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ON NON-ABELIAN GROUPS OF ORDER 2^n, $n \geq 4$ USING GAP

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Abstract: For every natural number $n \geq 4$ there are exactly 4 non-abelian groups (up to isomorphism) of order 2^n, with a subgroup of index 2. In this article, we are going to illustrate all of these groups properties and axioms using Groups, Algorithms and Programming GAP.

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1. Introduction

This article aims to: classify the family of all non-abelian 2-generator groups of order 2^n. By: describing the group structure, finding the order and the conjugacy classes, determining the derived subgroup and the nilpotency class of each group in this family. The Groups, Algorithms and Programming GAP will be used to support our illustrations.

For every natural number $n \geq 4$ there are exactly 4 non-abelian groups (up to isomorphism) of order 2^n, and they are:

- The dihedral group D_{2^n} is a non-abelian 2-generator group of order 2^n. This group presented by:

$$D_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = (r^k s)^2 = e \ , \ k = 1, 2, \cdots , n\rangle. \tag{1}$$

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Clearly, $\langle r\rangle \cong C_{2^{n-1}}$ is a subgroup of index 2. For more information about dihedral group of order 2^n one can see [2].

- The quasidihedral group QD_{2^n} is a non-abelian 2-generator group of order 2^n, simply presented by:

$$QD_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = e, rsr = r^{2^{n-2}}s\rangle. \tag{2}$$

Certainly, $\langle r\rangle \cong C_{2^{n-1}}$ is a subgroup of index 2.

- The generalized quaternion group GQ_{2^n} is a non-abelian 2-generator group of order 2^n. This group represented by:

$$GQ_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^4 = e, r^{2^{n-2}} = s^2, rs = sr\rangle. \tag{3}$$

Clearly, $\langle r\rangle \cong C_{2^{n-1}}$ is a subgroup of index 2.

- A 2-generator group G of order 2^n , $n \geq 4$, presented by:

$$\langle r, s \mid r^{2^{n-1}} = s^2 = e, sr = r^{2^{n-2}+1}s\rangle. \tag{4}$$

Which is isomorphic to the semidirect product of $C_{2^{n-1}}$ and C_2. Also, it has $C_{2^{n-1}}$ as a normal subgroup of index 2. We will call this group by Nondihedral group and denote it by ND_{2^n}.

2. Groups, Algorithms and Programming (GAP)

Group, Algorithms, Programming (GAP) is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects.

GAP have many build-in functions that help in group theory. The most attention is on the build-in functions used to find some required estimations for the groups studied in this research. The built-in GAP functions supposed to use the group structure and all of its elements to proceed the needed argument. This will need more time when groups have large size. The structure which contains only the build-in GAP functions well be called an ordinary GAP algorithm, which means a list of build-in functions sorting logically to obtain a certain result.

3. GAP's Groups structures

In this section, we will use GAP to introduce the group structure of D_{2^n}, QD_{2^n}, GQ_{2^n} and ND_{2^n} for $n \geq 4$. Then, using these structures to built GAP's codes that help to illustrate these groups properties, such as: Order classes, Conjugacy classes, nilpotency class and the derived subgroup of each group.

Certainly, all of the indicated groups are 2-generator 2-group. That are defined in GAP by a free group of two generators as:
`gap>F:=FreeGroup("r","s");;`

Then using `F` to define each group by the generators and relations using n that gives a group of size 2^n. The following are the GAP's codes of each group:

Algorithm 3.1 Dihedral group

```
gap> n:= ;;# set n.
gap> F:=FreeGroup("r","s");;
gap> D:=F/[F.1^(2^(n-1)),F.2^2,F.2*F.1*F.2*F.1];
```

Algorithm 3.2 Generalized Quaternion group

```
gap> n:= ;;# set n.
gap> F:=FreeGroup("r","s");;
gap> GQ:=F/[F.1^(2^(n-1)),
            F.2^4,
            F.1^(2^(n-2))*F.2^(-2),
            F.2^(-1)*F.1*F.2*F.1];
```

Algorithm 3.3 Quasidihedral group

```
gap> n:= ;;# set n.
gap> F:=FreeGroup("r","s");;
gap> QD:=F/[F.1^(2^(n-1)),
            F.2^2,
            F.2*F.1*F.2*F.1^(-(2^(n-2)-1))];
```

4. The Order Classes

A familiar concept in group theory will be produced in this section, that is the element order in a group. The order of x in G is the least positive integer k

Algorithm 3.4 Nondihedral group

```
gap> n:= ;;# set n.
gap> F:=FreeGroup("r","s");;
gap> ND:=F/[F.1^(2^(n-1)),
            F.2^2,
            F.2*F.1*F.2*F.1^(-(2^(n-2)+1))];
```

such that $x^k = e$. As a result of Lagrange's Theorem; any element x in a finite group G has a finite order $o(x)$ such that $o(x)$ divides $|G|$.

Let G be a finite group, and $x \in G$ with $o(x) = k$. Then, we need to count all $y \in G$ which has the same order as x. Precisely, we will determined a class of each order. The set of all of these classes is called the order classes of G, and denoted by OC_G. This can be obtained by: The set of all ordered pairs $[k, |O_k|]$ where k is the order of some elements in G and O_k is the set of all elements in G which have the order k. That is $O_k = \{x \in G \mid o(x) = k\}$. Then, the order classes of a group G can be written as:

$$OC_G = \{[k, |O_k|] \mid k = o(x)\,,\ \text{for}\ \ x \in G\}.$$

In this section we attend to find the order classes of all finite non-abelian 2-generator 2-group. This will done in two ways: Firstly, using the ordinary GAP's algorithm. Next, using a modified GAP's algorithm aims to improve the results and get them out easier.

Now we will use Algorithm 3.1, 3.2, 3.3 and 3.4 to find the order classes of all non-abelian 2-group of order 2^n , $n \geq 4$.

Algorithm 4.1 Order classes of dihedral group

```
gap> n:= ;;# set n.
gap> F:=FreeGroup("r","s");;
gap> D:=F/[F.1^(2^(n-1)),F.2^2,F.2*F.1*F.2*F.1];;
gap> x:=Elements(D);;
gap> Collected(List(x,i->Order(i)));
Runtime();
```

Using $n = 6$ (the first line of Algorithm 4.1), then this algorithm gives the order classes of the dihedral group of order $2^6 = 64$. The following is the result:

```
[ [1, 1], [2, 33], [4, 2], [8, 4], [16, 8], [32, 16] ]
655
```

The used time is 655 milliseconds, this time will grow fast for large n, which gives large group order. For $n = 10$, the order classes of D_{1024} were estimated in 13837 milliseconds. For $n = 13$ the time required is 1886551 milliseconds. But, for extra large n, there is no estimation and the result is "`exceed the memory size`".

To solve this issue, we will replace the ordinary GAP (Algorithm 4.1) by the next algorithm:

Algorithm 4.2 New GAP's code for the order classes of dihedral group of order 2^n

```
gap> n:= ;;# set n.
gap>Print("OC(D_(",2^n,"))=[1,1],[",2,",",2^(n-1)+1,"]");
    > for i in [3..n] do;
    >    Print(",[",2^(i-1),",",2^(i-2),"]");
    > od;
    > Print("\n");
    > Runtime();
```

Using $n = 6$ (the first line of Algorithm 4.2), this algorithm gives the order classes of the dihedral group of order $2^6 = 64$. The following are the results:

```
OC(D_(64))=[1,1],[2,33],[4,2],[8,4],[16,8],[32,16]
250
```

Clearly, Algorithm 4.2 not illustrate the set of the group elements to evaluate its order classes. It is only use n. This makes it save a lot of time and also gives the results even for large group size. The following table includes the time required for the order classes estimations of dihedral groups in both algorithms:

n	Group size	Time of Algorithm 4.1	Time of Algorithm 4.2
7	128	811	218
8	256	1294	222
11	2048	68266	260
12	4096	366040	265
20	1048576	`exceed the memory size`	272
30	1073741824	`exceed the memory size`	287

Table 1: The time (milliseconds) required to find the order classes of dihedral group by using Algorithm 4.1 and 4.2

It is clear that the new GAP's code depends only on n. which is not consider the group type and description or the elements structure, that is the reason which makes is very fast and does not produce any execution errors such as "`exceed the memory size`" when large n is used.

Similarly, we will replace the ordinary GAP's algorithms of the order classes of generalized quaternion, quasidihedral and nondihedral groups respectively by the following GAP's algorithms:

Algorithm 4.3 New GAP's code for the order classes of generalized quaternion group

```
gap> n:= ;;# set n.
gap> Print("[1,1],[2,1],[4,",2^(n-1)+2,"],");
    > for i in [3..n-1] do;
    >    Print("[",2^(i),",",2^(i-1),"],");
    > od;
    > Print("\n");
    > Runtime();
```

Algorithm 4.4 New GAP's code for the order classes of quasidihedral group

```
gap> n:= ;;# set n.
gap> Print("[1,1],[2,",2^(n-2)+1,"]",
             "[",4,",",2^(n-2)+2,"]");
    > for i in [3..n-1] do;
    >    Print(",[",2^(i),",",2^(i-1),"]");
    > od;
    > Print("\n");
    > Runtime();
```

Algorithm 4.5 New GAP's code for the order classes of nondihedral group

```
gap> n:= ;;# set n.
gap> Print("[1,1],[2,3],");
    > for i in [3..n] do;
    >    Print("[",2^(i-1),",",2^(i-1),"],");
    > od;
    > Print("\n");
    > Runtime();
```

It is worth to be mentioned, all of the new GAP's algorithms listed above are not consider the groups structure or the list of the groups elements. That makes their estimations done numerically rather than the algebraic calculations. This will save a lot of time even for large group size, which can not be estimated using the ordinary algorithms, for it needs a large size of computer memory. We are recommend to use these codes instead of the ordinary GAP's codes to find the order classes of this family of groups.

The first remark which can be obtained from the previous estimations, is: all of the groups illustrated in this study have distinct order classes for each n. This can show that, these groups are not isomorphic. See the following example:

Example 1. Let $n = 7$ and use Algorithm 4.5, 4.4, 4.3 and 4.2. Then we have:

- The order classes of D_{128} are:
  `[1,1],[2,65],[4,2],[8,4],[16,8],[32,16],[64,32]`
- The order classes of QD_{128} are:
  `[1,1],[2,33][4,34],[8,4],[16,8],[32,16],[64,32]`
- The order classes of GQ_{128} are:
  `[1,1],[2,1],[4,66],[8,4],[16,8],[32,16],[64,32]`
- The order classes of ND_{128} are:
  `[1,1],[2,3],[4,4],[8,8],[16,16],[32,32],[64,64]`

One can see that, the classes $[2, |O_2|]$ are distinct for this family. That is to say, the groups illustrated in this research have distinct numbers of elements of order 2.

5. Nilpotency Classes

The groups considered in this research are represent the family of all non-abelian group of order 2^n , $n \geq 4$. This section will introduce the nilpotency classes of this family corresponding to n.

Definition 1. ([4]) The lower central series of a group G is:

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \geq \gamma_c(G) \geq \cdots ,$$

where $\gamma_i(G) = [\gamma_{i-1}(G), G]$.

Definition 2. ([3]) A group G is called nilpotent, if $\gamma_c(G) = \{e\}$ for some positive integer c. The smallest such c is called the class of nilpotency of G.

Definition 3. ([3]) Fix a prime number p. A finite group whose order is a power of p is called a p-group.

Let G be a finite group. Then by Lagrange's Theorem, every element in G has an order that divides $|G|$. So if G is a p-group, then $|G|$ is a power of p; and so the order of any element in G is a power of p. In addition, a p-group is nilpotent [5].

Obviously, all of the groups studied in this article are 2-group. Therefore, they are nilpotent groups. The following theorems determine the nilpotency class of this family.

Lemma 1. *The derived subgroup of the groups D_{2^n} , QD_{2^n} and Q_{2^n} is of index 4 for $n \geq 4$.*

Proof. Let G be any of the groups listed above. Then, G is a 2-generator group, generated by r and s such that $\langle r \rangle$ is a normal subgroup of G. Thus, the commutator of any $x, y \in G$ is $[x, y] = r^{2k}$ for $k = 1, 2, \cdots, 2^{n-2}$. That is to say, the derived subgroup $G' = \{x \in G \mid x = r^{2k} \ , \ k = 1, 2, \cdots, 2^{n-2}\} = \langle r^2 \rangle$. Therefore, $[G : G'] = \dfrac{2^n}{2^{n-2}} = 4$. □

Theorem 1. *The groups D_{2^n} , QD_{2^n} and Q_{2^n} are nilpotent of class $n-1$ for all $n \geq 4$.*

Proof. Let G be any of the groups listed above. Using Lemma 1 we found that $[G : G'] = 4$. By Definition 2, $\gamma_{k+1}(G) = [\gamma_k(G), G] = \{e\}$ if $k = n-2$. Therefore, $\gamma_{n-1}(G)$ is the trivial group. Hence, $c = n-1$. □

Lemma 2. *The derived subgroup of $G = ND_{2^n}$ is $G' = \langle r^{n-2} \rangle$ for all $n \geq 4$.*

Proof. Let $G = ND_{2^n}$, which is generated by r and s. Let $x \in H = \langle r^{n-2} \rangle = \{e, r^{n-2}\}$. Then, $e \in G'$ and $r^{n-2} = [s, r]$. Thus, $H \subseteq G'$. Conversely, let $x \in G'$, then $x = [a, b]$ for $a, b \in G$. But, $a = r^i s$ and $b = r^j s$ for $i, j \in$

$\{1, 2, 3, \cdots, 2^{n-1}\}$. Thus,

$$\begin{aligned} x &= a^{-1}b^{-1}ab \\ &= (r^i s)^{-1}(r^j s)^{-1}(r^i s)(r^j s) \\ &= sr^{n-i}sr^{n-j}r^i sr^j s \\ &= r^{n-2(j-i)} \in H. \end{aligned}$$

Implies that $G' \subseteq H$. Hence, $G' = H = \langle r^{n-2} \rangle$. □

Theorem 2. *The group ND_{2^n} is nilpotent of class 2 for all $n \geq 4$.*

Proof. Let $G = ND_{2^n}$ for 2^n, $n \geq 4$ which is:

$$G = \langle r, s \mid r^{2^{n-2}} = s^2 = srsr^{2^{n-2}-1} \rangle$$

Using Lemma 2, we have $G' = \{e, r^{n-2}\}$, it is clear that $x^2 = 2$ for all $x \in G'$. Then, $\gamma_2(G) = [\gamma_1(G), G] = [\{e, r^{n-2}\}, G] = \{e\}$. Hence, G is nilpotent of class 2. □

6. Conjugacy Classes

Recall the definition of conjugacy; two elements x and y in a group G are conjugate if there exists $z \in G$ such that $y = zxz^{-1}$. The conjugacy class of $x \in G$ is the set of all elements in G that conjugate to x, and denoted by $Cl_G(x)$. The conjugation performs a partition for the group G, see [1]. So, if $Cl_G(x) \cap Cl_G(y) \neq \phi$, then $Cl_G(x) = Cl_G(y)$. The conjugacy classes of a group G is denoted by Cl_G and the number of such classes is $|Cl_G|$. Certainly, if G is an abelian group then $|Cl_G| = |G|$. Otherwise, $|Cl_G| < |G|$. In this section we will find the number of conjugacy classes of D_{2^n} , QD_{2^n} , Q_{2^n} and ND_{2^n}.

Lemma 3. *The centre of the group G for G is one of the groups D_{2^n} , QD_{2^n} or Q_{2^n} , $n \geq 4$ is $Z(G) = \{e, r^{n-2}\}$ for all $n \geq 4$.*

Theorem 3. *The number of the conjugacy classes of the groups D_{2^n} , QD_{2^n} and Q_{2^n} , $n \geq 4$ is $3 + 2^{n-2}$ for all $n \geq 4$.*

Proof. Let G be a group of the groups listed above of order 2^n. Then, G is a two generators group, generated by r and s such that $o(r) = 2^{n-1}$. Moreover,

$Cl_G(e) = \{e\}$ and $Cl_G(r^{2^{n-2}}) = \{r^{2^{n-2}}\}$, since $e, r^{2^{n-2}} \in Z(G)$. On the other hand, $Cl_G(s) = \{r^{2k}s \mid k = 1, 2, \cdots, 2^{n-2}\}$ and $Cl_G(rs) = \{r^{2k+1}s \mid k = 1, 2, \cdots, 2^{n-2}\}$. So there are two conjugacy classes for the set $\{r^m s \mid m = 1, 2, \cdots, 2^{n-1}\}$. Now, the other elements are all $x \in H = \langle r \rangle \setminus \{e, r^{2^{n-2}}\}$. This set contains $2^{n-1} - 2$ elements. Each element is conjugate to its inverse. That is $Cl_G(x) = \{x, x^{-1}\}$ for each $x \in H$. Therefore, there are $\frac{1}{2}\left(2^{n-1} - 2\right) = 2^{n-2} - 1$ conjugacy classes for H. Thus, the group G has $2 + 2 + 2^{n-2} - 1 = 3 + 2^{n-2}$ conjugacy classes. $\square$

Lemma 4. *The centre of the group $G = ND_{2^n}$, $n \geq 4$ is $Z(G) = \langle r^2 \rangle$ for all $n \geq 4$.*

This lemma concludes that, if $G = ND_{2^n}$ for $n \geq 4$, then $|Z(G)| = 2^{n-2}$. Clearly, $Cl_G(x) = \{x\}$ for all $x \in Z(G)$. This can help in the proof of the following theorem.

Theorem 4. *The number of the conjugacy classes of the group ND_{2^n} , $n \geq 4$ is $5(2^{n-3})$ for all $n \geq 4$.*

Proof. Let $G = ND_{2^n}$, $n \geq 4$. Then, G has 2^{n-2} conjugacy classes of the form $Cl_G(x) = \{x\}$ for all $x \in Z(G)$. Now we need to count the number of conjugacy classes of $x \in H = G \setminus Z(G)$. Clearly, $|H| = |G| - |Z(G)| = 2^n - 2^{n-2} = 3 \cdot 2^{n-2}$. For any $x \in H$, we have $|Cl_G(x)| = 2$. Thus, the number of the conjugacy classes which can be obtained from H is $|H|/2 = 3 \cdot 2^{n-3}$. Hence,

$$Cl_G = |Z(G)| + \frac{1}{2}|H| = 2^{n-2} + 3 \cdot 2^{n-3} = 2^{n-3}(2 + 3) = 5(2^{n-3}).$$

$\square$

7. Conclusions

Using arithmetic calculations instead of the algebraic calculations is more fast and easy. This research is interested in some essential estimations for the family of all 2-generator non-abelian groups of order 2^n for $n \geq 4$. We create new codes using GAP, to evaluate the order classes of these groups. The codes introduced in this research supposed to work numerically using only the group order. These estimations, save a lot of time and it is very easy to used rather than the ordinary algebraic estimations.

Similarly, as introduced in Section 4, one can establish new GAP's codes from Theorem 4, 3, 2 and 1 to find the number of conjugacy classes and the nilpotency classes for all 2-generator non-abelian groups of order 2^n , $n \geq 4$. These new codes are recommended to used rather than the ordinary algebraic estimations.

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