assumed that individuals who recovered from Lassa fever will never go back to susceptible class again (they remain recovered for life). Thus, the transition dynamic is given by:

$$\begin{aligned}\frac{dS_h(t,a_i)}{dt} &= \Lambda_h(a_i) \\ &- \sum_{i=0}^{L}\sum_{j=0}^{T}\left(\frac{\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\eta(a_i)\sigma_2(a_i)I_h(t,a_i)+\theta(e_j)+\kappa(a_i)}{N_h(t,a_i)}\right) \\ &\times S_h(t,a_i)(1-u_1(t)) - \mu_h(a_i)S_h(t,a_i), \quad (1)\end{aligned}$$

$$\begin{aligned}\frac{dE_h(t,a_i)}{dt} &= \\ &\sum_{i=0}^{L}\sum_{j=0}^{T}\left(\frac{\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\eta(a_i)\sigma_2(a_i)I_h(t,a_i)+\theta(e_j)+\kappa(a_i)}{N_h}\right) \\ &\times S_h(t,a_i)(1-u_1(t)) - (\gamma(a_i)\alpha_1(a_i)u_2(t)+\epsilon_h(a_i)+\mu_h(a_i))E_h(t,a_i), \quad (2)\end{aligned}$$

$$\frac{dI_h(t,a_i)}{dt} = \sum_{i=0}^{L} \epsilon_h(a_i)E_h(t,a_i) - (\psi(a_i)\alpha_2(a_i)u_2(t) + \mu_h(a_i) + \delta_h(a_i))I_h(t,a_i), \quad (3)$$

$$\frac{dR_h(t,a_i)}{dt} = \sum_{i=0}^{L}[\gamma(a_i)\alpha_1(a_i)u_2E_h(t,a_i) + \psi(a_i)\alpha_2(a_i)u_2(t)I_h(t,a_i)] - \mu_h(a_i)R_h(t,a_i), \quad (4)$$

$$\frac{dS_r(t,e_j)}{dt} = \Lambda_r(e_j) - \sum_{j=0}^{T}\left(\frac{\beta(e_j)\sigma_1(e_j)I_r(t,e_j) + \theta(e_j)}{N_r(t,e_j)}\right)(1-u_3(t))S_r(t,e_j) - (\mu_r(e_j) + \delta_r(e_j) + (1-u_3(t)))S_r(t,e_j), \quad (5)$$

$$\frac{dE_r(t,e_j)}{dt} = \sum_{j=0}^{T}\left(\frac{\beta(e_j)\sigma_1(e_j)I_r(t,e_j) + \theta(e_j)}{N_r(t,e_j)}\right)(1-u_3(t))S_r(t,e_j) - (\epsilon_r(e_j) + \mu_r(e_j) + \delta_r(e_j) + (1-u_3(t)))E_r(t,e_j), \quad (6)$$

$$\frac{dI_r(t,e_j)}{dt} = \sum_{j=0}^{T} \epsilon_r(e_j)E_r(t,e_j) - (\mu_r(e_j) + \delta_r(e_j) + (1-u_3(t)))I_r(t,e_j). \quad (7)$$

### 2.1. Existence of Solutions

First, we obtain boundedness of the state system given an optimal control set $\mathcal{U}$.

**Theorem 2.1.** *Given $u_1(t), u_2(t)$ and $u_3(t) \in \mathcal{U}$, the state equations (2.1)-(2.7) has a bounded positive solution on feasible invariant region $\mathcal{R}$ defined by* $\big\{S_h(t,a_i), E_h(t,a_i), I_h(t,a_i), R_h(t,a_i), S_r(t,e_j), E_r(t,e_j), I_r(t,e_j) \in \mathcal{R}^7 :$

$$\left.\begin{aligned} N_h(0,a_i) &\le N_h(t,a_i) \le \sum_{i=0}^{L} \frac{\Lambda_h(t,a_i)}{\mu_h(a_i)}, \\ N_r(0,e_j) &\le N_r(t,e_j) \le \sum_{j=0}^{T} \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3)} \end{aligned}\right\}$$

*with initial conditions* $S_h(0,a_i) \ge 0, E_h(0,a_i) \ge 0, I_h(0,a_i) \ge 0,\ R_h(0,a_i) \ge 0,$ $S_r(0,e_j) \ge 0, E_r(0,e_j) \ge 0, I_r(0,e_j) \ge 0.$

*Proof.* If the total human population size is given by $N_h(t,a_i) = S_h(t,a_i) + E_h(t,a_i)+I_h(t,a_i)+R_h(t,a_i)$ and the total size of rodent population is $N_r(t,e_j) = S_r(t,e_j) + E_r(t,e_j) + I_r(t,e_j)$, then the system (2.5)-(2.7) gives

$$\frac{dN_r(t,e_j)}{dt} \le \Lambda_r(e_j) - \sum_{j=0}^{T}(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t)))N_r(t,e_j), \tag{8}$$

and therefore the equation (2.8) leads to

$$\begin{aligned} &N_r(t,e_j)e^{(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t} \\ &\quad \le \sum_{j=0}^{T} N_r(0,e_j) + \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3)} e^{(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t} \\ &\qquad - \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))}, \end{aligned}$$

so that

$$\begin{aligned} N_r(t,e_j) \le \sum_{j=0}^{T} &N_r(0,e_j)e^{-(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t} \\ &+ \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))} \\ &- \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta(e_j)+(1-u_3(t))} e^{-(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t}. \end{aligned}$$

This implies

$$\begin{aligned} N_r(t,e_j) \le \sum_{j=0}^{T} &\frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))} \\ &\times (1-e^{-(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t}) \\ &+ N_r(0,e_j)e^{-(\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))t}. \end{aligned}$$

Taking the limit as $t \to \infty$ gives

$$N_r(t,e_j) \leq \sum_{j=0}^{T} \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))}.$$

Similarly, from equations (2.1)-(2.4) we obtain

$$\frac{dN_h(t,a_i)}{dt} \leq \Lambda_h(a_i) - \sum_{i=0}^{L} \mu_h(a_i) N_h(t,a_i), \tag{9}$$

and the inequality (2.9) gives

$$N_h(t,a_i)e^{\mu_h(a_i)t} \leq \sum_{i=0}^{L} N_h(0,a_i) + \frac{\Lambda_h(a_i)}{\mu_h(a_i)} e^{\mu_h(a_i)t} - \frac{\Lambda_h(a_i)}{\mu_h(a_i)},$$

so that

$$N_h(t,a_i) \leq \sum_{i=0}^{L} N_h(0,a_i)e^{-\mu_h(a_i)t} + \frac{\Lambda_h(a_i)}{\mu_h(a_i)} - \frac{\Lambda_h(a_i)}{\mu_h(a_i)} e^{-\mu_h(a_i)t}.$$

This implies

$$N_h(t,a_i) \leq \sum_{i=0}^{L} \frac{\Lambda_h(a_i)}{\mu_h(a_i)}(1-e^{-\mu_h(a_i)t}) + N_h(0,a_i)e^{-\mu_h(a_i)t}.$$

Taking the limit as $t \to \infty$ gives $N_h(t,a_i) \leq \sum_{i=0}^{L} \frac{\Lambda_h(t,a_i)}{\mu_h(a_i)}$. Thus, we have the following feasible region

$$\mathcal{R} = \{S_h(t,a_i), E_h(t,a_i), I_h(t,a_i), R_h(t,a_i), S_r(t,e_j), E_r(t,e_j), I_r(t,e_j) \in \mathcal{R}^7 :$$
$$N_r(t,e_j) \leq \sum_{j=0}^{T} \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3(t))},\ N_h(t,a_i) \leq \sum_{i=0}^{L} \frac{\Lambda_h(a_i)}{\mu_h(a_i)}\}. \qquad \square$$

### 2.2. Disease-Free Equilibrium Stable and Reproduction Number

The system (2.1)-(2.7) has a disease-free equilibrium given by

$$\pi_0 = \left( \frac{\Lambda_h(a_i)}{\mu_h(a_i)}, 0, 0, 0, \frac{\Lambda_r(e_j)}{\mu_r(e_j)+\delta_r(e_j)+(1-u_3)}, 0, 0 \right). \tag{10}$$

Using the next generation matrix operator approach, we obtain the basic reproduction number as,

$$\mathcal{R}_0(a) = \sum_{i=0}^{L} \frac{\eta(a_i)\sigma_2(a_i)\epsilon_h(a_i)}{(\gamma(a_i)\alpha_1(a_i)+\epsilon_h(a_i)+\mu_h(a_i))(\psi(a_i)\alpha_2(a_i)+\mu_h(a_i)+\delta_h(a_i))}. \tag{11}$$

### 2.3. Global Stability of Disease Free Equilibrium

In this sub-section, we show that disease free equilibrium is globally asymptotically stable (GAS) with respect to only the range of prevention control.

**Theorem 2.2.** *The disease-free equilibrium $\pi_0$ of the model (2.1)-(2.7), is globally asymptotically stable in $\mathcal{R}$ if $\mathcal{R}_0(a) \leq 1$ for all $a$ and $0 \leq u_1(t) \leq 1$ for all $t$.*

*Proof.* To determine global stability of the disease-free state, consider the following linear Lyapunov function

$$\begin{aligned}\mathcal{L} = &\frac{\epsilon_h(a_i)E_h(t,a_i)}{(\mu_r(e_j)+\delta_r(e_j))(\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2+\mu_h(a_i))q_t} \\ &+ \frac{I_h(t,a_i)}{(\mu_r(e_j)+\delta_r(e_j))(\psi(a_i)\alpha_2(a_i)u_2+\mu_h(a_i)+\delta_h(a_i))} \\ &+ \frac{E_r(t,e_j)}{\Lambda_r(e_j)(\beta(e_j)\sigma_1(e_j)+\theta(e_j))} + \frac{I_r(t,e_j)(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))}{(\beta(e_j)\sigma_1(e_j)+\theta(e_j))\Lambda_r(e_j)\epsilon_r(e_j)}, \end{aligned} \quad (12)$$

where $q_t = (\psi(a_i)\alpha_2(a_i)u_2 + \mu_h(a_i) + \delta_h(a_i))$.

The Lyapunov derivative (2.12) is given by

$$\begin{aligned}\dot{\mathcal{L}} = &\sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\epsilon_h(a_i)(\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j)+\kappa(a_i))(1-u_1)}{(\mu_r(e_j)+\delta_r(e_j))(\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2+\mu_h(a_i))q_t} \\ &+ \sum_{i=0}^{L} \frac{\eta(a_i)\sigma_2(a_i)\epsilon_h(a_i)I_h(t,a_i)(1-u_1)}{(\mu_r(e_j)+\delta_r(e_j))(\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2+\mu_h(a_i))q_t} \\ &- \frac{(\mu_r(e_j)+\delta_r(e_j))(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))I_r(t,e_j))}{(\beta(e_j)\sigma_1(e_j)+\theta(e_j))\Lambda_r(e_j)\epsilon_r(e_j)} \\ &- \frac{I_h(t,a_i)}{\mu_r(e_j)+\delta_r(e_j)} + \sum_{j=0}^{T} \frac{(\beta(e_j)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j))(1-u_3)}{\Lambda_r(e_j)(\beta(e_j)\sigma_1(e_j)+\theta(e_j))}, \end{aligned}$$

$$\begin{aligned}\dot{\mathcal{L}} = &\sum_{i=0}^{L}\sum_{j=0}^{T} \frac{\epsilon_h(a_i)(\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j)+\kappa(a_i))(1-u_1)}{(\mu_r(e_j)+\delta_r(e_j))(\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2+\mu_h(a_i))q_t} \\ &+ \frac{(1-u_1)\mathcal{R}_0(a)I_h(t,a_i)}{\mu_r(e_j)+\delta_r(e_j)} - \frac{I_h(t,a_i)}{\mu_r(e_j)+\delta_r(e_j)} \\ &+ \sum_{j=0}^{T} \frac{(\beta(e_j)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j))(1-u_3)}{\Lambda_r(e_j)(\beta(e_j)\sigma_1(e_j)+\theta(e_j))} \end{aligned}$$

$$- \frac{(\mu_r(e_j) + \delta_r(e_j))(\epsilon_r(e_j) + \mu_r(e_j) + \delta_r(e_j))I_r(t, e_j)}{(\beta(e_j)\sigma_1(e_j) + \theta(e_j))\Lambda_r(e_j)\epsilon_r(e_j)},$$

$$\dot{\mathcal{L}} \leq \frac{(1 - u_1)\mathcal{R}_0(a)I_h(t, a_i)}{\mu_r(e_j) + \delta_r(e_j)} - \frac{I_h(t, a_i)}{\mu_r(e_j) + \delta_r(e_j)}$$
$$\implies \dot{\mathcal{L}} \leq \frac{1}{\mu_r(e_j) + \delta_r(e_j)}((1 - u_1)\mathcal{R}_0(a) - 1)I_h(t, a_i).$$

There, $\dot{\mathcal{L}} \leq 0$ for $\mathcal{R}_0(a) \leq 1$ and $0 \leq u_1 \leq 1$. This is in sharp contrast with the results from many authors. Furthermore, $\dot{\mathcal{L}} = 0$ if and only if $I_h(t, a_i) = 0$. Thus $\pi_0$ is globally asymptotically stable in $\mathcal{R}$ if $R_0(a) \leq 1$ for all $a$ and $0 \leq u_1 \leq 1$ for all $t$.

### 2.4. Global Stability of Endemic Equilibrium

In this sub-section, we investigate the global stability of endemic equilibrium of the model (2.1)-(2.7) in the range of prevention and vector (rodent) control.

**Theorem 2.3.** *The unique endemic equilibrium, $E_e$, of the model (2.1)-(2.7) is globally asymptotically stable in $\mathcal{R}$ if $\mathcal{R}_0(a) > 1$, $0 \leq u_1(t) \leq 1$ and $0 \leq u_3(t) \leq 1$.*

*Proof.* Let $\mathcal{R}_0(a) > 1$, $0 \leq u_1(t) \leq 1$ and $0 \leq u_3(t) \leq 1$ so that a unique endemic equilibrium exists and consider the following nonlinear Lyapunov function

$$\begin{aligned}\mathcal{L} = S_h(t, a_i) - S_h^*(a_i) - S_h^*(a_i) \ln\left(\frac{S_h(t, a_i)}{S_h^*(a_i)}\right) \\ + E_h(t, a_i) - E_h^*(a_i) - E_h^*(a_i) \ln\left(\frac{E_h(t, a_i)}{E_h^*(a_i)}\right) \\ + \frac{(\gamma(a_i)\alpha_1(a_i)u_2 + \epsilon_h(a_i) + \mu_h(a_i))}{\epsilon_h(a_i)} \\ \times \left[I_h(t, a_i) - I_h^*(a_i) - I_h^*(a_i) \ln\left(\frac{I_h(t, a_i)}{I_h^*(a_i)}\right)\right] \\ + S_r(t, e_j) - S_r^*(e_j) - S_r^*(e_j) \ln\left(\frac{S_r(t, e_j)}{S_r^*(e_j)}\right) \\ + E_r(t, e_j) - E_r^*(e_j) - E_r^*(e_j) \ln\left(\frac{E_r(t, e_j)}{E_r^*(e_j)}\right)\end{aligned}$$

$$+\frac{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))}{\epsilon_r(e_j)} \times\left[I_r(t,e_j)-I_r^*(e_j)-I_r^*(e_j)\ln\left(\frac{I_r(t,e_j)}{I_r^*(e_j)}\right)\right]. \quad (13)$$

The Lyapunov derivative of (2.13) is given by

$$\begin{aligned}
\dot{\mathcal{L}} &= \sum_{i=0}^{L}\Lambda_h(a_i)\left(1-\frac{S_h^*(a_i)}{S_h(t,a_i)}\right)-\sum_{i=0}^{L}\mu_h S_h(t,a_i)\left(1-\frac{S_h^*(a_i)}{S_h(t,a_i)}\right)\\
&+\sum_{i=0}^{L}\sum_{j=0}^{T}(\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\eta(a_i)\sigma_2(a_i)I_h(t,a_i)+\theta(e_j)+\kappa(a_i))\\
&\qquad\times S_h^*(a_i)f(N_h)(1-u_1(t))\\
&-\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{(\rho(a_i)\sigma_1(e_j)I_r(t,e_j)+\eta(a_i)\sigma_2(a_i)I_h(t,a_i)+\theta(e_j)+\kappa(a_i))f_t}{E_h(t,a_i)}\\
&-\sum_{i=0}^{L}\frac{(\gamma(a_i)\alpha_1(a_i)u_2(t)+\mu_h(a_i)+\epsilon_h(a_i))q_t I_h(t,a_i)}{\epsilon_h(a_i)}
\end{aligned}$$

$$\begin{aligned}
&+\sum_{i=0}^{L}(\epsilon_h(a_i)+\mu_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2(t))E_h^*(a_i)\\
&+\sum_{i=0}^{L}\frac{(\gamma(a_i)\alpha_1(a_i)u_2(t)+\mu_h(a_i)+\epsilon_h(a_i))q_t I_h^*(a_i)}{\epsilon_h(a_i)}\\
&-\sum_{i=0}^{L}\frac{(\gamma(a_i)\alpha_1(a_i)u_2(t)+\mu_h(a_i)+\epsilon_h(a_i))I_h^*(a_i)E_h(t,a_i)}{I_h(t,a_i)}
\end{aligned}$$

$$\begin{aligned}
&+\sum_{j=0}^{T}\Lambda_r(e_j)\left(1-\frac{S_r^*(e_j)}{S_r(t,e_j)}\right)-\sum_{j=0}^{T}(\mu_r(e_j)+\delta_r(e_j))S_r(t,e_j)\left(1-\frac{S_r^*(e_j)}{S_r(t,e_j)}\right)\\
&+\sum_{j=0}^{T}(\beta(e_j)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j))S_r^*(e_j)f(N_r)(1-u_3(t))\\
&-\sum_{j=0}^{T}\frac{(\beta(e_j)\sigma_1(e_j)I_r(t,e_j)+\theta(e_j))S_r(t,e_j)E_r^*(e_j)f(N_r)(1-u_3(t))}{E_r(t,e_j)}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=0}^{T}\frac{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))(\mu_r(e_j)+\delta_r(e_j))I_r(t,e_j)}{\epsilon_r(e_j)}\\
&+\sum_{j=0}^{T}(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))E_r^*(e_j)\\
&-\sum_{j=0}^{T}\frac{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))I_r^*(e_j)E_r(t,e_j)}{I_r(t,e_j)}\\
&+\sum_{j=0}^{T}\frac{(\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j))(\mu_r(e_j)+\delta_r(e_j))I_r^*(e_j)}{\epsilon_r(e_j)},
\end{aligned}
\tag{14}
$$

where $q_t=(\psi(a_i)\alpha_2(a_i)u_2(t)+\mu_h(a_i)+\delta_h(a_i))$,
and $f_t=E_h^*(a_i)f(N_h)S_h(t,a_i)(1-u_1(t))$.

At for the endemic equilibrium, it is seen from (2.1)-(2.7) that

$$
\Lambda_h(a_i)=[A(1-u_1(t))S_h^*(a_i)f(N_h^*)+\mu_h(a_i)S_h^*(a_i)]
$$

$$
\epsilon_h(a_i)+\gamma(a_i)\alpha_1(a_i)u_2(t)+\mu_h(a_i)=\frac{A(1-u_1(t))S_h^*(a_i)f(N_h^*)}{E_h^*(a_i)}
$$

$$
\psi(a_i)\alpha_2(a_i)u_2(t)+\mu_h(a_i)+\delta_h(a_i)=\frac{\epsilon_h(a_i)E_h^*(a_i)}{I_h^*(a_i)}
$$

$$
\Lambda_r(e_j)=[B(1-u_3(t))S_r^*(e_j)f(N_r^*)+S_r^*(e_j)(\mu_r(e_j)+\delta_r(e_j))]
$$

$$
\begin{aligned}
&\epsilon_r(e_j)+\mu_r(e_j)+\delta_r(e_j)\\
&\qquad=\frac{(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))(1-u_3(t))S_r^*(e_j)f(N_r^*)}{E_r^*(e_j)}
\end{aligned}
$$

$$
\mu_r(e_j)+\delta_r(e_j)=\frac{\epsilon_r E_r^*(e_j)}{I_r^*(e_j)},
\tag{15}
$$

where:

$$
A=(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))
$$

and

$$B=(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j)).$$

Using (2.15) in (2.14), and then adding and subtracting the following systematically:

$$\sum_{i=0}^{L}\sum_{j=0}^{T}(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))S^0,$$

$$\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))c_t}{I_h^*(a_i)f(N_h)},$$

$$\sum_{j=0}^{T}(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f(N_r^*)(1-u_3(t)),$$

$$S^0=S_h^*(a_i)f(N_h^*)(1-u_1(t))$$

one gets

$$\begin{aligned}\dot{\mathcal{L}}=\sum_{i=0}^{L}\mu_h(a_i)S_h^*(a_i)\left(2-\frac{S_h^*(a_i)}{S_h(t,a_i)}-\frac{S_h(t,a_i)}{S_h^*(a_i)}\right)\\
+\sum_{i=0}^{L}\sum_{j=0}^{T}(\rho(a_i)\sigma_1(a_i)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)\\
+\theta(e_j)+\kappa(a_i))S_h^*(a_i)f(N_h^*)(1-u_1(t))\times\end{aligned}$$

$$\begin{aligned}&\left[4-\frac{S_h^*(a_i)}{S_h(t,a_i)}-\frac{E_h^*(a_i)S_h(t,a_i)f(N_h)}{E_h(t,a_i)S_h^*(a_i)f(N_h^*)}-\frac{I_h^*(a_i)E_h(t,a_i)}{E_h^*(a_i)I_h(t,a_i)}-\frac{I_h(t,a_i)f(N_h^*)}{I_h^*(a_i)f(N_h)}\right]\\
&+\sum_{i=0}^{L}\sum_{j=0}^{T}(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))f^0\\
&+\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))c_t}{I_h^*(a_i)f(N_h)}\\
&-\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))B_t}{I_h^*(a_i)}\end{aligned}$$

$$+(\mu_r(e_j)+\delta_r(e_j))S_r^*(e_j)\left(2-\frac{S_r^*(e_j)}{S_r(t,e_j)}-\frac{S_r(t,e_j)}{S_r^*(e_j)}\right)$$
$$+\sum_{j=0}^{T}(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f(N_r^*)(1-u_3(t))\times$$

$$\left[4-\frac{S_r^*(e_j)}{S_r(t,e_j)}-\frac{E_r^*(e_j)S_r(t,e_j)f(N_r)}{E_r(t,e_j)S_r^*(e_j)f^*}-\frac{I_r^*(e_j)E_r(t,e_j)}{E_r^*(e_j)I_r(t,e_j)}-\frac{I_r(t,e_j)f(N_r^*)}{I_r^*(e_j)f(N_r)}\right]$$
$$+\sum_{j=0}^{T}(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f(N_r)(1-u_3(t))$$
$$+\sum_{j=0}^{T}\frac{(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f^2(N_r^*)I_r(t,e_j)(1-u_3(t))}{I_r^*(e_j)f(N_r)}$$
$$-\sum_{j=0}^{T}\frac{(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f(N_r^*)I_r(t,e_j)(1-u_3(t))}{I_r^*(e_j)},$$

where

$$ct=S_h^*(a_i)f^2(N_h^*)I_h(t,a_i)(1-u_1(t)),$$
$$B_t=S_h^*(a_i)f(N_h^*)I_h(t,a_i)(1-u_1(t)),$$
$$f^0=S_h^*(a_i)f(N_h)(1-u_1(t)).$$

Further, algebraic manipulations give

$$\begin{aligned}\dot{\mathcal{L}}=&-\mathcal{L}_1-\mathcal{L}_2\\&-\sum_{i=0}^{L}\sum_{j=0}^{T}(\rho(a_i)\sigma_1(e_j)I_r^*(e_j)+\eta(a_i)\sigma_2(a_i)I_h^*(a_i)+\theta(e_j)+\kappa(a_i))\\&\times S_h^*(a_i)f(N_h^*)(1-u_1)\left[1-\frac{f(N_h)}{f(N_h^*)}+\frac{I_h(t,a_i)}{I_h^*(a_i)}-\frac{I_h(t,a_i)f(N_h^*)}{I_h^*(a_i)f(N_h)}\right]\\&-\mathcal{L}_3-\mathcal{L}_4-\sum_{j=0}^{T}(\beta(e_j)\sigma_1(e_j)I_r^*(e_j)+\theta(e_j))S_r^*(e_j)f(N_r^*)(1-u_3(t))\\&\times\left[1-\frac{f(N_r)}{f(N_r^*)}+\frac{I_r(t,e_j)}{I_r^*(e_j)}-\frac{I_r(t,e_j)f(N_r^*)}{I_r^*(e_j)f(N_r)}\right],\end{aligned}\tag{16}$$

where: $f(N_h) = \dfrac{1}{N_h(t,a_i)}$, $f(N_r) = \dfrac{1}{N_r(t,e_j)}$,

$$\mathcal{L}_1 = \sum_{i=0}^{L} \mu_h(a_i) S_h^* \left( \frac{S_h^*(a_i)}{S_h(t,a_i)} + \frac{S_h(t,a_i)}{S_h^*(a_i)} - 2 \right),$$

$$\mathcal{L}_2 = \sum_{i=0}^{L} \sum_{j=0}^{T} (\rho(a_i)\sigma_1(a_i) I_r^*(e_j) + \eta(a_i)\sigma_2(a_i) I_h^*(a_i) + \theta(e_j) + \kappa(a_i))$$

$$(1 - u_1(t)) S_h^*(a_i) f(N_h^*)$$

$$\times \left[ \frac{S_h^*(a_i)}{S_h(t,a_i)} + \frac{E_h^*(a_i) S_h(t,a_i) f(I_h)}{E_h(t,a_i) S_h^*(a_i) f(N_h^*)} + \frac{I_h^*(a_i) E_h(t,a_i)}{E_h^*(a_i) I_h(t,a_i)} + \frac{I_h(t,a_i) f(N_h^*)}{I_h^*(a_i) f(N_h)} - 4 \right],$$

$$\mathcal{L}_3 = \sum_{j=0}^{T} (\mu_r(e_j) + \delta_r(e_j)) S_r^* \left( \frac{S_r^*(e_j)}{S_r(t,e_j)} + \frac{S_r(t,e_j)}{S_r^*(e_j)} - 2 \right),$$

$$\mathcal{L}_4 = \sum_{j=0}^{T} (\beta(e_j)\sigma_1(e_j) I_r^*(e_j) + \theta(e_j)) S_r^*(e_j) f(N_r^*)(1 - u_3(t)) \times$$

$$\left[ \frac{S_r^*(e_j)}{S_r(t,e_j)} + \frac{E_r^*(e_j) S_r(t,e_j) f(N_r)}{E_r(t,e_j) S_r^*(e_j) f(N_r^*)} + \frac{I_r^*(e_j) E_r(t,e_j)}{E_r^*(e_j) I_r(t,e_j)} + \frac{I_r(e_j) f(N_r^*)}{I_r^*(e_j) f(N_r)} - 4 \right].$$

We need to show that $\mathcal{L}_1 \geq 0$, $\mathcal{L}_2 \geq 0$, $\mathcal{L}_3 \geq 0$ and $\mathcal{L}_4 \geq 0$. To do this, using the fact that the arithmetic mean is greater than or equal to the geometric mean (AM - GM inequality), we have

$$(S_h^*(a_i))^2 + (S_h(t,a_i))^2 - 2S_h^*(a_i) S_h(t,a_i) \geq 0$$

so that, $\left( \dfrac{S_h^*(a_i)}{S_h(t,a_i)} + \dfrac{S_h(t,a_i)}{S_h^*(a_i)} - 2 \right) \geq 0$. Hence, $\mathcal{L}_1 \geq 0$.

Further, let $x = \dfrac{S_h^*(a_i)}{S_h(t,a_i)}$, $y = \dfrac{E_h^*(a_i) f(N_h)}{E_h(t,a_i) f(N_h^*)}$, $z = \dfrac{I_h^*(a_i) f(N_h^*)}{I_h(t,a_i) f(N_h^*)}$.

Then,

$$\left[ \frac{S_h^*(a_i)}{S_h(t,a_i)} + \frac{E_h^*(a_i) S_h(t,a_i) f(N_h)}{E_h(t,a_i) S_h^*(a_i) f(N_h^*)} + \frac{I_h^*(a_i) E_h(t,a_i)}{E_h^*(a_i) I_h(t,a_i)} + \frac{I_h(t,a_i) f(N_h^*)}{I_h^*(a_i) f(N_h^*)} - 4 \right]$$

can be written as

$$f(x,y,z) = x + \frac{y}{x} + \frac{z}{y} + \frac{1}{z} - 4. \tag{17}$$

It suffices to show that $f(x,y,z) \geq 0$. Since $f_x = f_y = f_z = 0$ gives rise to $x = y = z$ and that $f_{xx} > 0$, $f_{yy} > 0$, $f_{zz} > 0$, one see that the minimum of $f(x,y,z)$ is attainable at $x = y = z$. In what follows, (2.17) is reduced to $(x-1)^2 \geq 0$ or $(y-1)^2 \geq 0$ or $(z-1)^2 \geq 0$ with equality if and only if $x = 1$ or $y = 1$ or $z = 1$ respectively. Hence, $\mathcal{L}_2 \geq 0$. The proof of $\mathcal{L}_3 \geq 0$ is similar to $\mathcal{L}_1 \geq 0$ while that of $\mathcal{L}_4 \geq 0$ is similar to $\mathcal{L}_2 \geq 0$, it follows from (2.16) that $\dot{\mathcal{L}} \leq 0$ with $\dot{\mathcal{L}} = 0$ if and only if $S_h(t,a_i) = S_h^*(a_i), E_h(t,a_i) = E_h^*(a_i), I_h(t,a_i) = I^*(a_i), S_r(t,e_j) = S_r^*(e_j), E_r(t,e_j) = E_r^*(e_j), I_r(t,e_j) = I_r^*(e_j)$, $0 \leq u_1(t) \leq 1$, $0 \leq u_3(t) \leq 1$. This further implies that

$$R_h(t,a_i) = \frac{\gamma(a_i)\alpha_1(a_i)E_h^{**}(a_i) + \psi(a_i)\alpha_1(a_i)I_h^{**}(a_i)}{\mu_h(a_i)} = R_h^{**}(a_i),$$

since $(S_h(t,a_i), E_h(t,a_i), I_h(t,a_i), S_r(t,e_j), E_r(t,e_j), I_r(t,e_j))$ tends to $(S_h^*(a_i), E_h^*(a_i), I_h^*(a_i), S_r^*(e_j), E_r^*(e_j), I_r^*(e_j))$ as $t \to \infty$. Therefore by LaSalle's principle, the largest compact invariant subset of the set where $\dot{\mathcal{L}} = 0$ is the endemic equilibrium point $E_e$. Thus, every solution in $\mathcal{R}$ approaches $E_e$ for $\mathcal{R}_0(a) > 1$, $0 \leq u_1(t) \leq 1$, $0 \leq u_3(t) \leq 1$ and $E_e$ is globally asymptotically stable. This complete the proof.

## 3. Analysis of Optimal Control

We define our objective (cost) functional as

$$J(u_1,u_2,u_3) = \int_0^T (A_1E_h(t,a_i) + A_2I_h(t,a_i) + A_3N_r(t,e_j) + B_1u_1^2(t) + B_2u_2^2(t) + B_3u_3^2(t))dt. \quad (18)$$

Here $A_1, A_2, A_3 > 0$ are weight constants of the exposed and infected classes respectively. $B_1, B_2, B_3 > 0$ represent the balancing cost factors for prevention, treatment and use of pesticide efforts respectively. It is assumed that the costs of prevention, treatment and use of pesticide are quadratic in the objective functional (3.8). The cost of treatment could come from cost of drug and other cost associated with other health conditions such as surveillance and follow up of drug management. Similarly, the cost to reduce number of rodent population is associated with cost of pesticide and public education.

We seek optimal controls $u_1^*(t), u_2^*(t), u_3^*(t)$ such that

$$J(u_1^*(t),u_2^*(t),u_3^*(t)) = \min_{u_1(t),u_2(t),u_3(t)} \left\{J(u_1(t),u_2(t),u_3(t)) : (u_1(t),u_2(t),u_3(t)) \in \mathcal{U}\right\}, \quad (19)$$

subject to the system (2.1)-(2.7), where

$$\mathcal{U} = \big\{(u_1(t), u_2(t), u_3(t)) : u_r(t) \text{ is piecewise continuous on } [0,T], \\ 0 \le u_r \le 1,\ r = 1,2,3\big\}. \quad (20)$$

The basic framework of this optimal control problem is to prove the existence of the optimal control in the case, where the optimal control has terminal value in the state variable and characterize the optimal control in the case where the optimal control does not has terminal value in the state variable through the optimality system and discuss the uniqueness of the optimality system.

**Theorem 3.1.** *Given an objective functional (3.1) subject to system (2.1)-(2.7) with initial conditions and the admissible control set (3.3), then there exists an optimal control* $u_1^*(t), u_2^*(t), u_3^*(t) \in \mathcal{U}$ *such that*

$$J(u_1^*(t), u_2^*(t), u_3^*(t)) = \min_{u_1(t),u_2(t),u_3(t)} J(u_1(t), u_2(t), u_3(t)),$$

*if the following conditions are satisfied:*
*(i) The sets of controls together with the corresponding state variables is nonempty;*
*(ii) The control set* $\mathcal{U}$ *is convex and closed;*
*(iii) The right hand side of the state system is bounded by a linear function in the state and control;*
*(iv) The integrand of the objective functional is convex on* $\mathcal{U}$ *and is bounded below by* $c_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\frac{\delta}{2}} - c_2 - c_3$, *where* $c_1, c_2, c_3 > 0$ *and* $\delta > 1$.

*Proof.* The result in Theorem 2.1 for the system (2.1)-(2.7) is used to give condition (i). The control set is closed and convex by definition. By Theorem 2.1, the right hand side of system (2.1)-(2.7) satisfies condition (iii). It is clear that $A_1E_h(t,a_i) + A_2I_h(t,a_i) + A_3N_r(t,e_j) + B_1u_1^2(t) + B_2u_2^2(t) + B_3u_3^2(t)$ is convex on $\mathcal{U}$. Furthermore, the variable states are bounded and there exists $c_1, c_2, c_3 > 0$ and $\delta > 1$ satisfying

$$A_1E_h(t,a_i) + A_2I_h(t,a_i) + A_3N_r(t,e_j) + B_1u_1^2(2) + B_2u_2^2(t) + B_3u_3^2(t) \\ \ge c_1(|u_1(t)|^2 + |u_2(t)|^2 + |u_3(t)|^2)^{\frac{\delta}{2}} - c_2 - c_3.$$

Therefore an optimal control exists.

### 3.1. The Optimal System

In this section, we shall establish optimal control in the case where optimal control does not has terminal value in the state variable. We use Pontrygin's maximum principle [17] to derive necessary condition for this optimal control to exist. With costate variables $\Gamma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$. We define our Lagrangian as follows:

$$\mathcal{F} = A_1 E_h(t, a_i) + A_2 I_h(t, a_i) + A_3 N_r(t, e_j) + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t)$$
$$+ \lambda_1 \left[ \Lambda_h(a_i) - \sum_{i=0}^{L} \sum_{j=0}^{T} Q S_h(t, a_i)(1 - u_1(t)) - \mu_h(a_i) S_h(t, a_i) \right]$$
$$+ \lambda_2 \left[ \sum_{i=0}^{L} \sum_{j=0}^{T} Q S_h(t, a_i)(1 - u_1(t)) - \gamma^0 \right]$$
$$+ \lambda_3 \left[ \sum_{i=0}^{L} \epsilon_h(a_i) E_h(t, a_i) - (\psi(a_i)\alpha_2(a_i) u_2(t) + \mu_h(a_i) + \delta_h(a_i)) I_h(t, a_i) \right]$$
$$+ \lambda_4 \left[ \sum_{i=0}^{L} \psi^0 - \mu_h(a_i) R_h(t, a_i) \right]$$
$$+ \lambda_5 \left[ \Lambda_r(e_j) - \sum_{j=0}^{T} Z(1 - u_3) S_r(t, e_j) - (\mu_r(e_j) + \delta_r(e_j) + r^0 \right]$$
$$+ \lambda_6 \left[ \sum_{j=0}^{T} Z(1 - u_3) S_r(t, e_j) - (\epsilon_r(e_j) + \mu_r(e_j) + \delta_r(e_j) + w^0 \right]$$
$$+ \lambda_7 \left[ \sum_{j=0}^{T} \epsilon_r(e_j) E_r(t, e_j) - (\mu_r(e_j) + \delta_r(e_j) + z^0 \right],$$

where

$$r^0 = (1 - u_3(t))) S_r(t, e_j), w^0 = (1 - u_3(t))) E_r(t, e_j),$$

$$z^0 = (1 - u_3(t))) I_r(t, e_j)$$

$$Z = \left( \frac{\beta(e_j)\sigma_1(e_j) I_r(t, e_j) + \theta(e_j)}{N_r(t, e_j)} \right),$$

$$\gamma^0 = (\gamma(a_i)\alpha_1(a_i) u_2(t) + \epsilon_h(a_i) + \mu_h(a_i)) E_h(t, a_i)$$

$$\psi^0 = [\gamma(a_i)\alpha_1(a_i) u_2 E_h(t, a_i) + \psi(a_i)\alpha_2(a_i) u_2(t) I_h(t, a_i)]$$

$$Q = \left( \frac{\rho(a_i)\sigma_1(e_j)I_r(t,e_j) + \eta(a_i)\sigma_2(a_i)I_h(t,a_i) + \theta(e_j) + \kappa(a_i)}{N_h(t,a_i)} \right).$$

**Theorem 3.2.** *Let be given optimal control $u_1^*, u_2^*, u_3^*$ and solution $S_h^*(a_i)$, $E_h^*(a_i)$, $I_h^*(a_i)$, $R_h^*(a_i), S_r^*(e_j)$, $E_r^*(e_j), I_r^*(e_j)$ of the corresponding state system (2.1)-(2.7) that minimizes $J(u_1,u_2,u_3)$ over $\mathcal{U}$. Then there exists adjoint (or costate) variables $\Gamma$ satisfying*

$$\frac{d\lambda_1}{dt} = \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{N^0}{N_h^2(t,a_i)}(N_h(t,a_i) - S_h(t,a_i))(\lambda_1 - \lambda_2) + \mu_h(a_i)\lambda_1, \tag{21}$$

$$\begin{aligned}\frac{d\lambda_2}{dt} &= -A_1 + (\epsilon_h(a_i) + \mu_h(a_i) + \gamma(a_i)\alpha_1(a_i)u_2(t))\lambda_2 \\ &\quad - \sum_{i=0}^{L}\epsilon_h(a_i)\lambda_3 - \sum_{i=0}^{L}\gamma(a_i)\alpha_1(a_i)u_2(t)\lambda_4,\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{d\lambda_3}{dt} &= \\ &- A_2 + \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{(1-u_1(t))S_h(t,a_i)[N_h(t,a_i)\eta(a_i)\sigma_2(a_i) - M](\lambda_1 - \lambda_2)}{N_h^2(a_i)} \\ &+ (\psi(a_i)\alpha_2 u_2(t)(a_i) + \mu_h(a_i) + \delta_h(a_i))\lambda_3 - \sum_{i=0}^{L}\psi(a_i)\alpha_2(a_i)u_2(t)\lambda_4,\end{aligned} \tag{23}$$

$$\frac{d\lambda_4}{dt} = \mu_h(a_i)\lambda_4, \tag{24}$$

$$\frac{d\lambda_5}{dt} = \sum_{j=0}^{T}\frac{P^0}{N_r^2(t,e_j)} + (\mu_r(e_j) + \delta_r(e_j) + (1 - u_3))\lambda_5 - A_3, \tag{25}$$

$$\frac{d\lambda_6}{dt} = (\epsilon_r(e_j) + \mu_r(e_j) + \delta_r(e_j) + (1 - u_3))\lambda_6 - \sum_{j=0}^{T} \epsilon_r(e_j)\lambda_7 - A_3, \quad (26)$$

$$\begin{aligned} \frac{d\lambda_7}{dt} &= \sum_{i=0}^{L}\sum_{j=0}^{T} \frac{(1-u_1(t))N_h(t,a_i)\rho(a_i)\sigma_1(e_j)S_h(t,a_i)(\lambda_1-\lambda_2)}{N_h^2(t,a_i)} - A_3 \\ &+ \left[\sum_{j=0}^{T} \frac{(N_r(t,e_j)\beta(e_j)\sigma_1(e_j)S_r(t,e_j) - Q_t S_r(t,e_j))(\lambda_5-\lambda_6)}{N_r^2(t,e_j)}\right] \\ &\times (1-u_3(t)) + (\mu_r(e_j) + \delta_r(e_j) + (1-u_3(t)))\lambda_7, \quad (27) \end{aligned}$$

*where*

$$M = (\rho(a_i)\sigma_1(e_j)I_r(t,e_j) + \eta(a_i)\sigma_2(a_i)I_h(t,a_i) + \kappa(a_i) + \theta(a_i)),$$

$$N^0 = (1-u_1(t))(\rho(a_i)\sigma_1(e_j)I_r(t,e_j) + \eta(a_i)\sigma_2(a_i)I_h(t,a_i) + \kappa(a_i) + \theta(e_j))$$

$$Q_t = (\beta(e_j)\sigma_1(e_j)I_r(t,e_j) + \theta(e_j))$$

$$P^0 = N_r(t,e_j) - S_r(t,e_j))(1-u_3(t))(\beta(e_j)\sigma_1(e_j)I_r(t,e_j) + \theta(e_j))(\lambda_5-\lambda_6)$$

*with the terminal conditions*

$$\begin{aligned} &\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0, \lambda_4(T) = 0, \\ &\lambda_5(T) = 0, \lambda_6(T) = 0, \lambda_7(T) = 0. \quad (28) \end{aligned}$$

*Furthermore,* $u_1^*, u_2^*, u_3^*$ *are given by*

$$\begin{aligned} u_1^* &= \max\left(0, \min\left(1, \frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T} \frac{MS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right]\right)\right), \\ u_2^* &= \max\left(0, \min\left(1, \frac{1}{2B_2}\sum_{i=0}^{L}[E_{0r}(\lambda_2-\lambda_4) + N(\lambda_3-\lambda_4)]\right)\right), \\ u_3^* &= \max\left(0, \min\left(1, \frac{1}{2B_3}\left[\sum_{i=0}^{L} ZS_r^*(e_j)(\lambda_5-\lambda_6) - D\right]\right)\right), \quad (29) \end{aligned}$$

*where* $N = \psi(a_i)\alpha_2(a_i)I_h^*(t,a_i)$ *and*

$$D = S_r(t,e_j)\lambda_5 + E_r(t,e_j)\lambda_6 + I_r(t,e_j)\lambda_7, E_{0r} = \gamma(a_i)\alpha_1(a_i)E_h^*(a_i).$$

*Proof.* The forms of the adjoint equation (3.4)-(3.10) and the terminal conditions (3.11 ) are standard results from Pontryagin's principle [15, 17]. To obtain the optimality condition (3.12). We differentiate the Lagrangian $\mathcal{F}$ with respect to $(u_1(t), u_2(t), u_3(t))$ and set the results equal to zero, we obtain

$$\begin{aligned} u_1^* &= \frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{MS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right], \\ u_2^* &= \frac{1}{2B_2}\sum_{i=0}^{L}\left[E_{0r}(\lambda_2-\lambda_4)+N(\lambda_3-\lambda_4)\right], \\ u_3^* &= \frac{1}{2B_3}\left[\sum_{i=0}^{L} ZS_r^*(e_j)(\lambda_5-\lambda_6)-D\right]. \end{aligned}$$

To determine an explicit expression for the optimal control $u_1^*$ a standard optimal technique is used. We consider the following cases:
(i) On the control set $\{u_1(t) : 0 < |u_1(t) - u^*| < 1, 0 \le t \le T\}$.
Hence

$$u_1^* = \frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{T_tS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right],$$

(ii) On the control set $\{u_1(t) : 0 < |u_1(t) - u^*| = 0, 0 \le t \le T\}$,
so, $u_1^* = \frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{T_tS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right]$,
which implies

$$\frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{T_tS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right] \le 0,$$

(iii) On the control set $\{u_1(t) : 0 < |u_1(t) - u^*| = 1, 0 \le t \le T\}$, so

$$u_1^* = \left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{T_tS*_h(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right] \ge 1,$$

where $T_t = (\rho(a_i)\sigma_1(e_j)I_r^*(t, e_j) + \eta(a_i)\sigma_2(a_i)I_h^*(a_i) + \theta(e_j) + \kappa(a_i))$.

From these three cases, the optimal control $u_1^*(t)$ is characterized as

$$u_1^* = \max\left(0, \min\left(1, \frac{1}{2B_1}\left[\sum_{i=0}^{L}\sum_{j=0}^{T}\frac{MS_h^*(a_i)(\lambda_2-\lambda_1)}{N_h^*(a_i)}\right]\right)\right),$$

following a similar argument, we obtain

$$u_2^* = \max\left(0, \min\left(1, \frac{1}{2B_2}\sum_{i=0}^{L}\left[E_{0r}(\lambda_2-\lambda_4)+N(\lambda_3-\lambda_4)\right]\right)\right),$$
$$u_3^* = \max\left(0, \min\left(1, \frac{1}{2B_3}\left[\sum_{i=0}^{L} ZS_r^*(e_j)(\lambda_5-\lambda_6)-D\right]\right)\right).$$

The optimality system consists of the system (2.1)-(2.7) with the initial conditions, $S_h(0,a_i)$, $E_h(0,a_i)$, $I_h(0,a_i)$, $R_h(0,a_i)$, $S_r(0,e_j)$, $E_r(0,e_j)$,$I_r(0,e_j)$, the costate system (3.4)-(3.10) with the terminal condition (3.11) and the optimality condition (3.12). Any optimal control $u_1^*$, $u_2^*$, $u_3^*$ must satisfy this optimality system.

### 3.2. Uniqueness of Optimality System

Using the bound for the state equations, the adjoint system has bounded coefficients and is linear in each adjoint variable. Hence the solution of the adjoint system are bounded for the final time sufficiently small. Due to this a priori boundedness of the state and adjoint functions and the Lipschitz structure of the optimality system, we obtain uniqueness of the optimal control for small $T$. The uniqueness of the optimal control pair follows from the uniqueness of the optimality system. The restriction of time interval is very common in optimal control problems (see [15, 16]).

## 4. Numerical Results

In this section, we discus the simulation results for optimal control model. Unless where stated otherwise, the parameters used in solving the model are presented in **Table 1**. The system is solved numerically using the Forward Backward sweep method in [16]. We first solve the initial value state system over the simulated time using a forward fourth order Runge Kutta scheme after making an initial guess for the optimal control function. Because of the terminal conditions (3.11), the adjoint system is solved by a Backward fourth order Runge Kutta method scheme using the current iteration solutions of the state equations. Then the control are updated by using a convex combination of the controls and the value from the characterization (3.12). This process is repeated and the iterations are stopped if the value of the unknowns at the previous iterations are very close to the ones at the present iteration. Using various combination of the three controls, one control at a time, we investigate and compare numerical results from simulations with the following scenarios: (i) $u_1 \neq 0, u_2 = 0, u_3 = 0$, (ii) $u_2 \neq 0, u_1 = u_2 = 0$, (iii) $u_3 \neq 0, u_1 = u_2 = 0$, (iv) $u_2 \neq 0, u_3 \neq 0, u_1 = 0$, (v) $u_1 \neq 0, u_3 \neq 0, u_2 = 0$, (vi) $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$. For numerical simulations we have used the following weight factors: $A_1 = 1, A_2 = 1, A_3 = 1,\ B_1 = 50, B_2 = 150, B_3 = 30$, initial state variables $S_h(0) = 500, E_h(0) = 20, I_h(0) = 0,\ R_h(0) = 10, S_r(0) = 2000, E_r(0) = 100, I_r(0) = 30$.

**Figure 1** represents the population of infected humans with and without control. The solid blue line denotes the population of infected individuals in system (2.1)-(2.7) without control while the black, pink, orange and red lines denotes the population of infected individuals in system (2.1)-(2.7) using various combination of the three controls. We see that the population of infected human with control is more sharply decreased than individuals without control. **Figure 2** represents the total number of rodent population in the system (2.1)-(2.7) with and without control. The solid blue line denotes the total population of rodent in system (2.1)-(2.7) without control while the black, pink, orange and red lines denotes the population of rodent in system (2.1)-(2.7) using various combination of the three controls. We see that the population of rodent with control is more sharply decreased than rodent without control.

## 5. Concluding Remarks

In this paper, we presented a Lassa fever model with multiple control using a deterministic system of differential equations and established that in the absence of disease (at the disease-free) and endemic equilibria the model is globally asymptotically stable if $R_0(a) \leq 1$, $R_0 > 1$ $0 \leq u_1 \leq 1$ and $0 \leq u_3 \leq 1$. This is different from what has been investigated by many authors in the literatures. The existence and uniqueness of an optimality system are discussed A characterization of the optimal control via adjoint variable was established. We use one control at a time and combination of two controls at a time while setting the other(s) to zero to investigate and compare the effect of control strategies on Lassa fever eradication. Our numerical results shows that the combination of the three (3) controls; personal protection, treatment and use of pesticide has the highest impact on the control of the disease.

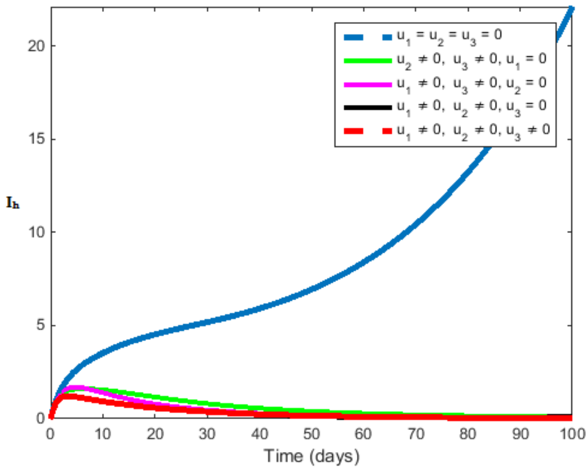


Figure 1: Simulation showing the effect of $u_1, u_2, u_3$ only on infectious human

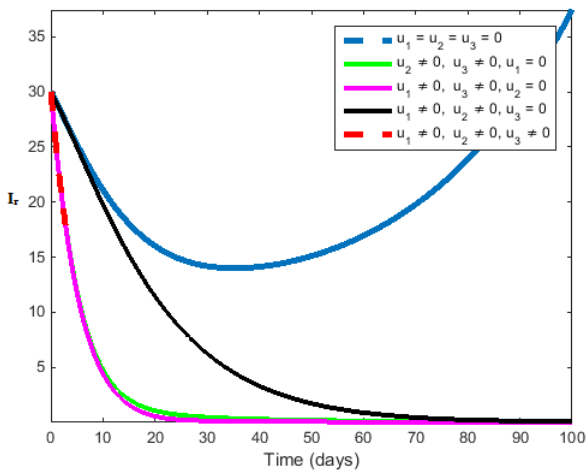


Figure 2: Simulation showing the effect of $u_1, u_2, u_3$ only on infectious rodent

| Parameter | Value |
|---|---|
| $\Lambda_h(a_i)$ | 0.038 |
| $\rho(a_i)$ | 0.6 |
| $\sigma_1(e_j)$ | 0.8 |
| $\eta(a_i)$ | 0.6 |
| $\sigma_2(a_i)$ | 0.56 |
| $\alpha_1(a_i)$ | 0.05 |
| $\alpha_2(a_i)$ | 0.9 |
| $\theta(e_j)$ | 0.022 |
| $\kappa(a_i)$ | 0.018 |
| $\epsilon_h(a_i)$ | 0.85 |
| $\gamma(a_i)$ | 0.9 |
| $\psi(a_i)$ | 0.45 |
| $\mu_h(a_i)$ | 0.02 |
| $\epsilon_r(e_j))$ | 0.85 |
| $\beta(e_j)$ | 0.75 |
| $\delta_h(e_j)$ | 0.2 |
| $\delta_r(e_j)$ | 0.3 |
| $\mu_r(e_j)$ | 0.6 |
| $\Lambda_r(e_j)$ | 0.56 |

Table 1. Description of the model parameters

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