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ON MULTIPLICITY OF RADIAL SOLUTIONS TO DIRICHLET PROBLEM INVOLVING THE p-LAPLACIAN ON EXTERIOR DOMAINS

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Abstract: In this paper we prove the existence of radial solutions having a prescribed number of sign change to the p-Laplacian $\Delta_p u + f(u) = 0$ on exterior domain of the ball of radius $R > 0$ centered at the origin in $\mathbf{R^N}$. The nonlinearity f is odd and behaves like $|u|^{q-1}u$ when u is large with $1 < p < q+1$ and $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) where $\beta > 0$. The method is based on a shooting approach, together with a scaling argument.

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1. Introduction and Statement Result

In this paper we deal with the existence and multiplicity of classical radial sign-changing solutions to the Dirichlet boundary problem involving the p-Laplacian

$$\Delta_p u + f(u) = 0 \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{in } \partial\Omega, \tag{2}$$

$$\lim_{|x|\to\infty} u(x) = 0. \tag{3}$$

Here $\Omega = \{x \in \mathbf{R}^N | \; |x| > R\}$ is the complement of the ball of radius $R > 0$ cen-

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tred at the origin with $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$ is the standard norm of $\mathbf{R}^N$. Also, $\Delta_p u$ is the p-Laplacian of the function u with $\Delta_p u = div\Big(|\nabla u|^{p-2}\,\nabla u\Big)$.

We will assume henceforth that the function f satisfies the following hypotheses:

(H1) $f : \mathbf{R} \to \mathbf{R}$ is odd and locally Lipschitzian,

(H2) $f(u) = |u|^{q-1}u + g(u)$ with $1 < p < q+1$ and

$$\lim_{|u|\to\infty} \frac{|g(u)|}{|u|^q} = 0,$$

(H3) There exists $\beta > 0$ such that

$$f(0) = f(\beta) = 0 \text{ where } f < 0 \text{ on } (0, \beta)\ ,\ f > 0 \text{ on } (\beta, \infty),$$

(H4) If $p > 2$ we also assume for some $\eta > 0$

$$\int_0^\eta \frac{1}{|F(u)|^{\frac{1}{p}}}\,\mathrm{d}u = \infty,$$

where $F(u) = \int_0^u f(s)\,\mathrm{d}s$.

As a consequence of the previous assumptions we have:

(i) $F(u) \to \infty$ as $|u| \to \infty$. F is even and bounded below by some $-F_0 < 0$ on $\mathbf{R}$, i.e.

$$F(u) \geq -F_0 \quad \forall u \in \mathbf{R}. \tag{4}$$

(ii) F is strictly increasing in (β, ∞) and decreasing in $(0, \beta)$.

(iii) F has a unique positive zero,$\gamma > \beta$ and $F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞).

Remark 1. If $1 < p \leq 2$ it follows from the fact that f is locally Lipschitzian the assumption (H4) also holds.

The radial symmetric solutions to (1)-(3) satisfy the problem

$$\Big(r^{N-1}\Phi_p(u')\Big)' + r^{N-1} f(u) = 0 \quad \text{if } R < r, \tag{5}$$

$$u(R) = 0 \quad \text{and} \quad u(r) \to 0 \quad \text{as} \quad r \to \infty, \tag{6}$$

where $\Phi_p(s) = |s|^{p-2}s$. Also $'$ denotes the derivative with respect to $r = |x| \geq 0$, $x \in \mathbf{R}^N$ and for radial functions as it is usual we shall write $u(x) = u(r)$ with $r = |x|$. We note that Φ_p is odd and differentiable on $\mathbf{R} \setminus \{0\}$ with $\Phi'_p(s) = (p-1)|s|^{p-2}$ and $\Phi_p^{-1} = \Phi_{p'}$, where p' the Hölder conjugate exponent of p. We will be interested only in classical solutions of (5)-(6) i.e., $u \in C^1([R,\infty),\mathbf{R})$ and $\Phi_p(u') \in C^1([R,\infty),\mathbf{R})$.

The research of radial solution of elliptic equations with zero Dirichlet boundary conditions (1)-(3) with the usual Laplace operator ($p = 2$) has widely studied by many authors via variational methods when Ω is bounded domain or the whole space $\mathbf{R}^N$, under different regularity and growth assumptions of nonlinearity of f, see for instance [1, 2], exploring the symmetry of the problem (1)-(3) to prove the existence of infinity radial solution by means the Mountain pass theorem. In particular when Ω is a ball and is f non increasing by an other argument well-know plane method to prove the existence and multiplicity of radial solution to this problem, see [4]. However, these arguments are quite difficult and provide no specific information of qualitative properties. Then it was an open question as to whether solutions exist with prescribed number of zeros. Jones and Küpper in [6] addressed this question using a dynamical systems approach and an application of the Conley index [6]. In [9] Mcleod, Troy and Weissler established the existence of sign changing bound state solutions by using the shooting techniques and a scaling argument when f satisfies appropriate sign conditions and is of subcritical growth. Pudipeddi [11] extended the previous result for the p-Laplacian where $1 < p < N$ and $\Omega = \mathbf{R}^N$ using the same approach when f is locally Lipschitz and odd and behaves like $|u|^{q-1}u$ for u sufficiently large with $p < q + 1 < \dfrac{Np}{N-p}$.

Recently on exterior domain, there has been an interest in studying this question if $p = 2$ we mention as instances [5, 7]. Here we use the shooting argument and a “simple” ordinary differential equation proof to establish that (1)-(3) has an infinite number of radial solutions with a prescribed number of zeros.

Our paper is organized as follows: in Section 2 we begin to establish some preliminary results concerning the existence and proprieties of radial solutions. In Section 3 we show that there are solutions with arbitrarily number large of zeros by using a scaling argument and finally, we shall prove the following “Main theorem”:

Theorem 2. *Assuming* (H1)–(H4) *and* $N \geq 2$. *Then for each nonnegative integer* k, *there exist two radially symmetric solutions* u_k *and* v_k *of problem*

(1)-(3) *which have exactly* k *zeros on* (R, ∞) *such that* $v_k'(R) < 0 < u_k'(R)$.

2. Preliminaries

To deal with the problem (5)-(6), we will use a shooting method and consider the initial value problem

$$\left(r^{N-1}\Phi_p(u')\right)' + r^{N-1}f(u) = 0 \quad \text{if} \quad r > R, \tag{7}$$

$$u(R) = 0 \quad \text{and} \quad u'(R) = a > 0. \tag{8}$$

To emphasize the dependence of the solution to (7) in the shooting parameter a, we will denote it u_a.

Lemma 3. *Assume* (H1) *and* (H2) *hold. Then* (7)-(8) *has a unique solution* u_a *defined on interval* $[R, \infty)$. *Moreover,* $a \to u_a$ *and* $a \to u_a'$ *are continuous on* $(0, \infty)$.

Proof. Let u be a solution of (7)-(8) and integrating (7) on $[R, r]$, we obtain

$$r^{N-1}\Phi_p(u') = R^{N-1}a^{p-1} - \int_R^r t^{N-1}f(u)\,\mathrm{d}t. \tag{9}$$

We rewrite this as

$$u'(r) = \left(\frac{R}{r}\right)^k \Phi_{p'}\left(a^{p-1} - \int_R^r \left(\frac{t}{R}\right)^{N-1} f(u)\,\mathrm{d}t\right), \tag{10}$$

where $k = \dfrac{N-1}{p-1}$. Integrating (10) on $[R, r]$ we obtain

$$u(r) = \int_R^r \left(\frac{R}{t}\right)^k \Phi_{p'}\left(a^{p-1} - \int_R^t \left(\frac{s}{R}\right)^{N-1} f(u)\,\mathrm{d}s\right)\mathrm{d}t.$$

Let $\epsilon > 0$. Denote $C^0[R, R+\epsilon]$ the Banach space of real continuous functions on $[R, R+\varepsilon]$ endowed with the sup norm $\|\,\|$. Let $a, \delta_0 > 0$ are fixed such that $\delta_0 < a^{p-1}$ and we define the complete metric space by

$$E := \{(u, v) \in \left(C^0[R, R+\epsilon]\right)^2 \quad | \quad u(R) = 0\,,\ v(R) = a^{p-1}\},$$

endowed with the distance $d(x_1,x_2) = \max(\|u_1 - u_2\|, \|v_1 - v_2\|)$ where $x_i = (u_i, v_i)$ for $i = 1, 2$.

Denote $B_\delta^\epsilon(a) := \{(u,v) \in E \mid \ d\Big((u,v),(0,a^{p-1})\Big) \leq \delta\}$ the closed ball of (E,d).

The existence and uniqueness for (7)-(8) result from the study of fixed point of the application $\mathit{\Gamma}_a : (u,v) \in E \to (\tilde{u},\tilde{v}) \in E$ where $\tilde{u}$ and $\tilde{v}$ are defined by

$$\tilde{u}(r) = \int_R^r \left(\frac{R}{t}\right)^k \Phi_{p'}(v)\,\mathrm{d}t,$$

$$\tilde{v}(r) = a^{p-1} - \int_R^r \left(\frac{t}{R}\right)^{N-1} f(u)\,\mathrm{d}t.$$

We will show that $\mathit{\Gamma}_a$ is a contraction mapping of $B_\delta^\epsilon(a)$ into itself for ϵ, δ small enough.

For all $(u,v) \in B_\delta^\epsilon(a)$ and $r \in [R, R+\epsilon]$ we have

$$|\tilde{u}(r)| \leq \epsilon\,\|v\|^{p'-1} \leq \epsilon\left(\delta_0 + a^{p-1}\right)^{p'-1}.$$

Therefore for ϵ small enough we have

$$\|\tilde{u}\| \leq \delta.$$

Furthermore, then

$$\begin{aligned} |\tilde{v}(r) - a^{p-1}| &\leq \int_R^r \left(\frac{t}{R}\right)^{N-1} |f(u(t))|\,\mathrm{d}t, \\ &\leq \frac{MR}{N}\Big((1+\frac{\epsilon}{R})^N - 1\Big) \leq \delta \quad \text{if} \quad \epsilon \to 0, \end{aligned}$$

where $M = \sup\limits_{|s|\leq\delta_0} |f(s)|$ and for ϵ small enough. Then it follows that $||\tilde{v} - a^{p-1}|| \leq \delta$, which implies that $(\tilde{u},\tilde{v}) \in B_\delta^\epsilon(a)$, for ϵ small enough. Now, let $x_i = (u_i, v_i) \in B_\delta^\epsilon(a)$ for $i = 1, 2$, then

$$d\Big(\mathit{\Gamma}_a(x_1), \mathit{\Gamma}_a(x_2)\Big) = \max\Big(\|\tilde{u_1} - \tilde{u_2}\|; \|\tilde{v_1} - \tilde{v_2}\|\Big).$$

For $r \in [R, R+\epsilon]$ fixed, thanks to the mean value theorem we obtain

$$\Phi_{p'}(v_1(r)) - \Phi_{p'}(v_2(r)) = \Phi'_{p'}(w)(v_1(r) - v_2(r)), \tag{11}$$

where $w = \alpha\, v_1(r) + (1-\alpha)\, v_2(r)$ for some $0 < \alpha < 1$ with $\Phi'_{p'}(w) = (p'-1)\,|w|^{p'-2}$. As $\|v_i - a^{p-1}\| \leq \delta \leq \delta_0$, then for each $i = 1, 2$ we have

$$a^{p-1} - \delta_0 \leq v_i(t) \leq a^{p-1} + \delta_0 \quad \forall t \in [R, R+\epsilon],$$

therefore there exists two constants $c_1 = a^{p-1} - \delta_0 > 0$ and $c_2 = a^{p-1} + \delta_0 > 0$ and for all $i = 1, 2$ we see that $c_1 \leq v_i(t) \leq c_2$ for all $t \in [R, R+\epsilon]$. Then it follows that

$$(p'-1)\,|w|^{p'-2} \leq C_p,$$

where $C_p = (p'-1)\, c_2^{p'-2}$ if $1 < p \leq 2$ and $C_p = (p'-1)\, c_1^{p'-2}$ if $p > 2$ and further from (11) we have

$$\|\Phi_{p'}(v_1) - \Phi_{p'}(v_2)\| \leq C_p \|v_1 - v_2\|.$$

Therefore

$$\|\tilde{u_1} - \tilde{u_2}\| \leq \epsilon\, C_p \|v_1 - v_2\|. \tag{12}$$

On the other hand, we see that

$$|\tilde{v_1}(r) - \tilde{v_2}(r)| \leq \int_R^r \left(\frac{t}{R}\right)^{N-1} |f(u_1) - f(u_2)|\,\mathrm{d}t.$$

By virtue of (H1), then there exists a constant $K > 0$ and for each $|s|$ and $|t| \leq \delta_0$ we have

$$|f(s) - f(t)| \leq K|s-t|.$$

Since $\|u_i\| \leq \delta \leq \delta_0$ for $i = 1, 2$ then it follows that

$$\|\tilde{v_1} - \tilde{v_2}\| \leq \lambda(\epsilon)\|u_1 - u_2\|,$$

where $\lambda(\epsilon) = \dfrac{KR}{N}\Big((1 + \dfrac{\epsilon}{R})^N - 1\Big)$. Thus, from (12) we have

$$d\Big(\mathit{\Gamma}_a(x_1), \mathit{\Gamma}_a(x_2)\Big) \leq \max\Big(\epsilon\, C_p, \lambda(\epsilon)\Big)\, d(x_1, x_2).$$

Since $\lambda(\epsilon) \to 0$ as $\epsilon \to 0$. Thus by the contraction mapping principle it follows that for ϵ small enough $\mathit{\Gamma}_a$ has a unique fixed point denoted $x_a = (u_a, v_a) \in B_\delta^\epsilon(a)$ and we have $u_a, v_a \in C^1([R, R+\varepsilon], \mathbf{R})$. In addition,

$$u_a' = \left(\frac{R}{r}\right)^k \Phi_{p'}(v_a) \quad \text{and} \quad \Phi_p(u_a') = \left(\frac{R}{r}\right)^{N-1} v_a.$$

Then it follows that $\Phi_p(u_a') \in C^1([R, R+\varepsilon], \mathbf{R})$. Hence for some $\epsilon > 0$, (7)-(8) has a unique solution u_a such that u_a and $\Phi_p(u_a') \in C^1([R, R+\varepsilon], \mathbf{R})$.

Next, we will show that the solution u_a can be extended on $[R, +\infty)$, we define the energy of solution to (7)-(8) as

$$E_a(r) = \frac{|u_a'|^p}{p'} + F(u_a) \quad \forall r \geq R. \tag{13}$$

Differentiating E_a and using (7), gives

$$E_a'(r) = -\frac{N-1}{r}|u_a'|^p \leq 0. \tag{14}$$

Which implies that E_a is non-increasing on $[R, +\infty)$, so

$$\frac{|u_a'|^p}{p'} + F(u_a) \leq \frac{a^p}{p'}.$$

From (4) we have

$$|u_a'| \leq \left(a^{p-1} + p'F_0\right)^{\frac{1}{p}} = M_a. \tag{15}$$

Then we have $|u_a'|$ is uniformly bounded on wherever it is defined, as $u_a(R) = 0$ thus it follows that $|u_a|$ is uniformly bounded on wherever it is defined. Hence the existence on all of $[R, +\infty)$ follows.

Now, we will show the continuous dependence of solutions on initial conditions. For this let $a > 0$ and $a_j \to a$ as $j \to \infty$ and denote $u_j(r) = u_{a_j}(r)$ for all j. In the following we shall prove that $u_j \to u_a$ and $u_j' \to u_a'$ as $j \to \infty$ on compact subsets of $[R, \infty)$.

Indeed, as the sequence (a_j) is bounded by some $A > 0$ and from (15) for all j, we get

$$|u_j'(r)| \leq \left(A^{p-1} + p'F_0\right)^{\frac{1}{p}} = M_2 \quad \forall r \geq R.$$

Therefore $u_j'(r)$ is uniformly bounded. Next, we will show that $u_j(r)$ is uniformly bounded.

Suppose by the way of contradiction that there exists a sequence $r_j \geq R$ such that $|u_j(r_j)| \to \infty$ as $j \to \infty$. Since $F(u) \to \infty$ as $|u| \to \infty$, then $F(u_j(r_j)) \to \infty$ as $j \to \infty$. As

$$E_{a_j}(r_j) = E_j(r_j) \geq F(u_j(r_j)),$$

then $E_j(r_j) \to \infty$ as $j \to \infty$. Since E_j is non-increasing and (a_j) is bounded then we have

$$E_j(r_j) \leq E_j(R) = \frac{a_j^p}{p'} \leq \frac{A^p}{p'} < \infty.$$

This contradict to $E_j(r_j) \to \infty$ as $j \to \infty$. Thus there exists $M_1 > 0$ and $M_2 > 0$ for all j such that

$$|u_j(r)| \leq M_1 \quad \text{and} \quad |u_j'(r)| \leq M_2 \quad \forall r \geq R.$$

This implies that u_j and u_j' are uniformly bounded and equicontinuous. Then by Arzela-Ascoli's theorem there exists subsequence still denoted u_j such that $u_j(r) \to u(r)$ as $j \to \infty$ uniformly on compact subsets of $[R, \infty)$. Therefore it follows from (H1) that $f(u_j) \to f(u)$ as $j \to \infty$ uniformly on compact subsets of $[R, \infty)$ and since $a_j \to a$, then we get

$$\begin{aligned} w_j(r) &= a_j^{p-1} - \int_R^r \left(\frac{t}{R}\right)^{N-1} f(u_j(t))\,\mathrm{d}t \\ &\to a^{p-1} - \int_R^r \left(\frac{t}{R}\right)^{N-1} f(u(t))\,\mathrm{d}t = w(r), \end{aligned}$$

uniformly on compact subsets of $[R, \infty)$. This implies that $\Phi_{p'}(w_j') \to \Phi_{p'}(w)$ as $j \to \infty$ uniformly on compact subsets of $[R, \infty)$. By virtue of (10) we obtain

$$u_j'(r) = \left(\frac{R}{r}\right)^k \Phi_{p'}(w_j'(r)) \to \left(\frac{R}{r}\right)^k \Phi_{p'}(w(r)) = v(r) \quad \text{as } j \to \infty,$$

uniformly on compact subsets of $[R, \infty)$. Furthermore, from

$$u_j(r) = \int_R^r u_j'(t)\,\mathrm{d}t \to \int_R^r v(t)\,\mathrm{d}t \quad \textit{as } j \to \infty,$$

pointwise on $[R, \infty)$ and $u_j(r) \to u(r)$, therefore it follows that u is differentiable and $u' = v$. Hence $u_j \to u$ and $u_j' \to u'$ uniformly on compact subsets of $[R, \infty)$ and finally, $a \to u_a$, $a \to u_a'$ are continuous on $(0, \infty)$. This completes the proof of Lemma 3. □

Remark 4. It is immediate that $u_a \in C^2([R, R+\varepsilon])$ for $1 < p \le 2$ and u is C^2 only at the point $r \in [R, R+\varepsilon]$ such that $u_a'(r) \neq 0$ for $p > 2$.

Proposition 5. *Assume* (H1)–(H3) *hold and* u_a *be solution of* (7)-(8).

(i) *Then* $r \to E_a(r)$ *is non-increasing and* $\lim_{r\to\infty} E_a(r)$ *is finite. In addition, if there exists* $r_0 > R$ *such that* $E_a(r_0) < 0$ *then* $u_a > 0$ *on* $[r_0, \infty)$.

(ii) *If* $\lim\limits_{r\to\infty} u_a(r) = \ell$ *exists, then* ℓ *is a zero of* f *and moreover*

$$\lim_{r\to\infty} u_a'(r) = 0 \quad , \lim_{r\to\infty} E_a(r) = F(\ell).$$

(iii) *Then* $u_a' > 0$ *on a maximal nonempty open interval* (R, M_a) *where either*

(a) $M_a = \infty, \lim\limits_{r\to\infty} u_a(r) = \beta$ *and* $0 \le u_a(r) < \beta$ *for all* $r \ge R$.
or

(b) M_a *is finite and* $u_a(M_a) \geq \beta$.

Proof. By virtue of (14) we have E_a is non-increasing and since $E_a(r) \geq -F_0$ implying that $E_a \to \zeta_a$ as $a \to \infty$. Now, let $r_0 > R$ such that $E_a(r_0) < 0$ by monotonicity of E_a we have $E_a(r) \leq E_a(r_0) < 0$ for $r \geq r_0$, if we suppose that there exists $r_1 > r_0$ zero of u_a, then it follows that $E_a(r_1) = \dfrac{|u_a'|^p}{p'} \geq 0$, which contradicts to $E_a(r_1) < 0$. Hence $u_a > 0$ on $[r_0, \infty)$.

For *(ii)* we suppose that $u_a(r) \to \ell$ as $r \to \infty$, then by continuity of F we have $\lim_{r\to\infty} F(u_a(r)) = F(\ell)$, so from (13) we have $\lim_{r\to\infty} |u_a'(r)| = \Big(p'(\xi_a - F(\ell))\Big)^{\frac{1}{p}} = m$. Assume to the contrary that $m > 0$, then for $0 < \epsilon < m$ there exists $r_0 > R$ such that $|u_a'| \geq m - \epsilon > 0$ for each $r \geq r_0$. Thus it follows that $|u_a(r) - u_a(r_0)| \geq (m-\epsilon)(r - r_0)$ which implies that $u_a(r) \to$ as $r \to \infty$. This is a contradiction. Hence $u_a'(r) \to 0$ as $a \to \infty$ and $\zeta_a = F(\ell)$. By (9) and applying L'Hôpital's rule we obtain

$$\begin{aligned} 0 &= \lim_{r\to\infty} \frac{\Phi_p(u_a'(r))}{r} = \lim_{r\to\infty} \frac{R^{N-1}a^{p-1}}{r^N} - \frac{\int_R^r t^{N-1} f(u_a)\,\mathrm{d}t}{r^N} \\ &= -\frac{f(\ell)}{N}. \end{aligned}$$

Therefore $f(\ell) = 0$.

Next for *(iii)*, as $u_a'(R) = a > 0$ and by continuity, there exists $\epsilon > 0$ such that $u_a' > 0$ on $(R, R+\epsilon)$. Denote (R, M_a) a maximal nonempty open interval where $u_a' > 0$. If $M_a = \infty$ then $u_a > 0$ is increasing and bounded above on $[R, \infty)$, therefore it follows that $u_a(r) \to \ell$ as $r \to \infty$. By virtue of *(ii)* we have $f(\ell) = 0$ and $u_a'(r) \to 0$ as $r \to \infty$. Thus $\ell = \beta$ and $0 \leq u_a < \beta$ on $[R, \infty)$. Hence *(a)* is proven. For *(b)*, if $M_a < \infty$ we must have $u_a'(M_a) = 0$ and $u_a' > 0$ on (r, M_a) for $R < r < M_a$. Assume to contrary that $0 < u_a(M_a) < \beta$ then by (H3) it follows $f(u_a) < 0$ on (r, M_a). Integrating (7) on (r, M_a) and using the fact $u_a'(M_a) = 0$ we get

$$r^{N-1}\Phi_p(u_a'(r)) = \int_r^{M_a} t^{N-1} f(u_a(t))\,\mathrm{d}t < 0.$$

This implies that $u_a' < 0$ on (r, M_a), this is a contradiction. Thus $u_a(M_a) \geq \beta$. Which completes the proof of Proposition 5. □

Lemma 6. *Assume* (H1)–(H3) *hold. Then*

1. *u_a has a maximum at $M_a > R$ for a sufficiently large. Moreover $|u_a|$ has a global maximum at M_a.*

2. *$\lim_{a\to\infty} u_a(M_a) = \infty$ and $\lim_{a\to\infty} M_a = R$.*

Proof. For (1), we suppose by the way of contradiction that $M_a = \infty$. Then $u'_a > 0$ on $[R, \infty)$ and by *(ii)-(iii)* of Proposition 5 we have $0 \le u_a(r) < \beta$, $u_a(r) \to \beta$ and $u'_a(r) \to 0$ as $r \to \infty$. Let $y_a = \dfrac{u_a(r)}{a}$. It is straightforward using (7)-(8) to show

$$\Big(r^{N-1}\Phi_p(y'_a)\Big)' + r^{N-1}\frac{f(ay_a)}{a^{p-1}} = 0 \quad \text{if} \quad r > R, \tag{16}$$

$$y_a(R) = 0 \quad \text{and} \quad y'_a(R) = 1. \tag{17}$$

Then it follows that

$$\Big(\frac{|y'_a|^p}{p'} + \frac{F(ay_a)}{a^{p-1}}\Big)' = -\frac{N-1}{r}\,|y_a|^p \le 0.$$

Therefore

$$\frac{|y'_a|^p}{p'} + \frac{F(ay_a)}{a^{p-1}} \le \frac{1}{p'}.$$

By virtue of (4) and for a sufficiently large we obtain

$$\frac{|y'_a|^p}{p'} \le \frac{1}{p'} + \frac{F_0}{a^{p-1}} \le \frac{1}{p'} + \frac{1}{p} = 1.$$

As $0 < y_a < \dfrac{\beta}{a}$ then for a sufficiently large we get $0 < y_a < 1$. It follows that $|y_a|$ and $|y'_a|$ are uniformly bounded if a is sufficiently large. Then by the Arzela-Ascoli Theorem we deduce that $y_a \to y$ uniformly as $a \to \infty$ on the compact sets of $[R, \infty)$, for some subsequence denoted the same by y_a with y is a continuous function on $[R, \infty)$ where $y(R) = 0$. Integrating (16) on $[R, r]$ gives

$$\Phi_p(y'_a) = \Big(\frac{R}{r}\Big)^{N-1} - \int_R^r \Big(\frac{t}{r}\Big)^{N-1}\frac{f(ay_a)}{a^{p-1}}\,\mathrm{d}t.$$

Since ay_a is bounded and f is continuous then $\dfrac{f(ay_a)}{a^{p-1}} \to 0$ uniformly on $[R, \infty)$ as $a \to \infty$. Therefore it follows that $\Phi_p(y'_a(r))$ converges to $\Big(\dfrac{R}{r}\Big)^{N-1}$ uniformly on compact subsets of $[R, \infty)$. This implies that y'_a converges uniformly as $a \to$

∞ on compact subsets of $[R, \infty)$ to a continuous function denoted $z(r) = \Big(\dfrac{R}{r}\Big)^k$ with $k = \frac{N-1}{p-1}$. Moreover z' exists and is continuous. Furthermore

$$y_a(r) = \int_R^r y_a'(t)\, \mathrm{d}t.$$

Letting $a \to \infty$, we obtain that

$$y(r) = \int_R^r z(t)\, \mathrm{d}t.$$

Then y_a is continuously differentiable and $y' = z$, thus $y_a' \to y'$ uniformly as $a \to \infty$ on the compact subsets of $[R, \infty)$. Since $0 < y_a < \dfrac{\beta}{a}$ so $y_a \to 0$ as $a \to \infty$, then $y \equiv 0$ would imply that $y' \equiv 0$. Which contradicts to $y'(R) = 1$. Thus u_a has a local maximum at $M_a > R$.

Next, we will show that $|u_a|$ has a global maximum at M_a. Otherwise, suppose that there exists $r_1 > M_a$ such that $|u_a(r_1)| > |u_a(M_a)|$. From (iii) of Proposition 5 we have $u_a(M_a) \geq \beta$, since F is even and increasing on (β, ∞) therefore it follows

$$E_a(r_1) = F(u_a(r_1)) = F(|u_a(r_1)|) > F(u_a(M_a)) = E_a(M_a).$$

By the monotonicity of E_a we have $r_1 \leq M_a$, which contradicts to $r_1 > M_a$.

For (2), we begin to claim that $u_a(M_a) \to \infty$ as $a \to \infty$. To proof this, assume to contrary that for a sufficiently large, there exists a constant $C > 0$ independent of a such that $|u_a(M_a)| \leq C$. As $|u_a|$ has a global maximum at M_a then $|u_a(r)| \leq C$ for all $r \geq R$.

Let $y_a = \dfrac{u_a}{a}$. We proceed in the same way as (1), we show that $y_a \to y$ with $y \equiv 0$ and $y'(R) = 1$ this is a contradiction. Hence $u_a(M_a) \to \infty$ as $a \to \infty$. In the following we want to show that $M_a \to R$, as $a \to \infty$.

From (1) and by monotonicity of E_a, for a sufficiently large we see that $E_a(r) \geq E_a(M_a) = F(u_a(M_a))$ on $[R, M_a]$. Denote $x_a = u_a(M_a)$, so $|u_a'| = u_a'(r) \geq (p')^{\frac{1}{p}}\Big(F(x_a) - F(u_a(r))\Big)^{\frac{1}{p}}$ for all $r \in [R, M_a]$. Integrating this on $[R, M_a]$ gives

$$\int_0^{x_a} \frac{ds}{\Big(F(x_a) - F(s)\Big)^{\frac{1}{p}}} = \int_R^{M_a} \frac{u_a'(r)}{\Big(F(x_a) - F(u_a(r))\Big)^{\frac{1}{p}}}\, dr,$$

then

$$\int_0^{x_a} \frac{ds}{\Big(F(x_a)-F(s)\Big)^{\frac{1}{p}}} \geq (p')^{\frac{1}{p}}(M_a - R) > 0. \tag{18}$$

First we estimate the integral on $[\frac{x_a}{2}, x_a]$ for a sufficiently large. From (H2) we have for x large enough that $f(x) \geq \frac{1}{2}x^q$ and since $u_a(M_a) = x_a \to \infty$ as $a \to \infty$ we therefore have for a large enough that

$$Q_a = \min_{[\frac{x_a}{2}, x_a]} f \geq \frac{1}{2^{q+1}}(x_a)^q.$$

As $p < q+1$ then it follows that

$$\frac{x_a^{1-\frac{1}{p}}}{Q_a^{\frac{1}{p}}} \leq 2^{\frac{q+1}{p}}(x_a)^{\frac{p-q-1}{p}} \to 0, \text{ as } \quad a \to \infty.$$

Thus

$$\lim_{a\to\infty} \frac{(x_a)^{1-\frac{1}{p}}}{Q_a^{\frac{1}{p}}} = 0. \tag{19}$$

Since $F(u)$ is increasing for u large enough, it follows that for $\frac{x_a}{2} \leq t \leq x_a$ we have by the mean value theorem for some c such that $\frac{x_a}{2} \leq t < c < x_a$:

$$F(x_a) - F(t) = f(c)\,(x_a - t) \geq Q_a\,(x_a - t). \tag{20}$$

Thus

$$\begin{aligned}\int_{\frac{x_a}{2}}^{x_a} \frac{dt}{\Big(F(x_a)-F(t)\Big)^{\frac{1}{p}}} &\leq (\frac{1}{Q_a})^{\frac{1}{p}} \int_{\frac{x_a}{2}}^{x_a} \frac{dt}{\Big(x_a - t\Big)^{\frac{1}{p}}} \\ &\leq \Big(\frac{p}{(p-1)\,2^{1-\frac{1}{p}}}\Big) \frac{(x_a)^{1-\frac{1}{p}}}{Q_a^{\frac{1}{p}}}.\end{aligned}$$

From (19) we therefore have

$$\lim_{a\to\infty} \int_{\frac{x_a}{2}}^{x_a} \frac{dt}{\Big(F(x_a)-F(t)\Big)^{\frac{1}{p}}} = 0. \tag{21}$$

Next we estimate the integral of left-hand side of (18) on $[0, \dfrac{x_a}{2}]$. Then we have $F(t) \le F(\dfrac{x_a}{2})$ for all $t \in [0, \dfrac{x_a}{2}]$ and a sufficiently large. Thus by (20) we have

$$F(x_a) - F(t) \ge F(x_a) - F(\frac{x_a}{2}) \ge Q_a \frac{x_a}{2}.$$

Then it follows that

$$\int_0^{\frac{x_a}{2}} \frac{dt}{\Big(F(x_a) - F(t)\Big)^{\frac{1}{p}}} \le 2^{\frac{1}{p}-1} \frac{(x_a)^{1-\frac{1}{p}}}{Q_a^{\frac{1}{p}}}.$$

Therefore by (19) we have

$$\lim_{a\to\infty} \int_0^{\frac{x_a}{2}} \frac{dt}{\Big(F(x_a) - F(t)\Big)^{\frac{1}{p}}} = 0. \tag{22}$$

Combining (21) and (22), we conclude that

$$\lim_{a\to\infty} \int_0^{x_a} \frac{dt}{\Big(F(x_a) - F(t)\Big)^{\frac{1}{p}}} = 0.$$

Hence from (18) we see that $M_a \to R$ as $a \to \infty$. This completes the proof of Lemma 6. □

Lemma 7. *Assume* (H1)–(H4) *hold. Then* $u_a > 0$ *on* (R, ∞) *for* $a > 0$ *sufficiently small.*

Proof. If $M_a = \infty$, then we have $u_a(r) > 0$ for each $r > R$ and so we are done in this case.

If $M_a < \infty$, there are two cases:

1. If $u_a(M_a) < \gamma$, since $E_a(M_a) = F(u_a(M_a)) < 0$ then by virtue of *(i)* in Proposition 5 we have $u_a > 0$ on $[M_a, \infty)$, as $u'_a > 0$ on $[R, M_a)$ so $u_a > 0$ on (R, ∞) then it follows we are done in this case as well.

2. If $u_a(M_a) \ge \gamma$, so there exists two real r_a and s_a with $R < r_a < s_a < M_a$ such that $u_a(r_a) = \beta$ and $u_a(s_a) = \frac{\beta+\gamma}{2}$. By the monotonicity of E_a we get
$$E_a(r) = \frac{|u'_a|^p}{p'} + F(u_a) \le \frac{a^p}{p'}, \quad \forall r \ge R.$$

Therefore

$$\frac{|u_a'|}{\Big(a^p - p'F(u_a)\Big)^{\frac{1}{p}}} \le 1 \quad \forall r \ge R. \tag{23}$$

As $u_a' > 0$ on $[R, r_a]$ and by integrating (23) on $[R, r_a]$, we obtain

$$\begin{aligned}\int_0^\beta \frac{dt}{\Big(a^p - p'F(t)\Big)^{\frac{1}{p}}} &= \int_R^{r_a} \frac{u_a'(r)}{\Big(a^p - p'F(u_a(r))\Big)^{\frac{1}{p}}}\, dr \\ &\le r_a - R.\end{aligned}$$

Using (H4) and Remark 1, then we see that

$$\int_0^\beta \frac{dt}{\Big(a^p - p'F(t)\Big)^{\frac{1}{p}}} \to \Big(\frac{p-1}{p}\Big)^{\frac{1}{p}} \int_0^\beta \frac{dt}{|F(t)|^{\frac{1}{p}}} = \infty \quad \text{as} \quad a \to 0^+.$$

Thus it follows that $r_a \to \infty$, also $s_a \to \infty$ as $a \to 0^+$. Next, by virtue of (13) and (14) we have

$$\Big(r^\alpha E_a(r)\Big)' = \Big(r^\alpha\Big)' F(u_a(r)), \tag{24}$$

where $\alpha = p'(N-1) > 1$ because $N \ge 2$ and $p' > 1$. Integrating the above equation on $[r_a, s_a]$ and using (H3), we obtain

$$\begin{aligned} s_a^\alpha E_a(s_a) - r_a^\alpha E_a(r_a) &= \int_{r_a}^{s_a} \Big(r^\alpha\Big)' F(u_a(r))\, dr \\ &\le F(\frac{\beta+\gamma}{2}) \int_{r_a}^{s_a} \Big(r^\alpha\Big)' dr \\ &\le F(\frac{\beta+\gamma}{2}) \Big(s_a^\alpha - r_a^\alpha\Big). \end{aligned}$$

As $F(u_a(r)) \le 0$ on $[R, r_a]$ and by (24) we have $r \to r^\alpha E_a(r)$ is decreasing. Integrating again (24) on $[R, r_a]$, we obtain

$$r_a^\alpha E_a(r_a) = R^\alpha E_a(r_a) + \int_R^{r_a} \Big(r^\alpha\Big)' F(u_a(r))\, dr \le R^\alpha \frac{a^p}{p'}.$$

Hence it follows that

$$s_a^\alpha E_a(s_a) \le R^\alpha \frac{a^p}{p'} + F(\frac{\beta+\gamma}{2}) \Big(s_a^\alpha - r_a^\alpha\Big). \tag{25}$$

Now, by means value theorem and since $\alpha > 1$ we therefore have,

$$s_a^\alpha - r_a^\alpha \geq \alpha\, r_a^{\alpha-1}\,(s_a - r_a). \tag{26}$$

By integrating (23) on $[r_a, s_a]$, we see that

$$\int_\beta^{\frac{\beta+\gamma}{2}} \frac{dt}{\Big(a^p - p'F(t)\Big)^{\frac{1}{p}}} = \int_{r_a}^{s_a} \frac{u_a'(r)}{\Big(a^p - p'F(u_a(r))\Big)^{\frac{1}{p}}}\,dr$$
$$\leq s_a - r_a.$$

Using (4), taking $0 < a < 1$ and for each $t \in [\beta, \frac{\beta+\gamma}{2}]$ we get

$$a^p - p'F(t) \leq 1 + p'F_0.$$

Therefore

$$s_a - r_a \geq \frac{\gamma-\beta}{2}\Big(1 + p'F_0\Big)^{-\frac{1}{p}}.$$

From (26) we deduce that

$$s_a^\alpha - r_a^\alpha \geq \frac{\gamma-\beta}{2}\Big(1 + p'F_0\Big)^{-\frac{1}{p}}\,\alpha\, r_a^{\alpha-1}.$$

As $\lim\limits_{a\to 0^+} r_a = \infty$ it follows that

$$\lim_{a\to 0^+} s_a^\alpha - r_a^\alpha = \infty.$$

Then by virtue of (25) and since $F(\frac{\beta+\gamma}{2}) < 0$ we have

$$\lim_{a\to 0^+} s_a^\alpha\, E_a(s_a) = -\infty.$$

Hence, for small enough positive a we get $E_a(s_a) < 0$. Thus by *(i)* in Proposition 5 it follows that $u_a > 0$ on $[s_a, \infty)$ if a is sufficiently small positive. As $u_a > 0$ increasing on $[R, s_a)$ which implies that $u_a > 0$ on $(R, s_a]$. Hence $u_a > 0$ on (R, ∞) for small enough positive a. This completes the proof of Lemma 7. ☐

Lemma 8. *Assume* (H1)–(H4) *hold. Then u_a has only simple zeros on $[R, \infty)$.*

Proof. The proof of this lemma is similar to [11, Lemma 2.6]. Assume to the contrary that u_a reaches a double zero at some point $r_0 > R$. Let

$$r_1 = \inf_{r>R} \left\{ r \quad | \quad u_a(r) = u_a'(r) = 0 \right\}.$$

As $u_a(r_0) = u_a'(r_0) = 0$ then $R \le r_1 < \infty$. First, we like to show that $r_1 = R$.

By contradiction we assume that $r_1 > R$ and let $r \in (\frac{R + r_1}{2}, r_1)$ fixed. Integrating (14) on (r, r_1) we see that

$$E_a(r_1) - E_a(r) = -\int_r^{r_1} \frac{(N-1)|u_a'|^p}{t} \, \mathrm{d}t.$$

From (13) and $E_a(r_1) = 0$ we see that

$$E_a(r) = \frac{|u_a'|^p}{p'} + F(u_a) = \int_r^{r_1} \frac{(N-1)|u_a'|^p}{t} \, \mathrm{d}t. \tag{27}$$

Denote $w = \int_r^{r_1} \frac{(N-1)|u_a'|^p}{t} \, \mathrm{d}t$ and differentiating gives

$$-\frac{N-1}{r} |u_a'|^p = w'.$$

Substituting in (27), we obtain

$$w' + \frac{\alpha}{r} w = \frac{\alpha}{r} F(u_a), \tag{28}$$

where $\alpha = p'(N-1) > 1$. Multiplying both sides by r^α we have

$$(r^\alpha w)' = \alpha r^{\alpha-1} F(u_a).$$

Integrating the above on $[r, r_1]$ we obtain

$$r_1^\alpha \, w(r_1) - r^\alpha \, w(r) = \alpha \int_r^{r_1} t^{\alpha-1} F(u_a) \, \mathrm{d}t.$$

As $u(r_1) = 0$, so for r sufficiently close to r_1 we have $|u_a| \le \beta$ and by (H3) implies that $F(u_a) < 0$ on (r, r_1). Since $w(r_1) = 0$, then it follows for r sufficiently close to r_1

$$w = \frac{\alpha}{r^\alpha} \int_r^{r_1} t^{\alpha-1} |F(u_a)| \, \mathrm{d}t.$$

Combining the equation above and (28), we therefore have, for r sufficiently close to r_1

$$|u_a'(r)|^p = p' \Big(|F(u_a(r))| + \frac{\alpha}{r^\alpha} \int_r^{r_1} t^{\alpha-1} |F(u_a)| \, \mathrm{d}t \Big). \tag{29}$$

Notice that for $r < r_1$ and r sufficiently close to r_1 then $u_a'(r) \neq 0$. Otherwise there exists $r_2 < r_1$ such that $u_a'(r_2) = 0$ then by (29) we deduce that $u_a \equiv 0$ on (r_2, r_1) and by continuity we have $u_a(r_2) = u_a'(r_2) = 0$. Which contradict the definition of r_1. Thus without loss of generality we assume that $u_a' < 0$ for $r < r_1$ with r is sufficiently close to r_1. Thus we have $0 < u_a \leq \beta$ on (r, r_1). Since F is decreasing on $[0, \beta]$, therefore it follows that $|F(u_a(r))| > |F(u_a(t))| > 0$ for each $r < t < r_1$. From (29) we therefore have

$$\begin{aligned} |u_a'|^p &\leq p'\Big(|F(u_a(r))| + \frac{|F(u_a(r))|}{r^\alpha}\,(r_1^\alpha - r^\alpha)\Big) = p'|F(u_a(r))|\Big(\frac{r_1}{r}\Big)^\alpha \\ &\leq p' 2^\alpha\,|F(u_a(r))| \quad (\text{since} \quad \frac{r_1}{r} < 2 - \frac{R}{r} < 2). \end{aligned}$$

Thus

$$|u_a'| \leq C_{p,\alpha}|F(u_a(r))|^{\frac{1}{p}},$$

where $C_{p,\alpha} = (p' 2^\alpha)^{\frac{1}{p}}$. Dividing by $|F(u_a(r))|^{\frac{1}{p}}$, integrating the above on $[r, r_1]$ and using (H4) we have

$$\infty = \int_0^{u_a(r)} \frac{1}{|F(s)|^{\frac{1}{p}}}\,\mathrm{d}s = \int_r^{r_1} \frac{|u_a'(t)|}{|F(u_a(t))|^{\frac{1}{p}}}\,\mathrm{d}t \leq C_{p,\alpha}\,(r_1 - r) < \infty.$$

This is impossible, therefore $r_1 = R$ implying $u_a'(R) = 0$. This is a contradiction with $u_a'(R) = a > 0$. Hence u_a has only simple zeros on $[R, \infty)$. This completes the proof of Lemma 8. □

3. Solutions with a prescribed number of zeros

In this section we are interested to study the behaviour of zeros of solution to (7)-(8) assuming (H1)–(H4) hypotheses. By virtue of Lemma 7, if a is small enough we saw that u_a has no zeros on (R, ∞) and by Lemma 8, we get u_a has only simple zeros. In the following as in [9, 11] we want to show that u_a has an arbitrary large number of zeros on (R, ∞) for a large enough. Which leads to the existence of sign changing solution. Let $\lambda_a^{\frac{p}{q-p+1}} = u_a(M_a)$ and we define

$$v_{\lambda_a}(r) = \lambda_a^{\frac{-p}{q-p+1}}\,u_a(M_a + \frac{r}{\lambda_a}) \quad \forall r \geq 0.$$

By (7), it is straightforward to show that

$$\Big((\lambda_a M_a + r)^{N-1}\,\Phi_p(v_{\lambda_a}')\Big)'$$

$$+\lambda_a^{\frac{-pq}{q-p+1}}\left(\lambda_a M_a + r^{N-1}\right)^{N-1} f(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a}) = 0, \tag{30}$$

$$v_{\lambda_a}(0) = 1 \quad , \quad v'_{\lambda_a}(0) = 0. \tag{31}$$

By Lemma 6 and $q - p + 1 > 0$ we have that

$$\lim_{a\to\infty} \lambda_a = \infty. \tag{32}$$

Lemma 9. *As $a \to \infty$, $v_{\lambda_a} \to v$, uniformly on compact subsets of $[0, \infty)$, and v satisfies*

$$\left(\Phi_p(v')\right)' + |v|^{q-1}v = 0, \tag{33}$$

$$v(0) = 1, \quad v'(0) = 0. \tag{34}$$

Proof. By virtue of (H2) we have

$$F(u) = \frac{1}{q+1}|u|^{q+1} + G(u),$$

where $G(u) = \int_0^u g(s)\,\mathrm{d}s$. Then it follows that

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^{q+1}} = \frac{1}{q+1} + \lim_{|u|\to\infty} \frac{G(u)}{|u|^{q+1}}.$$

By using the L'Hospital's rule we have,

$$\lim_{u\to\infty} \frac{G(u)}{u^{q+1}} = 0.$$

Since G is even then

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^{q+1}} = \frac{1}{q+1}. \tag{35}$$

Let

$$E_{\lambda_a}(r) = \frac{|v'_{\lambda_a}|^p}{p'} + \lambda_a^{\frac{-p(q+1)}{q-p+1}} F(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a}).$$

By differentiating we therefore have

$$E'_{\lambda_a}(r) = -\frac{(N-1)|v'_{\lambda_a}|^p}{r} \le 0.$$

Then E_{λ_a} is decreasing on $[0, \infty)$ which implies that

$$E_{\lambda_a}(r) \le E_{\lambda_a}(0) = \lambda_a^{\frac{-p(q+1)}{q-p+1}} F(\lambda_a^{\frac{p}{q-p+1}}).$$

By (32)–(35) we see that

$$\lambda_a^{\frac{-p(q+1)}{q-p+1}} F(\lambda_a^{\frac{p}{q-p+1}}) \to \frac{1}{q+1} \text{ as } a \to \infty.$$

Thus for a large enough we see that

$$E_{\lambda_a}(r) \leq \frac{2}{q+1} \quad \forall r \geq 0.$$

By (4) and (32) for a large enough, it follows that

$$\frac{|v'_{\lambda_a}|^p}{p'} \leq \frac{2}{q+1} + \lambda_a^{\frac{-p(q+1)}{q-p+1}} F_0 \leq \frac{3}{q+1} \quad \forall r \geq 0.$$

Hence we have $|v'_{\lambda_a}|$ is uniformly bounded on $[0,\infty)$ by $M_{p,q} = \Big(\frac{3p'}{q+1}\Big)^{\frac{1}{p}}$, for a sufficiently large. From Lemma 6 we obtain

$$|v_{\lambda_a}| \leq \lambda_a^{\frac{-p}{q-p+1}} u_a(M_a) = 1.$$

Thus $|v_{\lambda_a}|$ and $|v'_{\lambda_a}|$ are uniformly bounded. By Arzela-Ascoli's theorem, for some subsequence still denoted v_{λ_a}, we have $v_{\lambda_a} \to v$ uniformly on compact subsets of $[0,\infty)$ and v is continuous.

On the other hand, by integrating (30) on $[0,r]$ and using (31), we obtain

$$v'_{\lambda_a} = -\Phi_{p'}\Big(\int_0^r \Big(\frac{\lambda_a M_a + t}{\lambda_a M_a + r}\Big)^{N-1} \Big(|v_{\lambda_a}|^{q-1} v_{\lambda_a} + \lambda_a^{\frac{-pq}{q-p+1}} g(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a})\Big)\, dt\Big). \tag{36}$$

From (H2) and since g is continuous, then for all $\epsilon > 0$ there exists $C > 0$ such that

$$|g(u)| \leq C + \epsilon |u|^q \quad \forall u \in \mathbf{R},$$

it follows that

$$|g(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a})| \leq C + \epsilon\, \lambda_a^{\frac{pq}{q-p+1}} \quad \Big(\text{since } |v_{\lambda_a}| \leq 1\Big).$$

Thus

$$\lambda_a^{\frac{-pq}{q-p+1}} |g(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a})| \leq \lambda_a^{\frac{-pq}{q-p+1}} C + \epsilon.$$

By virtue of (32) we have $\lambda_a^{\frac{-pq}{q-p+1}} \to 0$ as $a \to \infty$. Then it follows that, for a sufficiently large

$$\lambda_a^{\frac{-pq}{q-p+1}} |g(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a})| \leq 2\epsilon,$$

which implies that $\lambda_a^{\frac{-pq}{q-p+1}} g(\lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a}) \to 0$ as $a \to \infty$ uniformly on $[0, \infty)$. From (32) we have $\lambda_a M_a \to \infty$ as $a \to \infty$, we therefore have $\Big(\dfrac{\lambda_a M_a + t}{\lambda_a M_a + r}\Big)^{N-1} \to 1$, as $a \to \infty$ uniformly on compact subsets of $[0, \infty)$. Since $v_{\lambda_a} \to v$ as $a \to \infty$ uniformly on compact subsets of $[0, \infty)$ and by (36) we deduce that

$$v'_{\lambda_a} \to -\Phi_{p'}\Big(\int_0^r |v|^{q-1} v\, dt\Big) \equiv w \quad \text{pointwise on } [0, \infty),$$

w is continuous. Furthermore from

$$v_{\lambda_a} = 1 + \int_0^r v'_{\lambda_a}\, dt,$$

since $v_{\lambda_a} \to v$ as $a \to \infty$ uniformly on compact subsets of $[0, \infty)$, $v'_{\lambda_a} \to w$ as $a \to \infty$ (pointwise) and $|v'_{\lambda_a}|$ is uniformly bounded by $M_{p,q}$, therefore by applying the dominated convergence theorem we have

$$v = 1 + \int_0^r w(t)\, dt.$$

Thus $v' = w$. Then it follows from (36) that

$$v' = -\Phi_{p'}\Big(\int_0^r |v|^{q-1} v\, dt\Big).$$

Hence $v \in C^1[0, \infty)$ and v satisfies (33)–(34). This ends the proof of Lemma 9. ☐

Lemma 10. *Let v be as in Lemma 9. Then v has a zero on $[0, \infty)$.*

Proof. Suppose by the way of contradiction that $v > 0$ on $[0, \infty)$. By integrating (33) on $[0, r]$ we obtain

$$-\Phi_p(v') = \int_0^r v^q\, dt > 0.$$

So $v' < 0$ and v is decreasing on $[0, \infty)$. It follows that

$$|v'(r)|^{p-1} = \int_0^r v^q\, dt \geq r\, v^q(r),$$

which implies that

$$\frac{-v'(r)}{v^{\frac{q}{p-1}}(r)} \geq r^{\frac{1}{p-1}}.$$

By integrating the above on $[0, r]$ we get

$$\frac{p-1}{q-p+1}\left(v^{-\frac{q-p+1}{p-1}}(r)-1\right) \geq \frac{p-1}{p}\, r^{\frac{p}{p-1}},$$

as $q-p+1>0$ we see that

$$v^{-\frac{q-p+1}{p-1}}(r) \geq 1+\frac{q-p+1}{p}\, r^{\frac{p}{p-1}} \geq \frac{q-p+1}{p}\, r^{\frac{p}{p-1}},$$

therefore

$$v^{q+1} \leq C_{p,q}\, r^{\frac{-p(q+1)}{q-p+1}},$$

where

$$C_{p,q} = \Big(\frac{p}{q-p+1}\Big)^{\frac{(q-p+1)(q+1)}{p-1}}.$$

Therefore it follows that

$$\int_1^\infty v^{q+1}(t)\, dt \leq C_{p,q} \int_1^\infty r^{\frac{-p(q+1)}{q-p+1}}\, dt < \infty.$$

By continuity of v we get v is bounded on compact set $[0,1]$ so

$$\int_0^\infty v^{q+1}(t)\, dt < \infty. \tag{37}$$

Let for each $r \geq 0$,

$$E(r) = \frac{|v'(r)|^p}{p'} + \frac{1}{q+1}|v(r)|^{q+1}.$$

From (33) we obtain $E'(r) = 0$ on $[0,\infty)$, it implies that $E \equiv E(0) = \dfrac{1}{q+1}$. Therefore it follows that

$$|v'(r)|^p = \frac{p'}{q+1}\Big(1-|v(r)|^{q+1}\Big). \tag{38}$$

Since $v>0$ and from (33) we see that

$$\Big(v\,\Phi_p(v')\Big)' = v'\,\Phi_p(v') - \Big(\Phi_p(v')\Big)' = |v'|^p - v^{q+1}.$$

Integrating this on $[0, r]$ and using (34) we have

$$v(r)\,\Phi_p(v'(r)) = \int_0^r |v'|^p - v^{q+1}\, dt,$$

since $v > 0$ and $v' < 0$, thus it follows that

$$\int_0^\infty |v'|^p \, dt \leq \int_0^\infty v^{q+1} \, dt < \infty. \tag{39}$$

By integrating (38) on $[0, r]$ we obtain

$$\int_0^r |v'|^p dt = \frac{p'}{q+1} \int_0^r (1 - v^{q+1}) \, dt = \frac{p'}{q+1}\Big(r - \int_0^r v^{q+1} \, dt\Big).$$

Letting $r \to \infty$ and using (37) we have

$$\int_0^\infty |v'|^p \, dt = \infty.$$

Which contradicts (39). This completes the proof of Lemma 10. □

Lemma 11. *If a is sufficiently large. Then u_a has an arbitrarily large number of zeros on (R, ∞).*

Proof. We begin to establish the following claim.

Claim: v has an infinite number of zeros on $[0, \infty)$.

Indeed, by Lemma 10 v has a zero $z_1 > 0$ such that $v > 0$ and $v' < 0$ on $(0, z_1)$. Next we want to show that v has a first local minimum on (z_1, ∞) denoted m_1. So, suppose by contradiction that v is decreasing on $[0, \infty)$. Since v is bounded and decreasing then $\lim\limits_{r\to\infty} v(r) = \ell$ exists. As in the proof of *(ii)* Proposition 5 we therefore have, $\lim\limits_{r\to\infty} v'(r) = 0$ and $\ell = 0$. From (38) and letting $r \to \infty$ we obtain

$$\lim_{r\to\infty} |v'(r)|^p = \frac{p'}{q+1} > 0.$$

This contradicts to $\lim\limits_{r\to\infty} v'(r) = 0$. Hence, v has a first minimum denoted $m_1 > z_1$ and $v_1 = v(m_1) < 0$. Thus v satisfies,

$$\Phi_p(v') = -\int_{m_1}^r |v|^{q-1} v \, dt, \quad \forall r > m_1$$
$$v(m_1) = v_1 \quad \text{and} \quad v'(m_1) = 0.$$

In a similar way of Lemma 10 we can show that v has a second zero at $z_2 > z_1 > R$ and has a second extremum $m_2 > z_2$. Continuing in this way we can get an arbitrarily large number of zeros for v on $[0, \infty)$. Which is complete the proof of claim. □

Now, since $v_{\lambda_a} \to v$ as $a \to \infty$ uniformly on compact subsets of (R, ∞). By virtue of the previous claim and (32), it follows that v_{λ_a} has an arbitrary large number of zeros for a large enough. Finally, as $u_a(M_a + \frac{r}{\lambda_a}) = \lambda_a^{\frac{p}{q-p+1}} v_{\lambda_a}(r)$, hence we can get as many zeros of u_a as desired on (R, ∞) for a sufficiently large. This ends the proof of Lemma 11.

4. Proof of Main Result

For $k \geq 0$ defined by set

$$S_k = \{a > 0 \quad | \quad u_a(r) \quad \text{has exactly } k \text{ zeros on } \ (R, \infty)\}$$

If a is sufficiently small it follows by Lemma 7 that $u_a > 0$ for any $r > R$, so $S_0 = \{a > 0 \quad | \quad u_a(r) > 0 \ \forall r > R\}$ is nonempty and by virtue of Lemma 11 we conclude that S_0 is bounded from above. Let

$$a_0 = \sup S_0.$$

Lemma 12. *$u_{a_0} > 0$ for all $r > R$.*

Proof. Assume to the contrary, that there exists a zero $z_0 > R$ of u_{a_0} . Then $u_{a_0}(z_0) = 0$ and $u_{a_0} > 0$ on (R, z_0) and by Lemma 8 we have $u'_{a_0}(z_0) < 0$. Thus $u_{a_0} < 0$ on $(z_0, z_0 + \epsilon)$ for some $\epsilon > 0$. If a close to a_0 with $a < a_0$ and by continuous dependence of solutions on initial conditions, it follows that $u_a \leq 0$ on $(z_0, z_0 + \epsilon)$. Which contradicts to the definition of a_0. □

Lemma 13. *$E_{a_0} \geq 0$ on $[R, \infty)$.*

Proof. By the definition of a_0, if $a > a_0$ then u_a has a zero denoted $z_a > R$. We begin to show the following result

$$\lim_{a \to a_0^+} z_a = \infty. \tag{40}$$

Indeed suppose by contradiction there exists a subsequence of z_a still denoted z_a converges to some $z \in [R, \infty)$, as $a \to a_0^+$. As u_a converges uniformly on a compact set $[R, z+1]$, Therefore it follows that

$$0 = \lim_{a \to a_0^+} u_a(z_a) = u_{a_0}(z).$$

Which contradicts to Lemma 12. Hence (40) is proven.

Now, assume to the contrary that there exists $r_0 > R$ such that $E_{a_0}(r_0) < 0$, then $E_a(r_0) < 0$ for a close to a_0^+. By the definition of a_0 and taking $a > a_0$, there exists a zero z_a of u_a such that $E_a(z_a) \geq 0 > E_a(r_0)$. From the monotonicity of E_a we have $z_a < r_0 < \infty$. Which contradicts (40). This ends the proof of Lemma 13. □

Lemma 14. *u_{a_0} has a local maximum at $M_{a_0} > R$.*

Proof. By the way of contradiction, suppose that $u'_{a_0} > 0$ on $[R, \infty)$. From *(iii)* of the Proposition 5 we see that $\lim_{r\to\infty} u_{a_0}(r) = \beta$ and by the definition of a_0 if $a > a_0$ then u_a has a zero denoted $z_a > R$. Next, from *(ii)* of Proposition 5 therefore it follows that $\lim_{r\to\infty} E_{a_0}(r) = F(\beta) < 0$. From Lemma 13 we see that $E_{a_0}(r) \geq 0$. Thus $0 \leq \lim_{r\to\infty} E_{a_0}(r) = F(\beta) < 0$, this is contradictory. End of the proof of Lemma 14. □

Lemma 15. *$u'_{a_0} < 0$ on (M_{a_0}, ∞).*

Proof. We argue by the contradiction. We distinguish two cases:

(1) If there exists $r_1 > M_{a_0}$ such that $u'_{a_0}(r_1) = 0$ and $u'_{a_0} < 0$ on (M_{a_0}, r_1), or

(2) If $u'_{a_0} = 0$ on $[M_{a_0}, \infty)$.

In the first case we have $0 < u_{a_0}(r_1) < \beta$. Indeed, assume to contrary that $u_{a_0}(r_1) \geq \beta$, then by (H3) we see that $f(u_{a_0}(r_1)) \geq 0$. Further from (10) and the fact that $u'_{a_0}(M_{a_0}) = 0$ we have

$$0 = -r^{N-1}\,\Phi_p(u'_{a_0}(r_1)) = \int_{M_{a_0}}^{r_1} t^{n-1}\, f(u_{a_0}(t))\, dt. \tag{41}$$

As u_{a_0} is non-increasing on $[M_{a_0}, r_1]$ then $u_{a_0}(t) \geq u_{a_0}(r_1)$ for all $t \in [M_{a_0}, r_1]$ and by using (H3) it follows that $f(u_{a_0}(t)) \geq f(u_{a_0}(r_1)) \geq 0$. From (41) we have $f(u_{a_0}) \equiv 0$ on $[M_{a_0}, r_1]$ implying that $u \equiv \beta$ on $[M_{a_0}, r_1]$, which contradicts to $u'_{a_0} < 0$ on (M_{a_0}, r_1). Thus $0 < u_{a_0}(r_1) < \beta$ implying $E_{a_0}(r_1) = F(u_{a_0}(r_1)) < 0$. Which contradicts to Lemma 13, then it follows we are done in this case. In the second case we must have that $u = c$ on $[M_{a_0}, \infty)$. Then by *(ii)* and *(iii)* of Proposition 5 it follows that $f(c) = 0$ and $c = \beta$. Thus $0 < u_{a_0} < \beta$ on (R, M_{a_0})

and by using (H3) we obtain $f(u_{a_0}) < 0$. From (10) and $u'_{a_0}(M_{a_0}) = 0$ we see that

$$r^{N-1}\,\Phi_p(u'_{a_0}(r)) = \int_r^{M_{a_0}} t^{n-1}\, f(u_{a_0}(t))\, dt < 0.$$

Therefore we have $u'_{a_0} < 0$ on (R, M_{a_0}), this a contradiction then it follows we are done in this case as well. End of the proof of Lemma 15. □

By Lemmas 14 and 15 then it follows that $\lim_{r\to\infty} u_a(r) = \ell$ exists and by *(ii)* of Proposition 5, we have that $f(\ell) = 0$ and $\lim_{r\to\infty} E_{a_0}(r) = F(\ell)$. Then either $\ell = 0$ or $\ell = \beta$.

If $\ell = \beta$ again by *(ii)* of Proposition 5 we therefore have $\lim_{r\to\infty} E_{a_0}(r) = F(\beta) < 0$. By Lemma 13 we have $E_{a_0}(r) \geq 0$ for each $r \geq R$, so it follows that $F(\beta) = \lim_{r\to\infty} E_{a_0}(r) \geq 0$. Which contradicts to $F(\beta) < 0$. Hence $\ell = 0$ and finally we have found a non-negative solution of (5)-(6).

Next by [11, Lemma 4.3], if $a > a_0$ and $a \to a_0$ then u_a has at most one zero on (R, ∞). From the definition of a_0 if $a > a_0$ we have u_a has at least one zero. Thus for $a > a_0$ close to a_0 the solution u_a has exactly one zero. Then it follows that S_1 nonempty and by lemma 11 we see that S_1 is bounded above. Let

$$a_1 = \sup S_1.$$

As in above lemmas by using a similar argument, we can show that u_{a_1} has one simple zero and $\lim_{r\to\infty} u_{a_1}(r) = 0$. Hence, it follows that there exists a solution of (5)-(6) which has exactly one sign change in (R, ∞).

Proceeding inductively we can show that, for each $k \in \mathbf{N}$ there exists a solution $u_{a_k} = u_k$ of (5)-(6) which has exactly k zeros on (R, ∞) with $u'_k(R) > 0$.

Now, in the case $a < 0$ we consider the problem

$$\begin{gathered} \left(r^{N-1}\Phi_p(v')\right)' + r^{N-1}f(v) = 0 \quad \text{if} \quad R < r, \\ v(R) = 0, \quad v'(R) = a < 0. \end{gathered} \tag{42}$$

We denote $w(r) = -v(r)$ on $[R, \infty)$, as f and Φ_p are odd, then it follows the problem (42) is equivalent to

$$\begin{gathered} \left(r^{N-1}\Phi_p(w')\right)' + r^{N-1}f(w) = 0 \quad \text{if} \quad R < r, \\ w(R) = 0, \quad w'(R) = -a > 0. \end{gathered}$$

Next, according to the case $a > 0$ we deduce that, for each $k \in \mathbf{N}$,the problem (5)-(6) has a solution w_k which has exactly k zeros on (R, ∞) with $w'_k(R) > 0$.

Hence, for each $k \in \mathbf{N}$ integer, (5)-(6) has a solution $v_k = -w_k$ which has k zeros on (R, ∞) and $v_k'(R) < 0$. This ends the proof of Theorem 2.

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